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ON THE EXTREMES OF A CLASS OF NONSTATIONARY PROCESSES WITH HEAVY TAILED INNOVATIONS

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Abstract

We consider a class of nonstationary time series defined by $Y_t = \mu_t + X_t$ and $X_t = \sum_{k=0}^{\infty} C_{t,k}\sigma_{t-k}\eta_{t-k}$ where $\{\eta_t; t \in \mathbb{Z}\}$ is a sequence of independent and identically random variables with regularly varying tail probabilities, σ_t is a scale parameter and $\{C_{t,k},\ t \in \mathbb{Z},\ k > 0\}$ an infinite array of random variables. In this article, we establish convergence of the normalized partial sum of X_t, and we deal with the asymptotic distribution for the normalized maximum. We also investigate, by Monte Carlo simulation, the goodness-of-fit of the limiting distribution.

Keywords: extreme value distributions, poisson random measure, regular varying function, nonstationary process.

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1. Introduction

In this article, we consider a class of nonstationary time series with the form.

(1) $$Y_t = \mu_t + X_t, \quad X_t = \sum_{k=0}^{\infty} C_{t,k}\eta_{t-k}\sigma_{t-k}$$

where $\{C_{t,k},\ t \in \mathbb{Z}; k \geq 0\}$ is an infinite array of positive random variables and $\{\eta_t; -\infty < t < \infty\}$ a sequence of independent and identically distributed random variables with regularly varying tail probabilities. Extreme value theory of nonstationary processes has been the purpose of investigations under certain conditions. Husler (1986) extended some results of the extreme-value theory of stationary random sequences to non-stationary random sequences.

Niu (1997) studied the limit theory for extreme values of a class of nonstationary time series with the following form

$$Y_t = \mu_t + X_t, \quad X_t = \sum_{k=0}^{\infty} c_k \eta_{t-k} \sigma_{t-k}, \tag{2}$$

where (c_k) is a sequence of real constants. In recent years, Kulik (2006) investigated the limit theory for moving average

$$X_t = \sum_{k=0}^{\infty} C_{t,k} Z_{t-k}, \tag{3}$$

where $\{C_{t,k},\ t \in \mathbb{Z}; k \geq 0\}$ is an infinite array of positive random variables.

In our purpose, we extend these two models (2) and (3) and consider nonstationary moving average process with random coefficients defined in (1).

This model is used very often in the field of environment, meteorology, hydrology, as it is able to successfully model phenomena such as extreme temperature, floods, storms and extreme ozone concentrations (see Coles [4], Eastoe and Tawn [10]).

We may give an example of model (1) for, say, ground-level ozone data $\{X_t\}$ defined by the following relation

$$X_t = \begin{cases} \phi_1 X_{t-1} + \sigma_{1t}\eta_t^{(1)}, & \text{if} \quad Y_{t-\delta} > \tau, \\ \phi_2 X_{t-1} + \sigma_{2t}\eta_t^{(2)}, & \text{if} \quad Y_{t-\delta} \leq \tau, \end{cases} \tag{4}$$

where τ and ϕ_i are non random constants and with threshold variable $Y_{t-\delta}$. Here $(\eta_t^{(1)})_{t\in\mathbb{Z}}$ and $(\eta_t^{(2)})_{t\in\mathbb{Z}}$ are sequences of iid random variables with regularly varying tail probabilities, and ϕ_1, ϕ_2 are constants parameters. We also assume that $(\eta_t^{(1)})_{t\in\mathbb{Z}}$ and $(\eta_t^{(2)})_{t\in\mathbb{Z}}$ are independent as random sequences.

The ground level ozone process has piecewise linear structure. It switches between two first order autoregressive process according to meteorological conditions, including daily temperature, relative humidity, wind speed and direction, which play an important role in determining the severity of ozone concentration.

In hydrology framework where the water level X_t is observed at a given location, $Y_{t-\delta}$ could be interpreted as threshold level upstream from that location and δ the delay (in terms of days, hours, for instance) for the raw wave to reach that location.

