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BAYESIAN ANALYSIS OF STRUCTURAL CHANGE IN A DISTRIBUTED LAG MODEL (KOYCK SCHEME)

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Abstract

Structural change for the Koyck Distributed Lag Model is analyzed through the Bayesian approach. The posterior distribution of the break point is derived with the use of the normal-gamma prior density and the break point, ν, is estimated by the value that attains the Highest Posterior Probability (HPP). Simulation study is done using R.

Given the parameter values $\phi = 0.2$ and $\lambda = 0.3$, the full detection of the structural change when $\sigma^2 = 1$ is generally attained at $\nu + 1$. The after one lag detection is due to the nature of the model which includes lagged variable. The interval estimate *HPP near* ν consistently and efficiently captures the break point ν in the interval $HPP_t \pm 5\%$ of the sample size. On the other hand, the detection of the structural change when $\sigma^2 = 2$ does not show any improvement of the point estimate of the break point ν.

Keywords: distributed lag model, posterior distribution, break point.

2010 Mathematics Subject Classification: 62C10, 62F15, 62P20, 47N30.

1. INTRODUCTION

When certain economic policy measures begin to take effect, economists are interested on when and how their effects will fully occur. Dependent variables often react to changes in one or more of the explanatory variables only after a lapse of time. This delayed reaction suggests the inclusion of lagged explanatory variables resulting in a dynamical model. One example of a dynamical model is the distributed lag model.

The general form of a linear distributed lag model (DLM) is

$$Y_t = \phi + \sum_{i=0}^{\infty} \alpha_i X_{t-i} + \epsilon_t, \tag{1}$$

where ϕ is constant, ϵ_t is the error term such that $\epsilon_t \sim N(0, \sigma_\epsilon^2)$, $t = 1, 2, \ldots$, and any change in X_t will affect $E[Y_t]$ in all the later periods. The term α_i is the ith reaction coefficient, and it is usually assumed that $\lim_{i\to\infty} \alpha_i = 0$ and $\sum_{i=0}^{\infty} \alpha_i = \alpha < \infty$.

There are many in the literatures that studied structural changes in generalized linear model through Bayesian approach. In 1996, Supe [5] assumed that when modeling time-series data, parameters are allowed to change with specific time point. He studied on structural change in AR(1) and AR(2) processes. In 2004, B. Western, *et al.* [6] studied on a Bayesian model that treats the change-point in a time series as a parameter to be estimated. In this model, inference for the regression coefficients reflects prior uncertainty about the location of the change point. In 2007, J.H. Park, *et al.* [4] introduced an efficient Bayesian approach to the multiple changepoint problem in the context of generalized linear models. In 2012, Chaturvedia [2] assumed structural changes in either the parameters of the regression model or the disturbances precision.

In this paper, the possible shifts in parameters of the distributed lag model, specifically the Koyck Scheme [3], is examined. This study derived the posterior distribution of the break point of a distributed lag model undergoing structural change and assuming normal-gamma prior. Also, in this study, the author developed a computer program that computes point estimates and construct credible sets for the values of the break point. Percentage of credible sets capturing the real value will be the basis for the accuracy of the estimates.

Definition [1]**.** Let $\mathbf{X} = (x_1, \ldots, x_n)$ be a random sample and $\Theta = (\theta_1, \theta_2, \ldots, \theta_k)$ be the parameter of interest and $\pi(\Theta)$ be the *prior distribution* associated with Θ, and $f(\mathbf{X}|\Theta)$ the distribution from which the sample was taken. Then the *posterior distribution* of Θ given $\mathbf{X}$, is defined as

$$\pi(\Theta|\mathbf{X}) = k \cdot \pi(\Theta) L(\Theta|\mathbf{X})$$

where $k = \dfrac{1}{\int \cdots \int \pi(\Theta) L(\mathbf{X}|\Theta) d\Theta}$. The likelihood function of the sample $\mathbf{X}$ given Θ is defined as $L(\Theta|\mathbf{x}) = L(\theta_1, \ldots, \theta_k | x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i|\theta_1, \ldots, \theta_k)$.

Definition [1]**.** (The Normal-Gamma). Let X be a real random variable and Y be a positive random variable, then X and Y are said to have a normal-gamma distribution if the density of X and Y is

$$f(x, y|\mu, \tau, \alpha, \beta) \propto y^{1/2} \exp\left[\frac{-\tau y}{2}(x-\mu)^2 y^{\alpha-1} e^{-y\beta}\right],$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, $y > 0$, $\tau > 0$, $\alpha > 0$, and $\beta > 0$.

