Discussiones Mathematicae
Probability and Statistics 37 (2017) 57–64
doi:10.7151/dmps.1193

SOME REMARKS ON THE LAW OF THE ITERATED LOGARITHM TYPE RESULTS FOR RANDOM POLYNOMIALS ON THE UNIT CIRCLE

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Abstract

We consider a complex random polynomials with independent and identically uniformly distributed random roots on the unit circle. For these random polynomials we prove some law iterated logarithm type results.

Keywords: random polynomial on the unit circle, log magnitude, strong convergence, LIL type theorems.

2010 Mathematics Subject Classification: 60F15.

1. Introduction

Random polynomials arise in many areas of mathematics, e.g. in spectral analysis of random matrices as characteristic polynomials. One of the main questions about the radom polynomials concerns the asymptotic behavior of roots of random polynomials (see e.g. [1, 5, 7, 8] and the references therein). Usually in these studies, a random polynomial is defined as an algebraic polynomial whose coefficients form a sequence of random variables (see [1]). However for some problems, e.g. in the study of the asymptotic behavior of log magnitude of the polynomial, construction of random polynomial from its roots is a more natural.

Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers and let $(z_k)_{k\in\mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) complex numbers. Then, above the pair of sequences determine the sequence of polynomials

$$P_N(z) = \prod_{k=1}^{N} (z - z_k)^{n_k} \tag{1}$$

with roots on the unit circle and their corresponding multiplicities. Using the above definition of random polynomial, and assuming that $(z_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. uniformly distributed unit magnitude complex numbers, Tucci and Whitting [9] in their considerations studied the asymptotic behavior of the logarithm of magnitude of random polynomials (1) on the unit circle i.e.,

$$L_N(\varphi) = \log\left|P_N\left(e^{i\varphi}\right)\right|^2 = \sum_{k=1}^{N} n_k \log\left(2\left(1-\cos\left(\varphi-\theta_k\right)\right)\right),$$

where $\varphi \in [0, 2\pi]$, and $\theta_k = \arg z_k$, $\theta_k \in [0, 2\pi]$ are arguments of the roots. Under some additional conditions, the authors proved a central limit theorem for $L_N(\varphi)$. In the same paper, the authors extended their univariate CLT for $L_N(\varphi)$ to the multivariate case. Furthermore, they showed that the log maximum magnitude of polynomials (1) converges weakly to strictly positive random variable. Motivated by the results obtained in [9] and using the same assumptions, we further study the asymptotic behavior of random polynomials on the unit circle. In this note, we prove some law of the iterated logarithm type results for the logarithm of magnitude of random polynomials (1) on the unit circle for the multivariate case.

2. Notation and useful lemmas

In this section we fix the notation and state some lemmas which will be needed in the subsequent sections. Throughout the note we will denote vectors, random vectors and matrices by the bold letters $\boldsymbol{a}$, $\boldsymbol{Y}$ and $\mathbf{A}$. The notations $\mathbf{0}$ and $\mathbf{I}_s$ stand for $s \times 1$ vector of zeros and $s \times s$ identity matrix. By $\|\cdot\|$ we will denote the the length of the vector in $\mathbb{R}^s$. As usual, we set $\lg x = \log(\max\{e, x\})$ and $\lg_2 = \lg(\lg x)$, and we denote by C_1 and C_2 generic positive numbers that are not necessarily the same at each appearance.

In the proofs of the main results the following lemmas will be needed.

Lemma 1 ([3])**.** *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N \in \mathbb{R}^s$ be independent random vectors with zero mean, and let*

$$\Lambda_i = E\left\|\boldsymbol{X}_i\right\|^2, \quad M_i = E\left\|\boldsymbol{X}_i\right\|^3, \quad \Lambda = \frac{1}{N}\sum_{i=1}^{N}\Lambda_i, \quad M = \frac{1}{N}\sum_{i=1}^{N} M_i,$$

and let

$$\boldsymbol{S}_N = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{X}_i.$$

Then there exist absolute constants K_1 *and* K_2 *such that for any* $x > 0$

$$P\left(\|\boldsymbol{S}_N\| \geq x\right) \leq 4\exp\left(-\frac{K_1 x^2}{\Lambda}\right) + K_2\frac{M}{\sqrt{N}x^3},$$

(*e.g. one may take* $K_1 = 1/24$ *and* $K_2 = 30000$).