When we define $\mathbb{I}_{1t} = \mathbb{I}_{\{Y_{t-\delta}>\tau\}}$, $\mathbb{I}_{2t} = 1 - \mathbb{I}_{1t}$, the model (4) can be written as

$$X_t = \phi_{(t)} X_{t-1} + Z_t \tag{5}$$

where

$$\phi_{(t)} = \phi_1 \mathbb{I}_{1t} + \phi_2 \mathbb{I}_{2t} \qquad \textit{and} \qquad Z_t = \sigma_{1t}\eta_t^{(1)}\mathbb{I}_{1t} + \sigma_{2t}\eta_t^{(2)}\mathbb{I}_{2t}.$$

The equation (5) is a stochastic difference equation where the pairs $(\phi_{(t)},\ Z_t)_t$ are sequences of independent and not identically distributed $\mathbb{R}^2$-valued random variables. Its solution can be written as

$$X_t = \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j-1} \phi_{(t-k)} \right) Z_{t-j}. \tag{6}$$

The rest of this paper is organized as follows. Section 2 contains background results and tools. In Section 3, we establish the asymptotic behavior of the partial sums. In Section 4, we establish the asymptotic behavior of the partial maxima. In Section 5, we propose to estimate the parameters of the model (4). In Section 6, we investigate, by Monte Carlo simulation, the goodness-of-fit of the limiting distribution of the normalized extremes.

2. Background results and tools

2.1. Point process

Let E be a state space taken to be a subset of compactified Euclidean space (such as $\mathbb{R}^d = [-\infty; +\infty]^d$). Let $\mathcal{E}$ be the Borel σ-algebra generated by open sets. For $x \in E$ and $A \in \mathcal{E}$, define the measure ε_x on $\mathcal{E}$ by

$$\varepsilon_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{7}$$

Let $\{x_i, i \geq 1\}$ be a countable collection of (not necessarily distinct) point of the space E. A point measure m_p is defined to be a finite measure on relatively compact subsets of E of the form $m_p = \sum_{i=1}^{\infty} \varepsilon_{x_i}$ which is nonnegative integer-valued. The class of point measures is denoted by $M_p(E)$ and $\mathcal{M}_p(E)$ is the smallest σ-algebra making the evaluation maps $m \to m(F)$ measurable where $m \in M_p(E)$ and $F \in \mathcal{E}$.

Let $\mathcal{C}_K^+$ be the set of all continuous, non-negative functions on the state E with compact support. If $N_n \in M_p(E)$ then N_n converges vaguely to N

($N_n \Rightarrow N$) if $N_n(f)$ converges to $N(f)$ for every $f \in \mathcal{C}_K^+$, where $N(f) = \int f dN$. A Poisson process on $(E, \mathcal{E})$ with mean measure μ is a point process N such that, for every $A \in \mathcal{E}$, $N(A)$ is a Poisson random variable with mean measure $\mu(A)$. A Poisson process or a Poisson random measure with mean measure μ is denoted by $PRM(\mu)$.

2.2. Assumptions and preliminary results

Under the following assumptions, Diop and Diouf ([9]) established the limit theorem for point processes based on the nonstationary time series (1).

We suppose that the absolute value of each weight $C_{t,k}$ has an upper endpoint c_k defined by

$$c_k = \sup\{c : \ \mathbb{P}(|C_{t,k}| \leq c) < 1\}, \quad k = 1, 2, \ldots. \tag{8}$$

We will use the following assumptions:

H$_1$ – The sequence of random variables $\{\eta_t, \ t \in \mathbb{Z}\}$ is a sequence of independent, identically distributed (iid) random variables and satisfies the condition of regularly varying tail probabilities with index $-\alpha$

$$\mathbb{P}(|\eta_1| > x) \sim x^{-\alpha} L(x), \quad x \to \infty, \tag{9}$$

where $\alpha > 0$ and L is a slowly varying function at infinity that is $\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1$, $\forall x > 0$ and tail balancing condition,

$$\lim_{x\to\infty} \frac{\mathbb{P}(\eta_1 > x)}{\mathbb{P}(|\eta_1| > x)} = \pi_0, \quad \lim_{x\to\infty} \frac{\mathbb{P}(\eta_1 < -x)}{\mathbb{P}(|\eta_1| > x)} = 1 - \pi_0. \tag{10}$$

where $0 < \pi_0 \leq 1$. Let a_n be the $1 - n^{-1}$ quantile of $|\eta_1|$:

$$a_n = \inf\left\{x : \mathbb{P}(|\eta_1| \leq x) \geq 1 - n^{-1}\right\}. \tag{11}$$

The condition of regularly varying tail probabilities satisfied by the sequence of random variables $\{\eta_t, \ t \in \mathbb{Z}\}$ is equivalent to this vague convergence

$$n\mathbb{P}(a_n^{-1}\eta_1 \in \cdot) \to \nu(\cdot), \tag{12}$$

where ν has density

$$\nu(dx) = \alpha\pi_0 x^{-\alpha-1} dx \mathbb{I}_{(0,\,\infty]}(x) + \alpha(1-\pi_0)(-x)^{-\alpha-1} dx \mathbb{I}_{[-\infty,\,0)}(x).$$

H$_2$ – The array $\{C_{t,k}, \ t \in \mathbb{Z}, \ k \geq 0\}$ is independent of $\{\eta_t, \ t \in \mathbb{Z}\}$.