2. Posterior analysis

Koyck [3] suggested a simplification of the model in (1). He assumes that the α_i's decrease exponentially over time, that is: $\alpha_i = \beta\lambda^i$ for all i with $0 < \lambda < 1$. Note that $\lim_{i\to\infty} \alpha_i = 0$ because $\lim_{i\to\infty} \lambda^i = 0$, $0 < \lambda < 1$. Also, note that $\sum_{i=1}^{\infty} \lambda^i = \frac{1}{1-\lambda}$, then $\sum_{i=1}^{\infty} \alpha_i = \beta \sum_{i=1}^{\infty} \lambda^i = \beta\frac{1}{1-\lambda} < \infty$. Thus, assumptions in the reaction coefficients α_i's for each lag explanatory variables in a DLM are satisfied. Using Koyck's assumptions, we simplify equation (1) as

$$Y_t = (1-\lambda)\phi + \beta X_t + \lambda Y_{t-1} + u_t,$$

where $u_t = \epsilon_t - \lambda\epsilon_{t-1}$ is the error of the resulting model. It can also be considered as a linear model with MA(1) error written as

$$Z_t = \beta_0 + \beta X_t + u_t,$$

where $Z_t = Y_t - \lambda Y_{t-1}$, $\beta_0 = (1-\lambda)\phi$, and $u_t = \epsilon_t - \lambda\epsilon_{t-1}$.

Let $(1, 2, \ldots, \nu, \nu+1, \ldots, n)$ be discrete time points. The structural change model to be considered is

$$Z_t = \begin{cases} \beta_0 + \beta_1 X_t + u_t, & t = 1, 2, \ldots, \nu \\ \beta_0 + \beta_2 X_t + u_t, & t = \nu+1, \ldots, n, \end{cases} \tag{2}$$

where $Z_t = Y_t - \lambda Y_{t-1}$, $\beta_0 = (1-\lambda)\phi$, $\beta_2 = \beta_1 + \Delta$, $\Delta > 0$, and $u_t = \epsilon_t - \lambda\epsilon_{t-1}$. In matrix form, we can rewrite (2) as

$$\mathbf{Z} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u},$$

where

$$\mathbf{Z}=\begin{bmatrix}\mathbf{Z_1}\\ \mathbf{Z_2}\end{bmatrix},\ \mathbf{X}=\begin{bmatrix}\mathbf{X_1} & \mathbf{0}\\ \mathbf{0} & \mathbf{X_2}\end{bmatrix},\ \boldsymbol{\beta}=\begin{bmatrix}\boldsymbol{\beta_1}\\ \boldsymbol{\beta_2}\end{bmatrix},\ \mathbf{X_1}=\begin{bmatrix}1 & X_1\\ \vdots & \vdots\\ 1 & X_\nu\end{bmatrix},\ \mathbf{X_2}=\begin{bmatrix}1 & X_{\nu+1}\\ \vdots & \vdots\\ 1 & X_n\end{bmatrix},$$

$$\mathbf{Z_1}=\begin{bmatrix}Z_1\\ \vdots\\ Z_\nu\end{bmatrix},\ \mathbf{Z_2}=\begin{bmatrix}Z_{\nu+1}\\ \vdots\\ Z_n\end{bmatrix},\ \boldsymbol{\beta_1}=\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix},\ \boldsymbol{\beta_2}=\begin{bmatrix}\beta_0\\ \beta_2\end{bmatrix},\quad \text{and}\quad \mathbf{u}=\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}.$$

Based on the Bayesian paradigm, to compute the posterior distribution of the break point, ν, we have to consider the following informations:

1) The prior distribution of ν is uniform over $1, 2, \ldots, n$.

2) The prior distribution of $\boldsymbol{\beta}$ given τ is $\mathbf{N}(\boldsymbol{\beta}^*, \tau^{-1}\mathbf{I})$. That is,

$$g_1(\boldsymbol{\beta}|\tau) \propto \tau^{1/2} \exp\left\{\frac{-\tau}{2}(\boldsymbol{\beta}-\boldsymbol{\beta}^*)'(\boldsymbol{\beta}-\boldsymbol{\beta}^*)\right\}.$$

3) The marginal distribution of τ is gamma. That is,

$$g_2(\tau) \propto \tau^{a-1}\exp\{-\tau b\}, \quad a>0, \quad b>0.$$

4) The joint prior distribution of the parameters is normal-gamma. That is,

$$h(\boldsymbol{\beta},\tau,\nu) \propto g_1(\boldsymbol{\beta}|\tau)\cdot g_2(\tau) = \tau^{a-1/2}\exp\left\{\frac{-\tau}{2}\left[2b+(\boldsymbol{\beta}-\boldsymbol{\beta}^*)'(\boldsymbol{\beta}-\boldsymbol{\beta}^*)\right]\right\}.$$

