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UNIFIED THEORY OF TESTING HYPOTHESES IN MIXED LINEAR MODELS

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Dedicated to Professor Roman Zmyślony on the occasion of this 70th Birthday

Abstract

In the paper an effective method of testing hypotheses for fixed effects and variance components in mixed linear normal models is presented. A new idea in the problem of testing hypotheses for parameters in mixed linear models was born in the construction of test for vanishing of single variance component (Michalski and Zmyślony, 1996) [9]. This test has been based on the decomposition of quadratic form of the locally best quadratic unbiased estimator of this components. The F-ratio test rejects null hypothesis if the ratio of positive and negative part of the corresponding estimator is sufficiently large. Although the construction of this exact test requires quite strong assumptions (covariance matrices commute after using the usual invariance procedure with respect to the group of translations), for many classic models, analysis of variance and regression models, we can successfully apply this idea to get the classical tests for testing hypotheses about the parameters of the corresponding model, often with larger values of the power function. In the case of testing the fixed effects is sufficient to replace linear hypothesis by the equivalent square hypothesis. Next, we consider the respective functions of the model parameters, which are squarely estimable connected with the null hypothesis. This article is a review of the most important results obtained by Michalski and Zmyślony (1996 [9], 1999 [10]) and by Gąsiorek *et al.* (2000) [2] using this idea.

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1. Introduction

Let us consider the general linear model

$$y = X\beta + \sum_{i=1}^{k} X_i\beta_i + e, \tag{1}$$

where X, $X_1, \ldots, X_k$ are known matrices, β is a p-vector of parameters corresponding to fixed effects, while $\beta_1, \ldots, \beta_k$ and e are stochastically independent random vectors normally distributed with zero mean and the covariance matrix $\sigma_i^2 I_{p_i}$ $(i = 1, \ldots, k)$ and $\sigma^2 I_n$, respectively. Under these assumptions we have the probabilistic structure for $n-$vector y of observations

$$E(y) = X\beta, \quad Var(y) = \sum_{i=1}^{k} \sigma_i^2 V_i + \sigma^2 I_n, \quad V_i = X_i X_i'. \tag{2}$$

In the paper of Michalski and Zmyślony (1996) [9], a test for vanishing a single variance component in the mixed linear normal model has been proposed. Next, under some assumptions, the authors have shown, that the traditional F-test can be derived from the best quadratic invariant unbiased estimator as a ratio of the positive and negative part of it. This approach gave rise to a unified theory of testing hypotheses both for fixed effects and variance components (see Michalski and Zmyślony, 1999 [10], Gąsiorek *et al.*, 2000 [2]). In the papers relationships between estimation and testing hypotheses for parameters of considered model have been presented. One can expect that estimators which have desirable statistical properties lead to a "good" statistical tests and a "good" confidence intervals.

2. The construction of test for variance components

Let us consider the following hypothesis

$$\mathrm{H}_0 : \sigma_{i_0}^2 = 0 \quad \text{vs} \quad \mathrm{H}_1 : \sigma_{i_0}^2 > 0, \tag{3}$$

where $i_0 \in \{1, 2, \ldots, k\}$. Olsen *et al.* (1976) [11] have derived a minimal sufficient statistics for the family of a maximal invariant statistic $t = By$ under the group $\mathcal{T}$ of translations $y \mapsto y + X\beta, \beta \in R^p$ with B being an $(n-p)$ x n-matrix such that $BB' = I_{n-p}$, $B'B = M$, $M = I - XX^+$, $p = \mathrm{rank}(X)$. The choice of a maximal invariant has no influence on the value of the optimal estimator, and properties of considered model may be presented both in terms of the subspace