H$_3$ – For each fixed m, the sequence $\{(C_{t,0}, \ldots, C_{t,m}), \ t \in \mathbb{Z}\}$ is strongly mixing.

H_4 – For some $\delta > 0$, $\sum_{k=1}^{\infty} c_k^{1-\delta} < \infty$, $\sum_{k=1}^{\infty} \sigma_k^{\alpha} c_k^{\delta\alpha} < \infty$.

We assume that there exist $M > 0$ and $\sum_{k=1}^{\infty} \mathbb{E} \mid \sigma_{t-k} C_{1,k} \mid^{\alpha} < M$.

Furthermore we assume that for fixed $k \geq 0$,

$$\frac{1}{n} \sum_{j=1}^{n} \sigma_{j-k}^{\alpha} \to \gamma_k^{\alpha}, \quad \text{as } n \to \infty, \tag{13}$$

where $\gamma_k > 0$, for all $k \geq 0$.

Now assume that the $\mathbb{R}^{\infty}$-valued random elements $\mathbf{C}_t = \{C_{t,k},\ k \geq 0\}$ form the stationary sequence $\{\mathbf{C}_t,\ t \geq 1\}$. Assume the $\mathbb{R}^{\infty}$-valued random elements $V_t = (V_{t,0}, V_{t,1}, \ldots)$, $t \in \mathbb{Z}$ has the same distribution as $\mathbf{C}_0$.

Diop and Diouf ([9]) established the following theorem, which discusses the weak convergence of the sequence of point processes based on $(a_n^{-1} X_k)_{k \in \mathbb{N}}$ to a function of a PRM.

Theorem 1. *Suppose that the non stationary sequence* (X_t) *is given by* (1). *Assume that the conditions* H_1–H_4 *hold. Then, in the space* $M_p([-\infty, \infty] \setminus \{0\})$,

$$N_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1} X_t} \Rightarrow N = \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} \varepsilon_{j_t V_{t,k}}, \tag{14}$$

where $\sum_{t=1}^{\infty} \varepsilon_{j_t}$ *is a PRM with density*

$$\mu(dx) = \gamma_0^{\alpha} \left(\pi_0 \alpha x^{-\alpha-1} dx \mathbb{I}_{(0,\infty]}(x) + (1-\pi_0) \alpha (-x)^{-\alpha-1} dx \mathbb{I}_{[-\infty,\, 0)}(x) \right).$$

The asymptotic tail behavior for X_t defined by (6) are given by the following theorem (see [8]).

Theorem 2. *Suppose that the conditions H_1–H_3 hold, then the tail behavior distribution of X_t defined in* (1) *is:*

$$\lim_{x \to \infty} \frac{\mathbb{P}\left(\mid \sum_{k=1}^{\infty} C_{t,k} \sigma_{t-k} \eta_{t-k} \mid > x \right)}{\mathbb{P}(\mid \eta_1 \mid > x)} = \sum_{k=1}^{\infty} \mathbb{E} \mid \sigma_{t-k} C_{1,k} \mid^{\alpha}. \tag{15}$$

3. ASYMPTOTIC BEHAVIOR OF THE PARTIAL SUMS

In the case $0 < \alpha < 1$, we establish convergence of the partial sums $S_n = \sum_{t=1}^{n} a_n^{-1} X_t$, where $\{X_t\}$ is given by (1).

Theorem 3. *Assume that $0<\alpha<1$, under assumptions of Theorem 1 we have*

$$S_n=\sum_{t=1}^{n} a_n^{-1}X_t \to^d S=\sum_{t=1}^{\infty}\sum_{k=1}^{\infty} j_tV_{t,k} \text{ as } n\to\infty \tag{16}$$

here $\to^d$ denotes convergence in distribution.