5) The conditional likelihood of $(\boldsymbol{\beta},\tau,\nu)$ given the sample observations $(\mathbf{X},\mathbf{Z})$ is $L = L(\boldsymbol{\beta},\tau,\nu|(\mathbf{X},\mathbf{Z}))$ given by

$$L \propto \begin{cases}\tau^{n/2}|\mathbf{\Lambda}|^{-1/2}\exp\left\{\frac{-\tau}{2}(\mathbf{Z}-\mathbf{X}\boldsymbol{\beta})'\mathbf{\Lambda}^{-1}(\mathbf{Z}-\mathbf{X}\boldsymbol{\beta})\right\}, & 1\le\nu\le n-1\\ \tau^{n/2}|\mathbf{\Lambda}|^{-1/2}\exp\left\{\frac{-\tau}{2}(\mathbf{Z_1}-\mathbf{X_1}\boldsymbol{\beta_1})'\mathbf{\Lambda}^{-1}(\mathbf{Z_1}-\mathbf{X_1}\boldsymbol{\beta_1})\right\}, & \nu=n,\end{cases}$$

where $\tau=\frac{1}{\sigma_u^2}$, $\sigma_u^2=\psi^{-1}(1+\lambda^2)$, and $\mathbf{\Lambda}$ is the precision matrix. These can be easily shown by the fact that $u_t \sim N(0,\psi^{-1}(1+\lambda^2))$ for $t=1,2,\ldots,n$ and $\mathbf{Z}\sim\mathbf{N}(\mathbf{X}\boldsymbol{\beta},\tau^{-1}\mathbf{\Lambda})$.

Theorem 1 (Posterior Probability Distribution of the Break Point)**.** *If the model* (2) *holds and ν, $\boldsymbol{\beta}$, and τ are unknown, and if ν is uniformly distributed over $1,2,\ldots,n$, the joint prior distribution of $\boldsymbol{\beta}$ and τ is such that: the conditional distribution of $\boldsymbol{\beta}$ given τ is normal with mean $\boldsymbol{\beta}^*$ and precision matrix $\tau^{-1}\boldsymbol{I}$*

($\tau > 0$) *where* $\boldsymbol{I}$ *is a given* $n \times n$ *identity matrix and* $\boldsymbol{\beta}^*$ *is a* 4×1 *constant vector, the prior distribution of* τ *is gamma with parameters* $a > 0$ *and* $b > 0$*, and* ν *is independent of* $(\boldsymbol{\beta}, \tau)$*, then the posterior distribution of* ν *given the sample observation* $(\boldsymbol{X}, \boldsymbol{Z})$ *is*

$$\pi(\nu|(\boldsymbol{X},\boldsymbol{Z})) = K \cdot \begin{cases} |\boldsymbol{\Lambda}|^{-1/2}|\boldsymbol{U}|^{1/2}(2/\boldsymbol{M})^{a+1/2}\Gamma(a+1/2), & 1 \le \nu \le n-1 \\ |\boldsymbol{\Lambda}|^{-1/2}|\boldsymbol{U}_1|^{1/2}(2/\boldsymbol{M}_1)^{a+1/2}\Gamma(a+1/2), & \nu = n, \end{cases}$$

where

$$K = \frac{1}{\displaystyle\int |\boldsymbol{\Lambda}|^{-1/2}|\boldsymbol{U}|^{1/2}(2/\boldsymbol{M})^{a+1/2}\Gamma(a+1/2)d\nu},$$

for $1 \le \nu \le n-1$*,*

$$\begin{aligned} \mathbf{U} &= X'\Lambda^{-1}X + I \\ \mathbf{V} &= X'\Lambda^{-1}Z + \boldsymbol{\beta}^* \\ \boldsymbol{W} &= 2b + \mathbf{Z}'\boldsymbol{\Lambda}^{-\mathbf{1}}\mathbf{Z} + \boldsymbol{\beta}^{*'}\boldsymbol{\beta}^* \\ \mathbf{M} &= -V'U^{-1}V + W \end{aligned}$$

and for $\nu = n$*,*

$$\begin{aligned} \mathbf{U_1} &= X_1'\Lambda^{-1}X_1 + I \\ \mathbf{V_1} &= X_1'\Lambda^{-1}Z_1 + \boldsymbol{\beta}_1^* \\ \boldsymbol{W}_1 &= 2b + \mathbf{Z_1'}\boldsymbol{\Lambda}^{-\mathbf{1}}\mathbf{Z_1} + \boldsymbol{\beta}_{\mathbf{1}}^{*'}\boldsymbol{\beta}_{\mathbf{1}}^* \\ \mathbf{M_1} &= -V_1'U_1^{-1}V_1 + W_1. \end{aligned}$$