$$\mathcal{V}_M = sp\{MV_1M, \ldots, MV_kM, M\}$$

and of the subspace

$$\mathcal{V}_B = sp\left\{BV_1B', \ldots, BV_kB', I_{n-p}\right\},$$

respectively. By Seely's (1971) [13] theorem, the statistical properties of the general linear model ensuring the existence of an uniformly best unbiased estimator of each estimable function $f'\sigma$ for $\sigma = (\sigma_1^2, \ldots, \sigma_k^2, \sigma^2)$ depend on the algebraic properties of the subspace $\mathcal{V}_M$ or those of the subspace $\mathcal{V}_B$, equivalently. Now, let $\mathcal{V}$ denote the smallest quadratic subspace generated by $\mathcal{V}_M = sp\{MV_1M, \ldots, MV_kM, M\}$ (the essential assumption). Hence, the exist orthogonal projections $E_1, \ldots, E_k, E_{k+1}$ which form its base. Next, for an invariant unbiased estimator of settled component $\sigma_{i_0}^2$, $i_0 \in \{1, \ldots, k\}$ in form $y'Ay$, where $A \in \mathcal{V}$, we perform a decomposition of this quadratic form into the positive and the negative part, i.e.,

$$A_+ = \sum_{j \in \mathcal{K}_1} c_j E_j \quad \text{and} \quad A_- = -\sum_{j \in \mathcal{K}_2} c_j E_j,$$

where

$$\mathcal{K}_1 = \{j : c_j > 0\} \quad \text{and} \quad \mathcal{K}_2 = \{j : c_j < 0\}$$

are disjoint subsets of $\{1, \ldots, k, k+1\}$ which in turn implies that $A_+A_- = 0$. Finally, the test statistic is built as the ratio $F = y'A_+y/y'A_-y$. The statistical properties of this test treat the Lemma 3.1 and Remark 3.1 given by Michalski and Zmyślony (1996) [9]. Moreover, from Theorem 3.1 we have that, " *If* $\mathrm{card}(\mathcal{K}_1) = 1$ *and* $\mathrm{card}(\mathcal{K}_2) = 1$, *then the statistic* F *under* H_0 *is F-distributed*". In Lemma 3.2 (1996 [9]) the authors proved that sufficient conditions guaranteeing the possession of the F-Snedecor probability distribution by the statistic F is so that both subspaces $\mathcal{V}$ and $\mathrm{sp}\{\{MV_1M, \ldots, MV_kM, M\} \setminus \{MV_{i_0}M\}\}$ are quadratic subspaces. Complementing these results is the theorem given by Fonseca *et al.* (2003) [1], which states that if $\mathcal{V}$ is Jordan algebra (a quadratic subspace) and the statistic F under H_0 is F-distributed then $\mathrm{sp}\{\{MV_1M, \ldots, MV_kM, M\} \setminus \{MV_{i_0}M\}\}$ is a commutative Jordan algebra.

3. APPLICATIONS

In this chapter we will summarize the most important examples of the application of this original method of constructing a test for testing hypotheses both for variance components and fixed effects. We will present tests of independence of normal variables, too.

3.1. Two variance components model

In this section we consider a case $k = 1$. Then the variance-covariance matrix of the model given by (1) reduces to $\sigma_1^2 V_1 + \sigma^2 I_n$ or $\sigma^2(\theta V_1 + I_n)$, where $V_1 = X_1 X_1'$ and $\theta = \sigma_1^2/\sigma^2$. It is easy to check that the quadratic subspace $\mathcal{V} = \{MV_1M, I_{n-p}\}$ is commutative (cf. Gnot and Kleffe, 1983) [3].