Lemma 2 ([6], Theorem 3.10)**.** *Let* $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ *be i.i.d. random vectors in a separable Banach space* $\mathbb{F}$ *and let* $\boldsymbol{S}_N = \sum_{i=1}^N \boldsymbol{X}_i$*, then for* $x > 0$

$$P\left(\max_{1\leq i\leq N} \|\boldsymbol{S}_i\| \geq 6x\right) \leq 4P\left(\|\boldsymbol{S}_N\| \geq x\right).$$

Lemma 3 ([6], Corollary 3.11)**.** *Let* $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ *be i.i.d. random vectors in separable Banach space* $\mathbb{F}$*,* $\boldsymbol{S}_N = \sum_{i=1}^N \boldsymbol{X}_i$ *and let* $0 \leq a_i \leq 1$*, then for* $x > 0$

$$P\left(\left\|\sum_{i=1}^N a_i \boldsymbol{X}_i\right\| \geq 6x\right) \leq 4P\left(\|\boldsymbol{S}_N\| \geq x\right).$$

Throughout the note, in Lemmas 2 and 3, we will assume that $\mathbb{F} = \mathbb{R}^s$.

3. Main results

Let $[\varphi_1, \ldots, \varphi_s]'$ be a vector of s distinct numbers on the interval $[0, 2\pi)$ and let

$$\boldsymbol{Y}_k = [Y_{k1}, \ldots, Y_{ks}]',$$

where $Y_{ki} = \log\left(2\left(1 - \cos\left(\varphi_i - \theta_k\right)\right)\right)$, $i = 1, \ldots, s$ be the corresponding random vector. As in [9], we assume that Y_{ki} are identically distributed according to $Y = \log\left(2\left(1 - \cos\left(2\pi U\right)\right)\right)$, where U is a uniform random variable on $[0, 1]$ and $E(Y) = 0$ and $Var(Y) = \sigma^2 \approx 3.292$. Further, we assume that for a given i, $i = 1, \ldots, s$, that $(Y_{ki})_{k\in\mathbb{N}}$ are independent. Finally, let $s_N = \sum_{k=1}^N n_k^2$.

In [9], it was pointed out that $(\boldsymbol{Y}_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. random vectors with mean vector $\boldsymbol{0}$ and covariance matrix $\boldsymbol{\Sigma}$ $(s \times s)$ determined by

$$\boldsymbol{\Sigma}_{ml} = K\left(|\varphi_m - \varphi_l|\right),$$

where

$$K(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \log\left(2\left(1 - \cos(\theta)\right)\right)\log\left(2\left(1 - \cos(\varphi + \theta)\right)\right) d\theta.$$

Theorem 4. *Let* $[\varphi_1, \ldots, \varphi_s]'$ *be a vector of distinct numbers on the interval* $[0, 2\pi)$ *and let* $(\boldsymbol{Y}_k)_{k\in\mathbb{N}}$ *be a corresponding sequence of i.i.d. random vectors with* $E\boldsymbol{Y}_1 = \boldsymbol{0}$ *and* $EY_{1i}^2 = \sigma^2$, *for* $i = 1, \ldots, s$. *Then*

$$\limsup_{N\to\infty} \frac{\left\| \sum_{k=1}^{N} \frac{n_k}{\max_{1\le k\le N} n_k} \boldsymbol{Y}_k \right\|}{\sqrt{s_N \lg_2 s_N}} \le 6\sqrt{24 s\sigma^2} \qquad a.s.$$

We prepare the proof of Theorem 4 with the following lemma.

Lemma 5. *Under the assumptions of Theorem* 4 *for a fixed s we have*

(i) $\Lambda = \frac{1}{N}\sum_{k=1}^{N} E\left\|\boldsymbol{Y}_k\right\|^2 = \sigma^2 s,$

(ii) $M = \frac{1}{N}\sum_{k=1}^{N} E\left\|\boldsymbol{Y}_k\right\|^3 \le s^{3/2}\left(EY^4\right)^{3/4} < \infty.$

Proof. (i) The proof of (i) follows from the assumptions and the definition of $\|\cdot\|$.

(ii) Since, $\boldsymbol{Y}_i$, $i = 1, \ldots, N$, are i.i.d., thus we have that $M = E\left\|\boldsymbol{Y}_1\right\|^3$. Hence, to prove (ii) it suffices to how that $E\left\|\boldsymbol{Y}_1\right\|^3 \le s^{3/2}\left(EY^4\right)^{3/4} < \infty$.