Proof. We prove the theorem for the case when $1 \le \nu \le n-1$. The posterior distribution of the parameters is

$$\begin{aligned} &\pi(\boldsymbol{\beta},\tau,\nu|(\mathbf{X},\mathbf{Z})) \;\propto\; L(\boldsymbol{\beta},\tau,\nu|(\mathbf{X},\mathbf{Z})) \cdot h(\boldsymbol{\beta},\tau,\nu) \\ &\propto \tau^{\frac{(2a+n-1)}{2}}|\boldsymbol{\Lambda}|^{-1/2}\exp\left\{\frac{-\tau}{2}[2b + (\boldsymbol{\beta}-\boldsymbol{\beta}^*)'(\boldsymbol{\beta}-\boldsymbol{\beta}^*) + (\mathbf{Z}-\mathbf{X}\boldsymbol{\beta})'\boldsymbol{\Lambda}^{-\mathbf{1}}(\mathbf{Z}-\mathbf{X}\boldsymbol{\beta})]\right\}. \end{aligned}$$

Algebraically, we can simplify the posterior distribution of parameters as

$$\pi(\boldsymbol{\beta},\tau,\nu|(\mathbf{X},\mathbf{Z})) \propto |\boldsymbol{\Lambda}|^{-1/2}\tau^{\frac{(2a+n-1)}{2}}\exp\left\{\frac{-\tau}{2}\left[(\boldsymbol{\beta}-\mathbf{U}^{-\mathbf{1}}\mathbf{V})'\mathbf{U}(\boldsymbol{\beta}-\mathbf{U}^{-\mathbf{1}}\mathbf{V})+\mathbf{M}\right]\right\},$$

letting $\mathbf{U} = \mathbf{X}'\boldsymbol{\Lambda}^{-1}\mathbf{X} + \mathbf{I}$, $\mathbf{V} = \mathbf{X}'\boldsymbol{\Lambda}^{-1}\mathbf{Z} + \boldsymbol{\beta}^*$, $\mathbf{W} = 2b + \mathbf{Z}'\boldsymbol{\Lambda}^{-1}\mathbf{Z} + \boldsymbol{\beta}^{*'}\boldsymbol{\beta}^*$, and $\mathbf{M} = -\mathbf{V}'\mathbf{U}^{-1}\mathbf{V} + \mathbf{W}$ for $1 \le \nu \le n-1$. Integrating out $\boldsymbol{\beta}$ and τ gives us the posterior distribution of ν below,

$$\pi(\nu|(\mathbf{X},\mathbf{Z})) \propto |\boldsymbol{\Lambda}|^{-1/2}|\mathbf{U}|^{1/2}(2/\mathbf{M})^{a+1/2}\Gamma(a+1/2), \quad 1 \le \nu \le n-1.$$

For the case when $\nu = n$, the posterior distribution of ν is given by

$$\pi(\nu|(\mathbf{X_1}, \mathbf{Z_1})) \propto |\mathbf{\Lambda}|^{-1/2}|\mathbf{U}_1|^{1/2}(2/\mathbf{M}_1)^{a+1/2}\Gamma(a+1/2), \quad \nu = n$$

where $\mathbf{U}_1 = \mathbf{X}_1'\mathbf{\Lambda}^{-1}\mathbf{X}_1 + \mathbf{I}$, $\mathbf{V}_1 = \mathbf{X}_1'\mathbf{\Lambda}^{-1}\mathbf{Z}_1 + \boldsymbol{\beta}_1^*$, $\mathbf{W}_1 = 2b + \mathbf{Z}_1'\mathbf{\Lambda}^{-1}\mathbf{Z}_1 + \boldsymbol{\beta}_1^{*'}\boldsymbol{\beta}_1^*$ and $\mathbf{M}_1 = -\mathbf{V}_1'\mathbf{U}_1^{-1}\mathbf{V}_1 + \mathbf{W}_1$. ■

3. Results and discussions

3.1. Structural change when $\sigma^2 = 1$

Table 1 gives a summary of detection results for $n = 10$, while Table 3 gives a summary for $n = 15$. For both tables, exact detection is made only when β_2 is twice β_1, but interval estimates (*HPP near* ν) consistently captures the break point. As change from β_1 to β_2 increases, point estimates improve while interval estimates give 100% capture of the break point.

Table 1. Simulation results using $n = 10$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0 = 0.14$, $\beta_1 = 1$, $\sigma^2 = 1$.