Let $\alpha_1 > \alpha_2 > \cdots > \alpha_{h-1} > \alpha_h = 0$ be the ordered sequence of different eigenvalues of $W = MV_1M$ with multiplicity $\nu_1, \ldots, \nu_{h-1}, \nu_h$, respectively. Thus, we have

$$W = MV_1M = \sum_{j=1}^{h-1} \alpha_j E_j \quad \text{and} \quad \mathcal{V} = \mathrm{sp}\{E_1, \ldots, E_{h-1}, E_h\},$$

where

$$E_h = I_{n-p} - \sum_{j=1}^{h-1} E_j \ \text{ with } \ \mathrm{trace}(E_j) = \nu_j.$$

Following Michalski and Zmyślony (1996, Lemma 4.1) [9] we can use a locally best unbiased quadratic invariant estimator (LBQIUE) of σ_1^2 at a point (σ_1^*, σ^*) and make the decomposition of quadratic form $y'Ay$ corresponding to given the point estimator. For comparison of different tests for verification of hypothesis

$$\text{(4)} \qquad \mathrm{H}_0 : \sigma_1^2 = 0 \quad \text{vs} \quad \mathrm{H}_1 : \sigma_1^2 > 0,$$

or equivalently

$$\text{(5)} \qquad \mathrm{H}_0(\theta) : \theta = 0 \quad \text{vs} \quad \mathrm{H}_1(\theta) : \theta > 0$$

we used F-test statistic based on Bayes LBQIUE of σ_1^2 at the point $(\sigma_1^*, \sigma^*) = (0, 1)$ such that $y'Ay$ is MINQUE for σ_1^2 (see Michalski and Zmyślony, 1996 [9] and Michalski, 2003 [7]). According to the method of construction described in Section 2 our test is based on the statistic

$$F_{\mathbf{A}_-^+} = \frac{y'A_+y}{y'A_-y} = \frac{\sum_{\alpha_i^*>0} \nu_i \alpha_i^* Z_i}{-\sum_{\alpha_i^*<0} \nu_i \alpha_i^* Z_i},$$

where $A = A_+ - A_-$ is the decomposition into the positive and the negative parts of the matrix A. Here $\alpha_i^* = \alpha_i - \mathrm{trace}(W)/\mathrm{rank}(W)$ and quadratic forms $Z_i = t'E_it/\nu_i$ are the minimal sufficient statistics for the family of distributions of a maximal invariant statistics $t = My$ under the group $\mathcal{T}$.

The hypothesis $H_0(H_0(\theta))$ against alternatives $H_1(H_1(\theta))$ is rejected for large values of the test statistics.

Remark 1. In the case $h = 2$ all tests under study are equivalent. It means that $\mathcal{V} = \text{sp}\{MV_1M, M\} = \text{sp}\{E_1, E_2\}$ is a quadratic subspace and in this case all tests are uniformly most powerful invariant (see Mathew, 1989) [6]. If $h > 2$ the uniformly most powerful invariant does not exist and in this case none test is dominating uniformly. Different tests proposed in literature based on various statistical and algebraic premises have different locally optimal properties (cf. Gnot and Michalski, 1994 [4], Michalski and Zmyślony, 1996 [9]).

Remark 2. In a comprehensive study Michalski (2009) [8] gave construction of exact confidence intervals for the variance component σ_1^2 and ratio $\theta = \sigma_1^2/\sigma^2$ in mixed linear model for the family of normal distributions $N_q(0, \sigma_1^2 W + \sigma^2 I_q)$, $q = n - p$ on the base of statistic $F_{A_-^+}$. In this way, in general case $h > 2$, he obtained a rich class of Bayesian interval estimators depending on a prior distribution on (σ_1^2, σ^2) (see also Michalski, 2003 [7]).

One of the most important examples of the application of the models with two variance components are the models corresponding to block designs.
A special case of the model (1) for $k = 1$ is a mixed two-way classification model

$$y_{ijl} = \beta_j + \tau_i + \varepsilon_{ijl}, \quad i = 1, \ldots, v; \quad j = 1, \ldots, b; \quad l = 1, \ldots, n_{ij},$$

corresponding to block design $BD(v, b, n, N)$, in which n experimental units are arranged in b blocks and treated by v treatments according to the incidence matrix $N = \Delta D'$ with possible unequal entries n_{ij}. The matrix form of the above model can be presented as follows