Proof. Using the same arguments as in the proof of Theorem 3.1 in [5]. For any Borel set $\mathbf{B}$ in $\mathbb{R}$ we define

$$S_n\mathbf{B}=\sum_{t=1}^{n} a_n^{-1}X_t\mathbb{I}_{\mathbf{B}}(a_n^{-1}\mid X_t\mid)$$

and

$$S\mathbf{B}=\sum_{t=1}^{\infty}\sum_{k=1}^{\infty} j_tV_{t,k}\mathbb{I}_{\mathbf{B}}(\mid j_tV_{t,k}\mid).$$

For every $\varepsilon>0$, we define this continuous function

$$\begin{aligned} T: M_p(\overline{\mathbb{R}}) &\to \mathbb{R}\\ \sum_{t=1}^{\infty}\varepsilon_{x_t} &\mapsto \sum_{t=1}^{\infty} x_t\mathbb{I}_{(\varepsilon,\infty)}(\mid x_t\mid). \end{aligned}$$

Applying the continuous mapping theorem to N_n and Theorem 1, we obtain:

$$\begin{aligned} S_n(\varepsilon,\infty) &= T(N_n)\\ &\to^d T(N)=S(\varepsilon,\infty) \end{aligned}$$

Using the same arguments as in the proof of Theorem 3.1 in [5], it follows that

$$S(\varepsilon,\infty)\to S(0,\infty)=\sum_{t=1}^{\infty}\sum_{k=1}^{\infty} j_tV_{t,k}, \quad \text{as } \varepsilon\to 0.$$

To prove (16), we only have to show now that $S_n(0,\varepsilon)\to^P 0$. In fact, by Theorem 2 we have that $\mid X_t\mid$ are random variables with regularly varying tail probabilities, therefore we can use the Theorem 2 of [11] (page 275) and get the following equivalence uniformly in t

$$\mathbb{E}\left(\mid X_t\mid \mathbb{I}_{(0,a_n\varepsilon)}(\mid X_t\mid)\right)\sim \frac{\alpha}{1-\alpha}a_n\varepsilon\mathbb{P}\left(\mid X_t\mid > a_n\varepsilon\right). \tag{17}$$

Let $\beta>0$, by Markov's inequality and (17), we have,

$$\mathbb{P}(\mid S_n(0,\varepsilon)\mid>\beta) \le \beta^{-1}\mathbb{E}\left|\sum_{t=1}^{n} a_n^{-1}X_t\mathbb{I}_{(0,\varepsilon)}(\mid a_n^{-1}X_t\mid)\right|$$

$$
\begin{aligned}
&\leq \beta^{-1} a_n^{-1} \sum_{t=1}^{n} \mathbb{E}\left(\mid X_t \mid \mathbb{I}_{(0,\varepsilon)}(\mid a_n^{-1} X_t \mid)\right) \\
&\sim \frac{\alpha}{1-\alpha}\beta^{-1}\varepsilon \sum_{t=1}^{n} \mathbb{P}(\mid X_t \mid > a_n\varepsilon) \\
&\sim \frac{\alpha}{1-\alpha}\beta^{-1}\varepsilon \sum_{t=1}^{n} \mathbb{P}(a_n^{-1} \mid \eta_1 \mid > \varepsilon) \sum_{k=1}^{\infty} \mathbb{E} \mid \sigma_{1-k} C_{1,k} \mid^{\alpha} \\
&\leq \frac{\alpha}{1-\alpha}\beta^{-1}\varepsilon n \mathbb{P}(a_n^{-1} \mid \eta_1 \mid > \varepsilon) M \\
&\to \frac{\alpha}{1-\alpha}\beta^{-1} M \varepsilon^{1-\alpha} \quad \text{as} \quad n \to \infty \to 0 \quad \text{as} \quad n \to 0.
\end{aligned}
$$

Then

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \mathbb{P}(\mid S_n(0,\varepsilon) \mid > \beta) = 0.$$

By Theorem 4.2 of [2], we have

$$S_n(0,\infty) = a_n^{-1} S_n \to^d \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} j_t V_{t,k}. \qquad \blacksquare$$

4. Asymptotic behavior of the partial maxima

Let $M_n = \max\{X_1, \ldots, X_n\}$, where the processes $(X_t)_t$ is defined by (1). In this section we present the main result concerning the asymptotic distribution for the suitably $(M_n)_n$ correctly normalized.