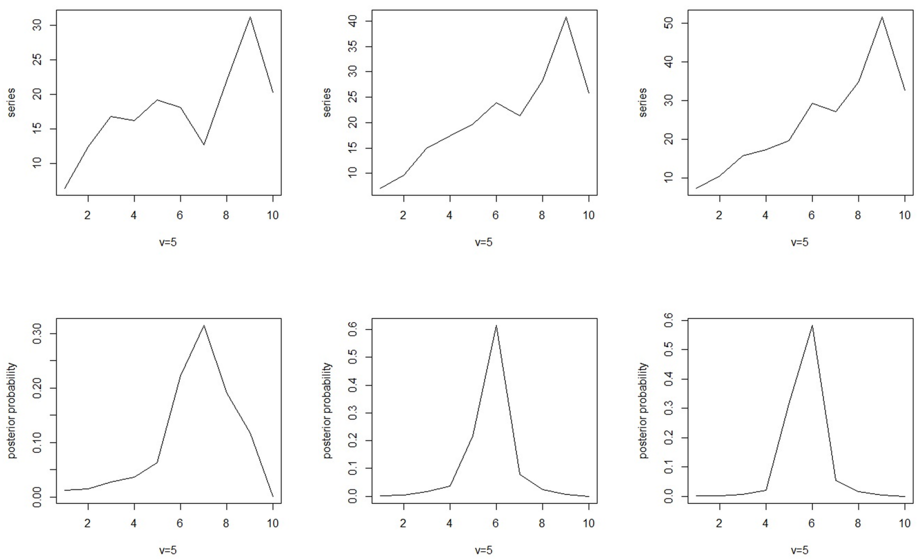
β_2	β_0^*	β_1^*	β_2^*	Break Point	HPP at ν	HPP near ν	Percentage near ν
1.2	0.14	1	1.2	5	0	49	98%
	0.14	1.3	1.4		0	48	96%
	0.14	1.5	1.6		0	49	98%
1.4	0.14	1	1.4	5	1	50	100%
	0.14	1.3	1.6		1	50	100%
	0.14	1.5	1.8		0	50	100%
1.6	0.14	1	1.6	5	0	50	100%
	0.14	1.3	1.8		1	50	100%
	0.14	1.5	2.0		1	50	100%
1.8	0.14	1	1.8	5	4	50	100%
	0.14	1.3	2.0		2	50	100%
	0.14	1.5	2.2		8	50	100%
2.0	0.14	1	2.0	5	15	50	100%
	0.14	1.3	2.2		18	50	100%
	0.14	1.5	2.4		13	50	100%

Table 2 is a sample posterior distribution of the break point ν. From this table, $\nu = 6$ gives a probability of .4986 while $\nu = 5$ gives a posterior probability of .3935. Thus the point estimate is $\nu^* = 6$ but *HPP near* ν includes $\nu = 5$, the actual break point. It is a pattern in the succeeding results that the point estimate *HPP at* ν tends to identify a value of ν which is *one lag* after the break point. This is because from the structure of the model, complete change in the model occurs after one lag.

Table 2. Posterior distribution of ν based on $n = 10$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0^* = 0.14$, $\sigma^2 = 1$, and $\Delta = 1.0$.

ν	1	2	3	4	5	6	7	8	9	10
pmf	.0013	.0025	.0085	.023	.3935	.4986	.0512	.0147	.0037	.0001

Figure 1 gives the time series plot of the simulated data and the posterior probability plot based on $n = 10$ and parameter values indicated in Table 1. We can see an improvement of the structural change of the plot as the value of Δ changes from 0.2 to 1.0. Also, the posterior probabilities tend to flock near the break point as the difference of the beta values increases.



(a) $\Delta = 0.2$ (b) $\Delta = 0.6$ (c) $\Delta = 1.0$

Figure 1.: Plot of the simulated data and the corresponding posterior probability plot based on $n = 10$, $\beta_0^* = 0.14$, and $\sigma^2 = 1$

Table 3. Simulation results using $n = 15$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0 = 0.14$, $\beta_1 = 1$, $\sigma^2 = 1$.

β_2	β_0^*	β_1^*	β_2^*	Break Point	HPP at ν	HPP near ν	Percentage near ν
1.2	0.14	1	1.2	8	0	40	80%
	0.14	1.3	1.4		0	43	86%
	0.14	1.5	1.6		1	39	78%
1.4	0.14	1	1.4	8	0	50	100%
	0.14	1.3	1.6		0	50	100%
	0.14	1.5	1.8		0	50	100%
1.6	0.14	1	1.6	8	7	50	100%
	0.14	1.3	1.8		7	50	100%
	0.14	1.5	2.0		6	50	100%
1.8	0.14	1	1.8	8	27	50	100%
	0.14	1.3	2.0		24	50	100%
	0.14	1.5	2.2		23	50	100%
2.0	0.14	1	2.0	8	47	50	100%
	0.14	1.3	2.2		48	50	100%
	0.14	1.5	2.4		47	50	100%

Table 4. Simulation results using $n = 30$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0 = 0.14$, $\beta_1 = 1$, $\sigma^2 = 1$.