$$y = D'\beta + \Delta'\tau + \varepsilon$$

It is assumed that $\tau = (\tau_1, \tau_2, \ldots, \tau_v)'$ is the vector of random treatment effects while $\beta = (\beta_1, \beta_2, \ldots, \beta_b)'$ is the vector of fixed block effects. Under usually assumption that $\tau \sim N(0, \sigma_1^2 I_v)$, $\varepsilon \sim N(0, \sigma^2 I_n)$ and $\text{E}(\varepsilon\tau')$=0 the expectation and the covariance matrix of y are, respectively

$$\text{E}(y) = D'\beta, \quad \text{Var}(y) = \sigma_1^2 \Delta'\Delta + \sigma^2 I. \tag{6}$$

The elements of Δ' and D' are 0 and 1 depending on the ordering of the components y. In each case $\Delta 1 = N1 = r$ is the vector of treatment replications and $D1 = N'1 = \kappa$ is the vector of block sizes. For the model given by (6) we have $\mathcal{V} = \text{sp}\{M\Delta'\Delta M, M\}$, where $M = I - D'(DD')^{-1}D$. In our model according to above considerations to construct test $F_{\mathbf{A}_-^+}$ just find it now the spectral decomposition of $W = M\Delta'\Delta M$, i.e., $W = \sum_{i=1}^h \alpha_i E_i$, where $\alpha_1 > \alpha_2 > \cdots > \alpha_{h-1} > \alpha_h = 0$ are the eigenvalues of $M\Delta'\Delta M$ with multiplicities $\nu_1, \nu_2, \ldots, \nu_{h-1}$ and $\nu_h + \text{rank}(D')$, respectively.

3.2. Testing hypotheses for linear functions of fixed effects

In the problem of testing hypotheses for fixed parameters it is the well known F-test which accepts null hypotheses if the ratio of quadratic length of residuals under null hypotheses and hypotheses recognizing the significance of the model is less than a given critical value. Let us consider the model (1) with probabilistic structure given by (2). The usual problem for such model is testing hypothesis for linear estimable functions $H'\beta$, where H is a (p x s)-matrix. It is known (e.g. see Rao, 1973 [12]) that the functions $H'\beta$ are estimable iff the exists a (p x s)-matrix Λ such that $H = X'X\Lambda$. Michalski and Zmyślony (1999, Lemma 2.1 [10]) proved that the following hypotheses

$$\text{H}_0 : H'\beta = 0 \quad \text{vs} \quad \text{H}_1 : H'\beta \neq 0, \tag{7}$$

$$\text{H}_0^* : \beta'HH'\beta = 0 \quad \text{vs} \quad \text{H}_1^* : \beta'HH'\beta > 0, \tag{8}$$

$$\text{H}_{0,\Lambda} : \beta'H(\Lambda'X'X\Lambda)^+H'\beta = 0 \quad \text{vs} \quad \text{H}_{1,\Lambda} : \beta'H(\Lambda'X'X\Lambda)^+H'\beta > 0 \tag{9}$$

are equivalent. Here C^+ stands for Moore-Penrose general inverse of matrix C. Let us consider now Gauss-Markow model as follows

$$\text{E}(y) = X\beta, \quad \text{Var}(y) = \sigma^2 I_n.$$

Let's assume that the matrix H satisfies estimability condition. It is known that for hypothesis (7) there exists F-test if $r = \text{rank}(X) < n$ and further if $H'\beta$ is estimable then the function $\beta'HH'\beta$ is quadratic estimable. Moreover, as follows Lemma 3.1. proved by Michalski and Zmyślony (1999) [10] a quadratic form $y'Ay$ is best quadratic unbiased estimator (BQUE) of $\beta'HH'\beta$ with $A_+ = X\Lambda\Lambda'X'$ and $A_- = \text{tr}(H\Lambda')M$, where $M = I_n - XX^+$. Theorem 3.2. (Michalski and Zmyślony, 1999 [10]) shows how to construct F-test for hypotheses (8) or (9) adopting the idea (see Section 2) introduced by Michalski and Zmyślony (1996) [9]. The test has a following form