Now, since $\boldsymbol{Y}_1'\boldsymbol{Y}_1$ is nonnegative random variable, thus applying the Lyapunov inequality (see [2], p. 643) and the partial moment inequality with $p = 2$ (see [2], p. 650) we get

$$E\left\|\boldsymbol{Y}_1\right\|^3 = E\left(\boldsymbol{Y}_1'\boldsymbol{Y}_1\right)^{3/2} \le \left(E\left(\boldsymbol{Y}_1'\boldsymbol{Y}_1\right)^2\right)^{3/4} = \left(E\left(\sum_{i=1}^{s} Y_{1i}^2\right)^2\right)^{3/4}$$

$$\le s^{3/4}\left(\sum_{i=1}^{s} EY_{1i}^4\right)^{3/4} = s^{3/2}\left(EY^4\right)^{3/4} < \infty,$$

since Y_{1i}, $i = 1, \ldots, s$, are identically distributed according to Y, and $EY^4 < \infty$ (≈ 123.3848) and $s \ge 1$ is fixed. This completes the proof. ■

***Proof of Theorem* 4.** Let us denote by $\chi(N) = \sqrt{s_N \lg_2 s_N}$, $N = 1, 2, 3, \ldots$, $a_k = \frac{n_k}{\max_{1\le k\le N} n_k}$, $k = 1, \ldots, N$ and

$$\boldsymbol{L}_N = \boldsymbol{L}_N(\varphi_1, \ldots, \varphi_s) = \sum_{k=1}^{N} a_k \boldsymbol{Y}_k.$$

In order to prove the theorem we show that for arbitrary small $\varepsilon > 0$ with probability 1 only finitely many events

$$\left\|\boldsymbol{L}_N\right\| > 6\sqrt{24 s\sigma^2}\,(1+\varepsilon)\,\chi(N)$$

occur.

Now, by the definition of s_N it follows that

$$s_N \to \infty, \text{ and } \frac{s_{N+1}}{s_N} \to 1 \qquad \text{as } N \to \infty. \tag{2}$$

Hence, it follows from (2) that for every τ there exist a nondecreasing sequence of integers N_k such that $N_k \to \infty$ as $k \to \infty$ and that

$$s_{N_k - 1} \le (1+\tau)^k < s_{N_k}. \tag{3}$$

Further, by (2) one can conclude that

$$s_{N_k} \sim (1+\tau)^k . \tag{4}$$

and in view of (2) and (3) one gets that $\chi(N_k)/\chi(N_{k-1}) < (1+2\tau)$.

Let $\{\boldsymbol{Y}_{N_k}, k \in \mathbb{N}\}$ be a subsequence of $\{\boldsymbol{Y}_N, N \in \mathbb{N}\}$. Applying Lemma 3 we have that

$$\begin{aligned}
&P\left(\|\boldsymbol{L}_{N_k}\| \ge 6\sqrt{24 s\sigma^2}\,(1+\varepsilon)\,\chi(N_k)\right) \\
&\le 4P\left(\left\|\sum_{i=1}^{N_k} \boldsymbol{Y}_i\right\| \ge \sqrt{24 s\sigma^2}\,(1+\varepsilon)\,\chi(N_k)\right) \\
&= 4P\left(\left\|\frac{1}{\sqrt{N_k}}\sum_{i=1}^{N_k} \boldsymbol{Y}_i\right\| \ge \sqrt{24 s\sigma^2}\,(1+\varepsilon)\,\frac{\chi(N_k)}{\sqrt{N_k}}\right).
\end{aligned}$$

Now, applying Lemma 1 with $K_1 = 1/24$ and $K_2 = 30000$, and Lemma 5 we get that

$$\begin{aligned}
&P\left(\|\boldsymbol{L}_{N_k}\| \ge 6\sqrt{24 s\sigma^2}\,(1+\varepsilon)\,\chi(N_k)\right) \\
&\le 4\exp\left(-\frac{(1+\varepsilon)^2\, 24 s\sigma^2 s_{N_k} \lg_2 s_{N_k}}{24\Lambda N_k}\right) + \frac{C_2\,(N_k)^{3/2}}{\sqrt{N_k}\,(s_{N_k}\lg_2 s_{N_k})^{3/2}} \\
&\le 4\exp\left(-\frac{(1+\varepsilon)^2\, s_{N_k} \lg_2 s_{N_k}}{N_k}\right) + \frac{C_2 N_k}{(s_{N_k}\lg_2 s_{N_k})^{3/2}} \\
&\le 4\exp\left(-(1+\varepsilon)^2 \lg_2 s_{N_k}\right) + \frac{C_2}{(s_{N_k})^{1/2}},
\end{aligned} \tag{5}$$

since, for sufficiently large k, $s_{N_k} \ge N_k$ and $\lg_2 s_{N_k} \ge 1$. Hence by (4), we obtain in (5) that

$$P\left(\|\boldsymbol{L}_{N_k}\| \ge 6\sqrt{24 s\sigma^2}\,(1+\varepsilon)\,\chi(N_k)\right) \le \frac{C_1}{k^{(1+\varepsilon)^2}} + \frac{C_2}{\left(\sqrt{1+\tau}\right)^k}.$$