β_2	β_0^*	β_1^*	β_2^*	Break Point	HPP at ν	HPP near ν	Percentage near ν
1.2	0.14	1	1.2	15	0	41	82%
	0.14	1.3	1.4		0	45	90%
	0.14	1.5	1.6		0	45	90%
1.4	0.14	1	1.4	15	0	50	100%
	0.14	1.3	1.6		0	50	100%
	0.14	1.5	1.8		0	50	100%
1.6	0.14	1	1.6	15	0	50	100%
	0.14	1.3	1.8		0	50	100%
	0.14	1.5	2.0		0	50	100%
1.8	0.14	1	1.8	15	0	50	100%
	0.14	1.3	2.0		0	50	100%
	0.14	1.5	2.2		0	50	100%
2.0	0.14	1	2.0	15	0	50	100%
	0.14	1.3	2.2		0	50	100%
	0.14	1.5	2.4		0	50	100%

Detection at the exact break point is hardly attained when we increase the sample size n to 30 and 50. This can be seen in Table 4 and Table 5 under the column *HPP at* ν. However break points are detected after *one lag* and can be seen in the posterior distribution of ν in Table 11 and Table 12 in Section 4. This can be explained by the fact that the full change can be detected after one lag because of the nature of the model which includes lagged variable. The interval estimate *HPP near* ν consistently captures the break point.

Figure 2 to Figure 4 in the Section 4 give a summary of the time series plots (first row) of the simulated data and the corresponding posterior probability plots (second row) of the break point, for different values of Δ. From these figures, it can be seen that as β_2 goes farther away from β_1, the structural change in the model becomes easier to distinguish and the posterior probabilities tend to flock near ν, the break point.

Table 5. Simulation results using $n = 50$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0 = 0.14$, $\beta_1 = 1$, $\sigma^2 = 1$.

β_2	β_0^*	β_1^*	β_2^*	Break Point	HPP at ν	HPP near ν	Percentage near ν
1.2	0.14	1	1.2	30	0	43	86%
	0.14	1.3	1.4		0	44	88%
	0.14	1.5	1.6		0	40	80%
1.4	0.14	1	1.4	30	1	50	100%
	0.14	1.3	1.6		0	50	100%
	0.14	1.5	1.8		0	50	100%
1.6	0.14	1	1.6	30	0	50	100%
	0.14	1.3	1.8		0	50	100%
	0.14	1.5	2.0		0	50	100%
1.8	0.14	1	1.8	30	0	50	100%
	0.14	1.3	2.0		0	50	100%
	0.14	1.5	2.2		0	50	100%
2.0	0.14	1	2.0	30	1	50	100%
	0.14	1.3	2.2		0	50	100%
	0.14	1.5	2.4		0	50	100%

3.2. Structural change when $\sigma^2 = 2$.

Table 6 to Table 9 show that the point estimate of the break point ν (*HPP at* ν) hardly detects the simulated break point when $\sigma^2 = 2$ as compared to the detection when $\sigma^2 = 1$ for all sample sizes. However, the highest posterior probability is attained after one lag, so the interval estimate will contain the simulated break point.

Table 6. Simulation results using $n = 10$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0 = 0.14$, $\beta_1 = 1$, $\sigma^2 = 2$.

β_2	β_0^*	β_1^*	β_2^*	Break Point	HPP at ν	HPP near ν	Percentage near ν
1.2	0.14	1	1.2	5	0	44	88%
	0.14	1.3	1.4		0	39	78%
	0.14	1.5	1.6		0	45	90%
1.4	0.14	1	1.4	5	0	49	98%
	0.14	1.3	1.6		0	49	98%
	0.14	1.5	1.8		0	50	100%
1.6	0.14	1	1.6	5	0	50	100%
	0.14	1.3	1.8		0	50	100%
	0.14	1.5	2.0		0	50	100%
1.8	0.14	1	1.8	5	0	50	100%
	0.14	1.3	2.0		0	50	100%
	0.14	1.5	2.2		0	50	100%
2.0	0.14	1	2.0	5	0	50	100%
	0.14	1.3	2.2		0	50	100%
	0.14	1.5	2.4		0	50	100%

Table 7. Simulation results using $n = 15$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0 = 0.14$, $\beta_1 = 1$, $\sigma^2 = 2$.

β_2	β_0^*	β_1^*	β_2^*	Break Point	HPP at ν	HPP near ν	Percentage near ν
1.2	0.14	1	1.2	8	0	25	50%
	0.14	1.3	1.4		0	27	54%
	0.14	1.5	1.6		0	25	40%
1.4	0.14	1	1.4	8	0	49	98%
	0.14	1.3	1.6		0	48	96%
	0.14	1.5	1.8		0	48	96%
1.6	0.14	1	1.6	8	0	50	100%
	0.14	1.3	1.8		0	50	100%
	0.14	1.5	2.0		0	50	100%
1.8	0.14	1	1.8	8	0	50	100%
	0.14	1.3	2.0		0	50	100%
	0.14	1.5	2.2		0	50	100%
2.0	0.14	1	2.0	8	0	50	100%
	0.14	1.3	2.2		0	50	100%
	0.14	1.5	2.4		0	50	100%

Table 8. Simulation results using $n = 30$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0 = 0.14$, $\beta_1 = 1$, $\sigma^2 = 2$.