$$F = F_{A_-^+} = \frac{y'A_+^*y}{y'A_-^*y} = \frac{\hat{\beta}'X'X\Lambda(\Lambda'X'X\Lambda)^+\Lambda'X'X\hat{\beta}}{\text{rank}(H)\hat{\sigma}^2} = \frac{y'X\Lambda(\Lambda'X'X\Lambda)^+\Lambda'X'y}{\text{rank}(H)\hat{\sigma}^2},$$

where $\hat{\beta} = (X'X)^+X'y$ and $\hat{\sigma}^2 = y'My/(n - \text{rank}(X))$.

Remark 3. Since $y'A_+^*y/\sigma^2$ under hypothesis $\text{H}_{0,\Lambda}$ is chi-square distributed with $\nu_1 = \text{rank}(H)$ degrees of freedom (d.f.) and $y'My/\sigma^2 = (n-r)\hat{\sigma}^2/\sigma^2$ is chi-square distributed with $\nu_2 = n - \text{rank}(X)$ d.f. and $A_+A_- = 0$, where $A_- = \frac{\text{rank}(H)}{n-r}M$ then $F = F_{A_-^+}$ is F-distributed with ν_1 d.f. for numerator and ν_2 d.f. for denominator. It can be proven that this test is equivalent to the well known ANOVA F-test.

Theorem 4.1. in Michalski and Zmyślony (1999) [10] gives the sufficient conditions for existence of F-test in the problem of testing hypothesis (9) in general linear model given by (1) and (2), i.e, the vector of observations $y \sim N_n(X\beta, \Sigma_{i=1}^k \sigma_i^2 V_i + \sigma^2 I_n)$.

3.3. Statistical inference for two normal samples

Example 1. Test for means or difference means $\mu_1 - \mu_2$.

Let us consider a simple normal model $y_{ij} = \mu_i + e_{ij}$, $i = 1, 2$; $j = 1, \dots, n_i$; $n_1 + n_2 = n$ and $e_{ij} \sim N(0, \sigma^2)$. The matrix form of this model can be presented as follows

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{n_1} & 0 \\ 0 & \mathbf{1}_{n_2} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + e$$

with the probabilistic structure

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim N \left\{ \begin{bmatrix} \mathbf{1}_{n_1} & 0 \\ 0 & \mathbf{1}_{n_2} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \ \sigma^2 I_n \right\}.$$

We are interested in testing the following hypothesis

$$\mathrm{H}_0 : \mu_1 - \mu_2 = 0 \quad \text{vs} \quad \mathrm{H}_1 : \mu_1 - \mu_2 \neq 0 \tag{10}$$

or equivalently

$$\mathrm{H}_0^* : (\mu_1 - \mu_2)^2 = 0 \quad \text{vs} \quad \mathrm{H}_1^* : (\mu_1 - \mu_2)^2 > 0 \tag{11}$$

Since for quadratic estimators we have $y'Ay = \mathrm{trace}(Ayy') =< A, yy' >$ it is important for unbiased estimators to know the expectation yy'. We calculate that

$$\mathrm{E}(yy') = \begin{bmatrix} \mathbf{1}_{n_1}\mathbf{1}'_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \mu_1^2 + \begin{bmatrix} 0 & \mathbf{1}_{n_1}\mathbf{1}'_{n_2} \\ \mathbf{1}_{n_2}\mathbf{1}'_{n_1} & 0 \end{bmatrix} \mu_1\mu_2 + \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{1}_{n_2}\mathbf{1}'_{n_2} \end{bmatrix} \mu_2^2 + \sigma^2 I_n$$