Theorem 4. *Let $(X_t)_t$ be the process defined by the equation* (1). *Assume that the conditions* H1–H4 *hold. Then for all $x > 0$, as $n \to \infty$*

$$\mathbb{P}(a_n^{-1} M_n \leq x) \to \exp\left\{- \left[\gamma_0^{\alpha} \pi_0 \mathbb{E}(V^+)^{\alpha} + \gamma_0^{\alpha}(1-\pi_0)\mathbb{E}(V^-)^{\alpha}\right] x^{-\alpha}\right\}, \tag{18}$$

with

$$V^+ = \max_k V_{t,k} \mathbb{I}_{\{V_{t,k}>0\}} \quad \textit{and} \quad V^- = \max_k V_{t,k} \mathbb{I}_{\{V_{t,k}<0\}}.$$

Proof. Using the definition of N_n, we note that $\{a_n^{-1} M_n \leq x\}$ is equivalent to $(N_n(x,\infty] = 0)$. Applying the continuous mapping theorem to the next function

$$
\begin{aligned}
T : M_p([0,\infty) \times \overline{\mathbb{R}} \setminus \{0\}) &\to \mathbb{D}(0,\infty) \\
\sum_{k=1}^{\infty} \varepsilon_{(t_k,j_k)} &\mapsto \sup\{j_k, t_k \leq \cdot\}
\end{aligned}
$$

where $\mathbb{D}(0,\infty)$ is the Skorokhod space of cadlag functions on $(0,\infty)$ and using Theorem 1, we obtain

$$\mathbb{P}(a_n^{-1}M_n \le x) = \mathbb{P}(N_n(x,\infty] = 0) \to \mathbb{P}(N(x,\infty] = 0).$$

Note that the event $\{N(x,\infty] = 0\}$ is equivalent to none of the points of the set $\{j_t V_{t,k}, k \ge 1, t \ge 1\}$ exceeding x, what is still equivalent to

$$\bigcap_{t=1}^{\infty} \left\{ j_t \bigvee_{k=1}^{\infty} V_{t,k} \le x \right\}.$$

Since $\{j_t \bigvee_{k=1}^{\infty} V_{t,k},\, t \ge 1\}$ are the points of PRM on $\overline{\mathbb{R}} \setminus \{0\}$ of mean measure

$$\mu(x) = \left(\gamma_0^\alpha \pi_0 \mathbb{E}(V^+)^\alpha + \gamma_0^\alpha (1-\pi_0)\mathbb{E}(V^-)^\alpha\right) x^{-\alpha}.$$

This set corresponds to

$$\left\{\max(j_t V^+, -j_t V^-) \le x\right\}.$$

Then

$$\mathbb{P}(a_n^{-1}M_n \le x) \to \exp\left\{-\left(\gamma_0^\alpha \pi_0 \mathbb{E}(V^+)^\alpha + \gamma_0^\alpha (1-\pi_0)\mathbb{E}(V^-)^\alpha\right) x^{-\alpha}\right\}. \qquad \blacksquare$$

5. Estimation Methods

5.1. Estimation of the parameters of the model

In this section, we consider the following threshold autoregressive model for (X_t):

$$X_t = \begin{cases} \phi_1 X_{t-1} + \sigma_t \eta_t, & \text{if} \quad Y_{t-\delta} \le \tau, \\ \phi_2 X_{t-1} + \sigma_t \eta_t, & \text{if} \quad Y_{t-\delta} > \tau, \end{cases} \tag{19}$$

where $\{\eta_t\}$ are sequences of iid random variables with regularly varying tail propbabilities, τ and ϕ_i are non random constants and with threshold variable $Y_{t-\delta}$. Specifically, we assume in the sequel that

$$\mathbb{P}\{|\eta_1| > x\} = 1 - exp(-x^{-\alpha}), \qquad \alpha > 0. \tag{20}$$

The scale parameter σ_t is modeled as a nonlinear function of covariables of the form

$$\sigma_t = exp\left\{ a_0 + \sum_{j=1}^{m} a_j x_{tj} \right\}. \tag{21}$$

We propose to estimate the parameters of the model (19)–(21) assuming that δ and τ are known parameters. The parameters are estimated by the Maximum Likelihood Estimation method. We choose this method for its simplicity and its good asymptotic properties. Other estimation methods can be considered. In the introduction we showed that the model (19) can be rewritten under this relation:

$$X_t - \phi_{(t)} X_{t-1} = \sigma_t \eta_t. \tag{22}$$

Let

$$\phi = (\phi_1, \phi_2), a = (a_0, \dots, a_m), \theta = (\alpha, \phi, a), X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n).$$

The likelihood function for the model (19) is given by

$$L(\theta, X|Y) = p(X) L^*(\theta, X|Y)$$

where $L^*(\theta, X|Y)$ is the conditional likelihood function and $p(X)$ is the joint density of n variables $(X_1, \dots, X_n)$. The conditional likelihood function is then given by

$$\begin{aligned} L^*(\theta, X|Y) &= \alpha^{n-1} \prod_{t=2}^{n} \left[\left| X_t - \phi_{(t)} X_{t-1} \right|^{-\alpha-1} \sigma_t^{\alpha} \right] \\ &\times \exp\left\{ - \sum_{t=2}^{n} \left(\frac{X_t - \phi_{(t)} X_{t-1}}{\sigma_t} \right)^{-\alpha} \right\}. \end{aligned}$$