β_2	β_0^*	β_1^*	β_2^*	Break Point	HPP at ν	HPP near ν	Percentage near ν
1.2	0.14	1	1.2	15	0	16	32%
	0.14	1.3	1.4		0	12	24%
	0.14	1.5	1.6		0	19	38%
1.4	0.14	1	1.4	15	0	46	92%
	0.14	1.3	1.6		0	45	90%
	0.14	1.5	1.8		0	43	86%
1.6	0.14	1	1.6	15	0	50	100%
	0.14	1.3	1.8		0	50	100%
	0.14	1.5	2.0		0	50	100%
1.8	0.14	1	1.8	15	0	50	100%
	0.14	1.3	2.0		0	50	100%
	0.14	1.5	2.2		0	50	100%
2.0	0.14	1	2.0	15	0	50	100%
	0.14	1.3	2.2		0	50	100%
	0.14	1.5	2.4		0	50	100%

Table 9. Simulation results using $n = 50$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0^* = 0.14$, $\beta_1 = 1$, $\sigma^2 = 2$.

β_2	β_0^*	β_1^*	β_2^*	Break Point	HPP at ν	HPP near ν	Percentage near ν
1.2	0.14	1	1.2	30	0	17	34%
	0.14	1.3	1.4		0	18	36%
	0.14	1.5	1.6		0	22	44%
1.4	0.14	1	1.4	30	0	48	96%
	0.14	1.3	1.6		0	46	92%
	0.14	1.5	1.8		0	45	90%
1.6	0.14	1	1.6	30	0	50	100%
	0.14	1.3	1.8		0	50	100%
	0.14	1.5	2.0		0	50	100%
1.8	0.14	1	1.8	30	0	50	100%
	0.14	1.3	2.0		0	50	100%
	0.14	1.5	2.2		0	50	100%
2.0	0.14	1	2.0	30	0	50	100%
	0.14	1.3	2.2		0	50	100%
	0.14	1.5	2.4		0	50	100%

4. Tables and graphs

Table 10. Posterior distribution of ν based on $n = 15$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0^* = 0.14$, $\sigma^2 = 1$, and $\Delta = 1.0$.

ν	1	2	3	4	5	6	7	8	9	10
pmf	.0008	.0018	.0045	.0088	.0171	.0396	.0738	.4533	.3597	.0228

ν	11	12	13	14	15
pmf	.0089	.0045	.0028	.0014	.0001

Table 11. Posterior distribution of ν based on $n = 30$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0^* = 0.14$, $\sigma^2 = 1$, and $\Delta = 1.0$.

ν	1	2	3	4	5	6	7	8	9	10
pmf	.0000	.0001	.0001	.0002	.0003	.0005	.0006	.0009	.0027	.0028

ν	11	12	13	14	15	16	17	18	19	20
pmf	.0032	.0047	.0084	.0276	.3029	.5339	.0878	.0121	.0041	.0022

ν	21	22	23	24	25	26	27	28	29	30
pmf	.0015	.0010	.0007	.0005	.0004	.0003	.0002	.0001	.0000	.0000

Table 12. Posterior distribution of ν based on $n = 50$, $\phi = 0.2$, $\lambda = 0.3$, $\beta_0^* = 0.14$, $\sigma^2 = 1$, and $\Delta = 1.0$.

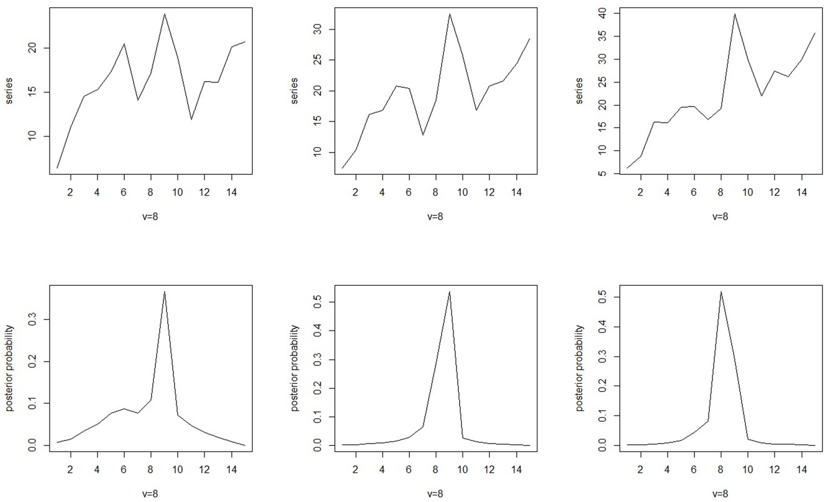
ν	1	2	3	4	5	6	7	8	9	10
pmf	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002	.0002	.0002