Thus, the series $\sum_{k=1}^{\infty} P\left(\|\boldsymbol{L}_{N_k}\| \geq 6\sqrt{24s\sigma^2}\,(1+\varepsilon)\,\chi\,(N_k)\right)$ is convergent for every $\varepsilon > 0$ and according to Borel-Cantelli lemma

$$\limsup_{k\to\infty} \frac{\|\boldsymbol{L}_{N_k}\|}{\sqrt{s_{N_k} \mathrm{lg}_2 s_{N_k}}} \leq 6\sqrt{24s\sigma^2} \qquad a.s.$$

Now, let ε be an arbitrary, positive number. Further, let $\gamma < \varepsilon$ be another positive constant. Finally, let the positive constant τ be such that $(1+\varepsilon)\,(1+2\tau)^{-1/2} > 1+\gamma$. In order to prove that the whole sequence behaves properly, let us denote by $A_k\,(\varepsilon)$ the event

$$\max_{N_k \leq j \leq N_{k+1}} \|\boldsymbol{L}_j\| \geq 6\,(1+\varepsilon)\,\sqrt{24s\sigma^2}\chi\,(N_k)\,.$$

By the Borel-Cantelli lemma, with probability 1, at most finitely many of the events $A_k\,(\varepsilon)$ will occur, once we show that

$$\sum_{k=1}^{\infty} P\,(A_k\,(\varepsilon)) < \infty$$

for arbitrarily small $\varepsilon > 0$.

Since, $\chi(N_k) > (1+2\tau)^{-1/2}\,\chi\,(N_{k+1})$, we have

$$\begin{aligned} P\,(A_k\,(\varepsilon)) &\leq P\left(\max_{1\leq j\leq N_{k+1}} \|\boldsymbol{L}_j\| \geq 6\,(1+\varepsilon)\,\sqrt{24s\sigma^2}\chi\,(N_k)\right) \\ &\leq P\left(\max_{1\leq j\leq N_{k+1}} \left\| \frac{\max_{1\leq j\leq N_{k+1}} n_i}{\max_{1\leq j\leq N_{k+1}} n_i} \sum_{j=1}^{i} \boldsymbol{Y}_j \right\| \geq 6\,(1+\varepsilon)\,\sqrt{24s\sigma^2}\chi\,(N_k)\right) \\ &\leq P\left(\max_{1\leq i\leq N_{k+1}} \left\| \sum_{j=1}^{i} \boldsymbol{Y}_j \right\| \geq 6\,(1+\varepsilon)\,(1+2\tau)^{-1/2}\,\sqrt{24s\sigma^2}\chi\,(N_{k+1})\right) \\ &\leq P\left(\max_{1\leq i\leq N_{k+1}} \left\| \sum_{j=1}^{i} \boldsymbol{Y}_j \right\| \geq 6\,(1+\gamma)\,\sqrt{24s\sigma^2}\chi\,(N_{k+1})\right). \qquad (6) \end{aligned}$$

For sufficiently large k, $s_{N_{k+1}} \geq N_{k+1}$ and $\mathrm{lg}_2 s_{N_{k+1}} \geq 1$. By Lemmas 2, 1 and 5 we obtain in (6) that

$$\begin{aligned}
P\left(A_k\left(\varepsilon\right)\right) &\le 4P\left(\left\|\sum_{i=1}^{N_{k+1}} \boldsymbol{Y}_i\right\| \ge (1+\gamma)\sqrt{24s\sigma^2}\chi\left(N_{k+1}\right)\right) \\
&= C_1 P\left(\left\|\frac{1}{\sqrt{N_{k+1}}}\sum_{i=1}^{N_{k+1}} \boldsymbol{Y}_i\right\| \ge (1+\gamma)\sqrt{24s\sigma^2}\frac{\chi\left(N_{k+1}\right)}{\sqrt{N_{k+1}}}\right) \\
&\le C_1 \exp\left(-\frac{(1+\gamma)^2 s_{N_{k+1}} \mathrm{lg}_2 s_{N_{k+1}}}{\sqrt{N_{k+1}}}\right) + C_2 \frac{\left(N_{k+1}\right)^{3/2}}{\sqrt{N_{k+1}}\left(s_{N_{k+1}} \mathrm{lg}_2 s_{N_{k+1}}\right)^{3/2}} \\
&\le \frac{C_1}{k^{(1+\gamma)^2}} + \frac{C_2}{\left(\sqrt{1+\tau}\right)^k},
\end{aligned}$$