The parameter θ can be estimated by maximizing the conditional log-likelihood:

$$\widehat{\theta} = arg \max_{\theta} log L^*(\theta, X|Y).$$

5.2. Estimation of the tail balancing coefficients

The coefficient π_0, in the tail balancing condition (10), plays a very important role in the interpretation of peaks observed in the trajectory of a time series when the underlying distribution has fat tails. Indeed the higher this value π_0 is close to the unity, more there is presence of large positive values and in contrario, more this value is close to zero, more the occurrence of minima is important. This coefficient is defined here by:

$$\pi_0 = \lim_{x \to \infty} \frac{\mathbb{P}\{\eta_1 > x\}}{\mathbb{P}\{|\eta_1| > x\}} \quad \text{and} \quad 1 - \pi_0 = \lim_{x \to \infty} \frac{\mathbb{P}\{\eta_1 < -x\}}{\mathbb{P}\{|\eta_1| > x\}}.$$

The probability that η_1 and $|\eta_1|$ exceed a threshold x fixed, can be estimated by the following frequencies:

$$\widehat{\mathbb{P}}\{\eta_1 > x\} = \frac{card\{t, \widehat{\eta}_t > x\}}{n} \quad \text{and} \quad \widehat{\mathbb{P}}\{|\eta_1| > x\} = \frac{card\{t, |\widehat{\eta}_t| > x\}}{n},$$

where $\widehat{\eta}_t$ is the residual il the estimation of the model, n the sample size. Then, the coefficient π_0 can be estimated by the mean of r ratios.

$$\widehat{\pi}_0 = \frac{1}{r} \sum_{i=1}^{r} \frac{\widehat{\mathbb{P}}\{\eta_1 > x_i\}}{\widehat{\mathbb{P}}\{|\eta_1| > x_i\}}.$$

5.3. Estimation of γ_0

The scale parameter σ_t is modeled by:

$$\sigma_t = exp \left\{ a_0 + \sum_{j=1}^{m} a_j x_{tj} \right\}.$$

The parameter γ_0 defined by the following limit

$$\gamma_0^{\alpha} = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \sigma_t^{\alpha}$$

can be estimated by

$$\widehat{\gamma_0}^{\widehat{\alpha}} = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \widehat{\sigma}_t^{\widehat{\alpha}}$$

where

$$\widehat{\sigma}_t = exp \left\{ \widehat{a}_0 + \sum_{j=1}^{m} \widehat{a}_j x_{tj} \right\}$$

and $\widehat{\alpha}$, $\widehat{a}_0$, $\widehat{a}_j$ are the values of α, a_0 and a_j obtained by the maximum likelihood estimation method.

6. Simulation study

6.1. Method of simulation

For the simulation, the realizations of the random variable η_t are generated by using the following representation: $\eta_t = \Gamma_t V_t$.

(Γ_t) is a sequence of independent and identically distributed random variables such as

$$\mathbb{P}\{\Gamma_t = 1\} = \pi_0 \quad \text{and} \quad \mathbb{P}\{\Gamma_t = -1\} = 1 - \pi_0,$$

π_0 is the coefficient of tails balancing defined by (10), if $\pi_0 = \frac{1}{2}$ then the distribution is symmetric. We can verify easily that Γ_t can be written as $\Gamma_t = \mathbb{I}_{\{U_t \le \pi_0\}} - \mathbb{I}_{\{U_t > \pi_0\}}$, where (U_t) are random variables uniformly distributed in $(0, 1)$, $\mathbb{I}_A$ is the indicator function of the set A.

(V_t) a another sequence of independent and identically distributed random variables and satisfies the following conditions:

$$\begin{aligned}\mathbb{P}\{V_1 < x\} = \phi_\alpha(x) &= \exp(-x^{-\alpha}), \quad \alpha > 0, \quad \text{for } x > 0,\\ &= 0 \quad \text{for } x \leq 0.\end{aligned}$$

The random variables V_t are generated using the following function $(-logU_t)^{-1/\alpha}$ where $U_t \sim U(0,1)$. The random variables Γ_t and V_t are assumed to be independent.