$$= V_1\mu_1^2 + V_2\mu_1\mu_2 + V_3\mu_2^2 + V_4\sigma^2.$$

It is easy check that $\mathcal{V} = \mathrm{sp}\{V_1, V_2, V_3, V_4\}$ is quadratic space. Let the parameters a vector $\eta = (\mu_1^2, \mu_1\mu_2, \mu_2^2, \sigma^2)'$ and a matrix $W = [w_{ij}] = [\mathrm{tr}(V_iV_j)]$ for $i, j = 1, 2, 3, 4$ and a vector $w = (\mathrm{tr}(V_1yy'), \mathrm{tr}(V_2yy'), \mathrm{tr}(V_3yy'), \mathrm{tr}(V_4yy'))$. Now we solve the system of normal equations for η, i.e., $W\eta = w$ and we obtain

$$\hat{\eta} = \begin{bmatrix} \hat{\mu_1^2} \\ \widehat{\mu_1\mu_2} \\ \hat{\mu_2^2} \\ \hat{\sigma^2} \end{bmatrix} = W^{-1} \begin{bmatrix} (y_{1.})^2 \\ 2y_{1.}y_{2.} \\ (y_{2.})^2 \\ (y_{..})^2 \end{bmatrix}, \quad \text{where } W = \begin{bmatrix} n_1^2 & 0 & 0 & n_1 \\ 0 & 2n_1n_2 & 0 & 0 \\ 0 & 0 & n_2^2 & n_2 \\ n_1 & 0 & n_2 & n_1 + n_2 \end{bmatrix},$$

and

$$W^{-1} = \begin{bmatrix} \frac{s+1}{sn_1^2} & 0 & \frac{1}{sn_1n_2} & \frac{-1}{sn_1} \\ 0 & \frac{1}{2n_1n_2} & 0 & 0 \\ \frac{1}{sn_1n_2} & 0 & \frac{s+1}{sn_2^2} & \frac{-1}{sn_2} \\ \frac{-1}{sn_1} & 0 & \frac{-1}{sn_2} & \frac{1}{s} \end{bmatrix}, \quad \text{where} \quad s = n_1 + n_2 - 2.$$

We will use classical notation $y_{1.} = \sum_{j=1}^{n_1} y_{ij}$, $y_{2.} = \sum_{j=1}^{n_2} y_{ij}$ and $y_{..} = \sum_{i,j} y_{ij}$.

We can check that the unbiased estimator of the function $(\mu_1 - \mu_2)^2$ is

$$\overbrace{(\mu_1 - \mu_2)^2} = \overbrace{\mu_1^2} + \overbrace{\mu_2^2} - 2\overbrace{\mu_1\mu_2} = \sum_{i=1}^{4} \alpha_i w_i = y'Ay,$$

where $w_i = \mathrm{tr}(V_i yy') = y'V_i y$.

The result of decomposition of the quadratic form $y'Ay = y'A_+y - y'A_-y$ is as follows

$$y'A_+y = \sum_{i=1}^{4} \alpha_i^+ w_i = (\overline{y_{1.}} - \overline{y_{2.}})^2 \frac{n_1n_2}{n_1+n_2}, \quad y'A_-y = \sum_{i=1}^{4} \alpha_i^- w_i = \widehat{\sigma^2} = \overline{SSE}$$

where

$$\overline{SSE} = \frac{\sum_{j=1}^{n_1}(y_{1j} - \overline{y_{1.}})^2 + \sum_{j=1}^{n_2}(y_{2j} - \overline{y_{2.}})^2}{n_1 + n_2 - 2}$$

and $\overline{y_{1.}}$, $\overline{y_{2.}}$ are the averages in samples 1 and 2, respectively.

Now according to the idea introduced by Michalski and Zmyślony (1996) [9] the test statistic for testing hypothesis (11) is as follows

$$F = \frac{y'A_+y}{y'A_-y}$$

and under null hypothesis $\mathrm{H}_0^* : (\mu_1 - \mu_2)^2 = 0$ has F-Snedecor distribution with 1 degree of freedom for the numerator and $s = n_1 + n_2 - 2$ d.f. for the denominator.