ν	11	12	13	14	15	16	17	18	19	20
pmf	.0003	.0003	.0003	.0004	.0004	.0005	.0005	.0007	.0007	.0008

ν	21	22	23	24	25	26	27	28	29	30
pmf	.0009	.0012	.0015	.0020	.0035	.0058	.0072	.0172	.1246	.1878

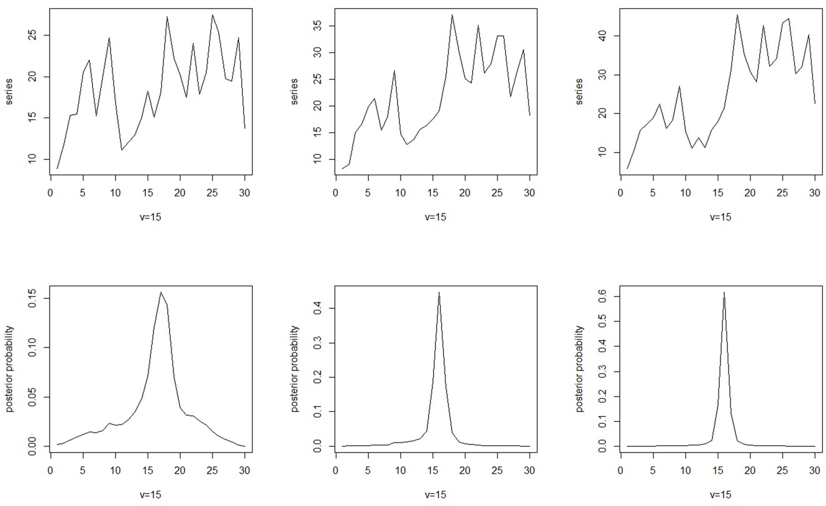
ν	31	32	33	34	35	36	37	38	39	40
pmf	.3058	.2099	.0889	.0162	.0072	.0046	.0025	.0017	.0015	.0010

ν	41	42	43	44	45	46	47	48	49	50
pmf	.0007	.0005	.0004	.0004	.0003	.0002	.0002	.0001	.0000	.0000



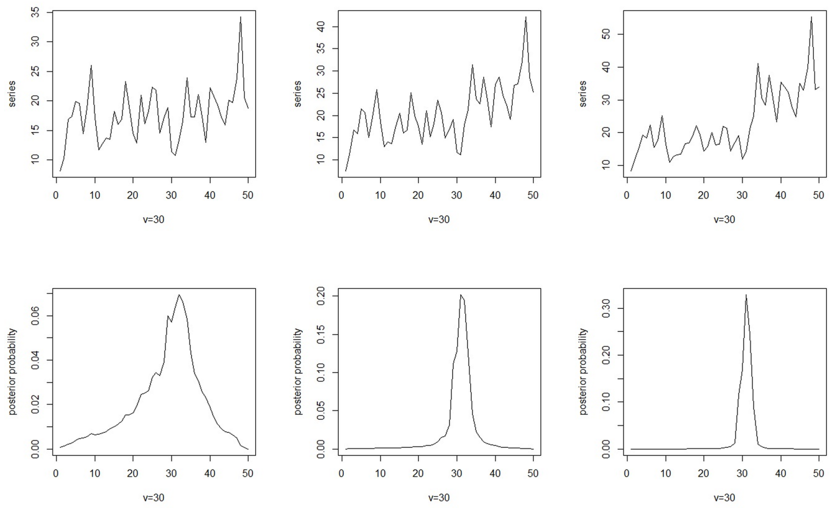
(a) $\Delta = 0.2$ (b) $\Delta = 0.6$ (c) $\Delta = 1.0$

Figure 2.: Plot of the simulated data and the corresponding posterior probability plot based on $n = 15$, $\beta_0^* = 0.14$, and $\sigma^2 = 1$



(a) $\Delta = 0.2$ (b) $\Delta = 0.6$ (c) $\Delta = 1.0$

Figure 3.: Plot of the simulated data and the corresponding posterior probability plot based on $n = 30$, $\beta_0^* = 0.14$, and $\sigma^2 = 1$



(a) $\Delta = 0.2$ (b) $\Delta = 0.6$ (c) $\Delta = 1.0$

Figure 4.: Plot of the simulated data and the corresponding posterior probability plot based on $n = 50$, $\beta_0^* = 0.14$, and $\sigma^2 = 1$

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