Thus, the series $\sum_{k=1}^{\infty} P\left(A_k\left(\varepsilon\right)\right)$ is convergent for every $\varepsilon > 0$ and according to the Borel–Cantelli lemma,

$$\limsup_{N\to\infty} \frac{\|\boldsymbol{L}_N\|}{\sqrt{s_N \mathrm{lg}_2 s_N}} \le 6\sqrt{24s\sigma^2} \qquad a.s.$$

This completes the proof. ■

Now, let us assume that $n_k = 1$ for $k = 1, 2, \ldots, N$. Then $s_N = N$ and by Theorem 4 we get the following corollary.

Corollary 6. *Let $[\varphi_1, \ldots, \varphi_s]'$ be a vector of distinct numbers on the interval $[0, 2\pi)$. Then, under the assumptions of Theorem 4*

$$\limsup_{N\to\infty} \frac{\left\|\sum_{k=1}^{N} \boldsymbol{Y}_k\right\|}{\sqrt{N \mathrm{lg}_2 N}} \le 6\sqrt{24s\sigma^2} \qquad a.s.$$

In case when $n_k = 1$, for $k = 1, 2, \ldots, N$, we can characterize the a.s. limit set for the sequence $\left(\sum_{k=1}^{N} \boldsymbol{Y_k}\right) / \sqrt{2N \mathrm{lg}_2 N}$.

Theorem 7. *Let $[\varphi_1, \ldots, \varphi_s]'$ be a vector of s distinct numbers on the interval $[0, 2\pi)$ and $(\boldsymbol{Y}_k)_{k\in\mathbb{N}}$ be a corresponding sequence of i.i.d. random vectors with $E\boldsymbol{Y}_1 = \boldsymbol{0}$ and $E\left(\boldsymbol{Y}_1\boldsymbol{Y}_1'\right) = \boldsymbol{\Sigma} > 0$. Let*

$$\widetilde{\boldsymbol{L}}_N = \widetilde{\boldsymbol{L}}_N\left(\varphi_1, \varphi_2, \ldots, \varphi_s\right) = \frac{\sum_{k=1}^{N} \boldsymbol{Y_k}}{\sqrt{2N \mathrm{lg}_2 N}}.$$

Then, the a.s. limit set of $(\widetilde{\boldsymbol{L}}_N)$ is

$$K = \left\{\boldsymbol{\Sigma a} : \boldsymbol{a}'\boldsymbol{\Sigma a} \le 1\right\} = \left\{\boldsymbol{\Sigma}^{1/2}\boldsymbol{w} : \|\boldsymbol{w}\| \le 1\right\}.$$

Proof. By the assumptions and by the Cholesky decomposition we have that $\mathbf{\Sigma}^{1/2}$ and its inverse exist. Further, one can find matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{Z} = \boldsymbol{A}\boldsymbol{Y}_1$ is random vector with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}_s$, and that $\boldsymbol{Y}_1 = \mathbf{B}\boldsymbol{Z}$ a.s. Thus, it follows that $K = \mathbf{B}K_{\boldsymbol{Z}} = B\overline{B\,(0,1)}$, and $\widetilde{\boldsymbol{L}}_N = \mathbf{B}\boldsymbol{T}_N$ a.s., where

$$\boldsymbol{T}_N = \sum_{k=1}^{N} \mathbf{A}\boldsymbol{Y}_k.$$

Let $\mathbf{A} = \mathbf{\Sigma}^{-1/2}$ and $\mathbf{B} = \mathbf{\Sigma}^{1/2}$. Then, by the generalized law of the iterated logarithm [4, Lemma 2] it follows that $(\boldsymbol{T}_N)$ is relatively compact and the set of its limit points is $\overline{B(0,1)}$, and

$$C\left(\boldsymbol{L}_N\right) = \mathbf{B}C\left(\frac{\boldsymbol{T}_N}{\sqrt{2N\lg_2 N}}\right) = \mathbf{B}K_{\boldsymbol{Z}} = K.$$

This completes the proof. ■

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Received 25 November 2016
Accepted 29 June 2017