The tail balancing condition (10) is verified by the random variables (η_t). Indeed

$$\begin{aligned}\mathbb{P}\{\eta_t < x\} &= \mathbb{P}\{\Gamma_t V_t < x\}, \quad \text{with} \quad x < 0\\ &= \mathbb{E}\left(\mathbb{P}\{\Gamma_t V_t < x \mid \Gamma_t\}\right)\\ &= \pi_0 \mathbb{P}\{V_t < x\} + (1-\pi_0)\mathbb{P}\{-V_t < x\}\\ &= (1-\pi_0)\mathbb{P}\{-V_t < x\},\end{aligned}$$

then

$$\mathbb{P}\{\eta_t < -x\} = (1-\pi_0)\mathbb{P}\{V_t > x\}, \quad \text{for} \quad x > 0.$$

Thus we obtain

$$\frac{\mathbb{P}\{\eta_t < -x\}}{\mathbb{P}\{|\eta_t| > x\}} = \frac{(1-\pi_0)\mathbb{P}\{V_t > x\}}{\mathbb{P}\{V_t > x\}} = 1 - \pi_0.$$

Then

$$\begin{aligned}\mathbb{P}\{\eta_t > x\} &= \mathbb{P}\{\Gamma_t V_t > x\}, \quad \text{with} \quad x > 0\\ &= \mathbb{E}\left(\mathbb{P}\{\Gamma_t V_t > x \mid \Gamma_t\}\right)\\ &= \pi_0 \mathbb{P}\{V_t > x\} + (1-\pi_0)\mathbb{P}\{-V_t > x\}\\ &= \pi_0 \mathbb{P}\{V_t > x\}.\end{aligned}$$

Finally

$$\frac{\mathbb{P}\{\eta_t > x\}}{\mathbb{P}\{|\eta_t| > x\}} = \frac{\pi_0 \mathbb{P}\{V_t > x\}}{\mathbb{P}\{V_t > x\}} = \pi_0.$$

6.2. Models studied in simulation

For the simulation we consider the threshold autoregressive model defined by:

$$X_t = \begin{cases} \phi_1 X_{t-1} + \sigma_t \eta_t, & \text{si} \quad y_t \leq \tau,\\ \phi_2 X_{t-1} + \sigma_t \eta_t, & \text{si} \quad y_t > \tau, \end{cases} \tag{23}$$

where σ_t is a function of t defined by:

$$\sigma_t = \exp(a_0 + a_1 t),$$

and η_t is a sequence of independent random variables identically distributed such as

$$\mathbb{P}\{|\eta_1| > x\} = 1 - exp(-x^{-\alpha}), \quad \alpha > 0,$$

τ and ϕ_i are real constants and y_t is the threshold variable.

All of the simulations involve one of the following models summarized in Table 1. The first one is stationary and the others nonstationary. In all these models, the threshold variable Y_t is uniformly distributed in $(0,1)$.

For simplicity, we propose to expose in this subsection only the results of simulation of **Model 1**, **Model 2** and **Model 3**. The results of **Model 4**, **Model 5** and **Model 6** show that the parameter α and the threshold τ have no influence on the results of simulation, hence they are not presented.

	α	ϕ_1	ϕ_2	τ	a_0	a_1
Model 1	0.5	0.3	0.7	0.5	0	0
Model 2	0.5	0.3	0.7	0.5	0	1.3
Model 3	0.5	0.3	0.7	0.5	0.5	1.3
Model 4	1	0.3	0.7	0.2	0	1.3
Model 5	1	1.2	0.8	0.8	0	1.3
Model 6	1.5	1.2	0.8	0.5	0	1.3

Table 1. Data generating processes.

6.3. Numerical illustration

In our simulation we choose $\pi_0 = 0.5$. We simulate $s = 1000$ realizations of length $n = 1000$ for the process (X_t) defined in (23). We obtain s estimates values for each parameter. Example, for α, we obtain $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_s$ and we calculate

$$\widehat{\alpha} = \frac{1}{s}\sum_{i=1}^{s}\widehat{\alpha}_i, \quad RMSE = \left(\frac{1}{s}\sum_{i=1}^{s}(\widehat{\alpha}_i - \alpha_0)^2\right)^{1/2}, \quad MAE = \frac{1}{s}\sum_{i=1}^{s}\left|\widehat{\alpha}_i - \alpha_0\right|.$$

	$\hat{\alpha}$	$\hat{\phi}_1$	$\hat{\phi}_2$
Mean	0.493	0.298	0.656
RMSE	0.054	0.014	0.161
MAE	0.040	0.002	0.047

Table 2. Estimated values for the **Model 1**.