Remark 4. Note that for the well known statistic

$$t = \frac{\overline{y_{1.}} - \overline{y_{2.}}}{\sqrt{\overline{SSE}\frac{n_1+n_2}{n_1n_2}}}$$

which under the null hypothesis $\mathrm{H}_0 : \mu_1 - \mu_2 = 0$ is t-distributed with $n_1 + n_2 - 2$ degrees of freedom there is equality $t^2 = F \sim^{\mathrm{H}_0} F_{1,s}$. Besides, we have the quantiles equality, i.e., $F_{1,s}(\alpha) = t_s^2(\alpha/2)$, where α denotes order of quantile of the probability distribution. This clearly proves that both tests are equivalent.

Example 2. Test of independence of variables or test for correlation coefficient $\rho(X, Y)$.

Let $y_i = (y_{1i}, y_{2i})'$ for $i = 1, \dots, n$ will be two-dimensional iid random variables according to the normal distribution $N(\mu, \Sigma)$ as follows

$$\mathrm{E}(y_i) = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1^2 & cov \\ cov & \sigma_2^2 \end{bmatrix}.$$

Here $cov = \mathrm{Cov}(Y_1, Y_2)$ denotes the covariance of random variables Y_1 and Y_2. The correlation coefficient $\rho(Y_1, Y_2) = \frac{\mathrm{Cov}(Y_1,Y_2)}{\sigma_1\sigma_2}$ because $\rho = 0 \longleftrightarrow cov = 0$. And consequently the hypotheses

$$\mathrm{H}_0 : \rho = 0 \quad \text{vs} \quad \mathrm{H}_1 : \rho \neq 0 \tag{12}$$

and

$$\mathrm{H}_0 : cov = 0 \quad \text{vs} \quad \mathrm{H}_1 : cov \neq 0 \tag{13}$$

are equivalent. It is proven that there exists the best unbiased estimator (BUE) $\widehat{cov}$ for cov (see Zmyślony, 1976 [14] and 1980 [15], cf. also Gąsiorek *et al.*, 2000 [2]). It is given by

$$\widehat{cov(Y_1, Y_2)} = \frac{1}{4n}\sum_{i=1}^{n}[(y_{1i}+y_{2i})-(\overline{y_{1.}}+\overline{y_{2.}})]^2 - \frac{1}{4n}\sum_{i=1}^{n}[(y_{1i}-y_{2i})-(\overline{y_{1.}}-\overline{y_{2.}})]^2 = y'Ay,$$

where $y = (y_1, y_2)$ and $y'Ay = y'A_+y - y'A_-y$ with matrices determined as follows

$$A = \begin{bmatrix} 0 & \frac{1}{2n}(I_n - \frac{\mathbf{1}_n\mathbf{1}'_n}{n}) \\ \frac{1}{2n}(I_n - \frac{\mathbf{1}_n\mathbf{1}'_n}{n}) & 0 \end{bmatrix},$$

$$A_+ = \frac{1}{4n}\begin{bmatrix} I_n - \frac{\mathbf{1}_n\mathbf{1}'_n}{n} & I_n - \frac{\mathbf{1}_n\mathbf{1}'_n}{n} \\ I_n - \frac{\mathbf{1}_n\mathbf{1}'_n}{n} & I_n - \frac{\mathbf{1}_n\mathbf{1}'_n}{n} \end{bmatrix}, \quad A_- = \frac{1}{4n}\begin{bmatrix} I_n - \frac{\mathbf{1}_n\mathbf{1}'_n}{n} & -I_n + \frac{\mathbf{1}_n\mathbf{1}'_n}{n} \\ -I_n + \frac{\mathbf{1}_n\mathbf{1}'_n}{n} & I_n - \frac{\mathbf{1}_n\mathbf{1}'_n}{n} \end{bmatrix}.$$

It is easy to check that $A_+\Sigma A_- = 0 \Leftrightarrow \sigma_1^2 = \sigma_2^2$.

Because $\mathrm{E}(y'A_+y) =^{\mathrm{H}_0^*} \mathrm{E}(y'A_-y)$, $\mathrm{E}(y'A_+y) >^{\mathrm{H}_1^*} \mathrm{E}(y'A_-y)$ $(cov > 0)$ and $\mathrm{E}(y'A_+y) <^{\mathrm{H}_1^*} \mathrm{E}(y'A_-y)$ $(cov < 0)$ and according to the idea given by Michalski and Zmyślony (1996) [9] so we have the following test statistic

$$F = \frac{y'A_+y}{y'A_-y} \sim^{\mathrm{H}_0^*} F_{n-1,n-1}, \tag{14}$$

i.e., F under H_0 has $F - Snedecor$ distribution with $n-1$ d.f. both for the numerator and the denominator and the test rejecting H_0 if $F > F_\alpha$ is unbiased (F_α is a critical value of the test at significance level α). Moreover, we can check that $\mathrm{tr}(A_+\Sigma){=}(n-1)(1+\rho)\sigma^2$ and $\mathrm{tr}(A_-\Sigma){=}(n-1)(1-\rho)\sigma^2$ therefore for any fixed ρ we have

$$\mathrm{F}(\rho) = \frac{1+\rho}{1-\rho} \cdot \frac{y'A_+y}{y'A_-y} \sim^{H_1^*} F_{n-1,n-1}. \tag{15}$$

Finally, the acceptance region of a level $\alpha-$ test for testing

$$\mathrm{H}_0 : \rho = \rho_0 \quad \text{vs} \quad \mathrm{H}_1 : \rho \neq \rho_0 \tag{16}$$

has the following form

$$\frac{1+\rho_0}{1-\rho_0} F_{1-\alpha/2,n-1,n-1} < F < \frac{1+\rho_0}{1-\rho_0} F_{\alpha/2,n-1,n-1}.$$

Thus we can construct $100\%(1-\alpha)$ confidence interval for correlation coefficient ρ (see Gąsiorek *et al.*, 2000 [2]).

Remark 5. Recall that well known the $t - Student$ test based on statistics

$$t = \frac{R}{\sqrt{1-R^2}}\sqrt{n-2},$$

where R is the sample correlation coefficient reject null hypothesis (12) if $|t| > t_{\alpha/2,n-2}$. It can be proved that the F-test is most powerful invariant with respect the group three transformations: translation, scale and orthogonal transformation on minimal sufficient statistics (the t-Student test is also invariant). Thus the power of our test is at least equal to the power of the classic $t - Student$ test (cf. Lehmann, 1986, p. 219 [5]). The simulation study in Gąsiorek *et al.* (2000) [2] shows that for the small sizes of samples (n<10) and large $|\rho|$ the F-test is much better than the $t - Student$ test. However, for larger samples sizes ($n \geq 10$) the power functions for both tests almost coincide.

Remark 6. Gąsiorek *et al.* (2000) [2] constructed a more powerful test than the *t-Student* test even when the variances are unknown but their ratio is known to be equal to a constant k. The authors examined also the behaviour of F-test in the surroundings $(k-\varepsilon, k+\varepsilon)$, i.e., examined, in the some sense, a robustness of this test to ε-disturbances of normal distribution. It is interesting that in this paper they have obtained adaptive test $F = \frac{1+R}{1-R}$ putting in F-statistics depended on k the ratio of sample variances, i.e., $k = \frac{s_2^2}{s_1^2}$, which is equivalent to the classical *t-Student* test.

Conclusion

The presented examples show how on the ground of the idea proposed by Michalski and Zmyślony you can elegantly and systematically construct tests with good statistical properties and at the same time reproduce familiar parametric tests for different linear models related to the variance analysis or regression analysis.

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