	$\hat{\alpha}$	$\hat{\phi}_1$	$\hat{\phi}_2$	$\hat{a}_1$
Mean	0.504	0.300	0.699	1.280
RMSE	0.048	0.003	0.014	0.301
MAE	0.037	0.001	0.001	0.028

Table 3. Estimated values for the **Model 2**.

	$\hat{\alpha}$	$\hat{\phi}_1$	$\hat{\phi}_2$	$\hat{a}_0$	$\hat{a}_1$
Mean	0.482	0.298	0.664	0.437	1.303
RMSE	0.096	0.044	0.144	0.067	0.035
MAE	0.052	0.010	0.045	0.040	0.011

Table 4. Estimated values for the **Model 3**.

6.4. Goodness of fit test

It is known according to the Theorem 4 that the distribution of the normalized maximum of the process $(X_t)_t$ is well approximated by the Fréchet's distribution. Now we investigate, by simulation experiments, the goodness-of -fit of the limiting distribution. We generate $N = 250000$ realizations from the process $(X_t)_t$ and we use the blocks method dividing the data into $m = 625$ blocks of observations of length $n = 400$. Let $M_n^{(j)} = \max(X_1^{(j)}, X_2^{(j)}, \ldots, X_n^{(j)})$ be the maximum of the n observations of the block j. The normalized maxima are then defined by $a_n^{-1}M_n^{(1)}, \ldots, a_n^{-1}M_n^{(m)}$ with $a_n = \left(log\frac{n}{n-1}\right)^{-1/\alpha}$.

We first use a graphical tool in order to compare the two distributions. In Figure 1, the corresponding qq-plot shows a satisfactory fitting.

This result is confirmed by Kolmogorov-Smirnov test at 5% level under the null hypothesis that the distribution of the normalized maxima follows the law given in the Theorem 4. The Statistics of Kolmogorov (KS) and p-values are given in the Table 5.

α	0.5	1	1.5
KS	0.0525	0.0425	0.0401
p-value	0.6399	0.8186	0.8629

Table 5. Kolmogorov-Smirnov Statistics and p-values between the empirical law of the normalized maxima and the limit laws.

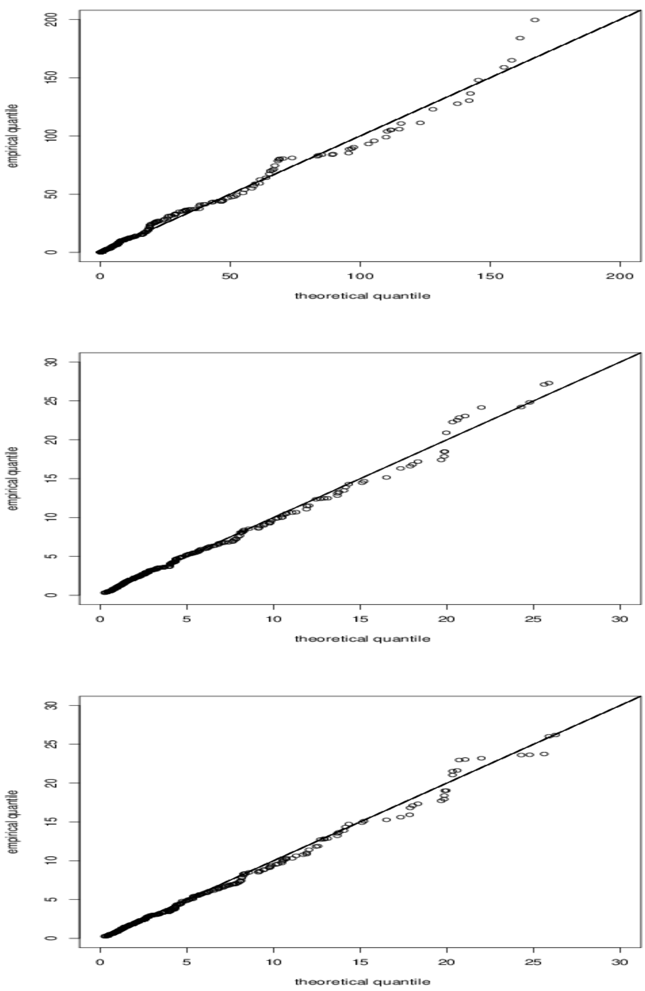

Figure 1. QQ-plot of maxima normalized block maxima against the theoretical limiting distribution for the process(23) with $\alpha = 0.5$ (top), $\alpha = 1$ (middle), $\alpha = 1.5$ (bottom), with $\phi_1 = 0.3$, $\phi_2 = 0.7$.

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