A WEIGHTED VERSION OF GAMMA DISTRIBUTION

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Abstract

Weighted Gamma (WG), a weighted version of Gamma distribution, is introduced. The hazard function is increasing or upside-down bathtub depending upon the values of the parameters. This distribution can be obtained as a hidden upper truncation model. The expressions for the moment generating function and the moments are given. The non-linear equations for finding maximum likelihood estimators (MLEs) of parameters are provided and MLEs have been computed through simulations and also for a real data set. It is observed that WG fits better than its submodels (WE), Generalized Exponential (GE), Weibull and Exponential distributions.

Keywords: gamma distribution, weight function, hazard function, maximum likelihood estimator, Akaike Information criterion.

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1. Introduction

In observational studies for human, wild-life, insect, plant or fish population, it is not possible to select sampling units with equal probabilities. In such cases, there are no well-defined sampling frames and recorded observations are biased. These observations don't follow the original distribution and hence their modelling gives birth to the theory of weighted distributions. The idea of weighted distributions was conceptualized by Fisher [7] and studied by Rao [18] in a unified manner who pointed out that in many situations, the recorded observations cannot be considered as a random sample from the original distribution.This can be due to one or the other reason viz. non-observability of some events, damage caused to original observations and adoption of unequal probability sampling.

For a non-negative random variable X with density function $f(x)$ and a non-negative weight function $w(x)$ with finite non-zero expectation, the weighted random variable X^w has the density function

$$(1.1) \qquad\qquad f^w(x) = \frac{w(x)f(x)}{E[w(X)]}.$$

The distribution of X^w is called the weighted distribution corresponding to X.

The weighted distribution with $w(x) = x$ is called the length-biased (size-biased) distribution which finds various applications in biomedical areas such as early detection of a disease. Rao [18] used this distribution in the study of human families and wild-life populations. Various other important weighted distributions and their properties have been discussed by Mahfoud and Patil [15], Gupta and Kirmani [9], Jain *et al.* [12], Nanda and Jain [16], Patil [17] and Gupta and Kundu [11]. It is, therefore, important to study the stochastic orderings and ageing properties of the weighted random variables with respect to the original random variables.

Let X and Y be two random variables with probability density functions $f(x)$ and $g(x)$. The corresponding distribution functions are denoted by $F(x)$ and $G(x)$. $h_X(x)$ and $h_Y(x)$ are the failure rate functions and $\mu_X(x)$ and $\mu_Y(x)$ are the mean residual life functions of X and Y, respectively (Barlow and Proschan, [5]).

Definitions of the few ageing concepts and the partial orderings used in the paper are given below:

(a) INCREASING (DECREASING) FAILURE RATE (IFR (DFR)):
F is said to be Increasing (Decreasing) Failure Rate if failure rate $h_X(x)$ is increasing (decreasing) monotonically in x or equivalently, $-\log \overline{F}(x)$ is convex (concave).

(b) UPSIDE-DOWN BATHTUB (UBT) SHAPED DISTRIBUTION:
F is said to be upside-down bathtub shaped distribution if the failure rate $h_X(x)$

initially increases for $x \in (0, x_0)$, then becomes constant and eventually decreases for $x > x_0$.

(c) STOCHASTIC (ST) ORDERING: X is said to be smaller than Y in the stochastic order ($X \underset{\leq}{^{ST}} Y$) if $F(x) \geq G(x)$ for every $x \geq 0$ and with a strict inequality for some x.

(d) FAILURE RATE (FR) ORDERING: X is said to be smaller than Y in the failure rate order ($X \underset{\leq}{^{FR}} Y$) if $h_X(x) \geq h_Y(x)$ for every $x \geq 0$ or equivalently, if $\overline{F}(x)/\overline{G}(x)$ is decreasing in x.

(e) LIKELIHOOD RATIO (LR) ORDERING: X is said to be smaller than Y in the likelihood ratio order ($X \underset{\leq}{^{LR}} Y$) if $f(x)/g(x)$ is decreasing in x.

(f) MEAN RESIDUAL LIFE ORDERING: X is said to be smaller than Y in the mean residual life order ($X \underset{\leq}{^{MRL}} Y$) if $\mu_X(x) \leq \mu_Y(x)$ for all $x \geq 0$ or equivalently, if

$$\frac{\int_x^\infty \overline{F}(u)du}{\int_x^\infty \overline{G}(u)du} \quad \text{is decreasing in } x.$$

We have the following chain of implications between the various partial orderings discussed above:

$$Y \underset{\leq}{^{LR}} X \Rightarrow Y \underset{\leq}{^{FR}} X \Rightarrow Y \underset{\leq}{^{ST}} X$$

$$\Downarrow$$

(1.2)
$$Y \underset{\leq}{^{MRL}} X$$

For the above definitions, one may refer to Lai and Xie [14] and Shaked and Shanthikumar [19]. Gamma distribution is a popular model in reliability studies and communication engineering and is considered as a generalization of the Exponential distribution (Johnson *et al.* [13]). This distribution has been widely used in many areas such as reliability engineering, survival analysis, queuing systems, hydrological analysis, climatology and insurance sector. The lifetime of a mechanical system, the load on the web server, the flow of items through manufacturing and distribution processes, the amount of rainfall accumulated in a reservoir and the size of loan defaults or aggregate insurance claims can be modelled by Gamma distribution. When engineers design space shuttles with two fuel pumps – one active and the other in reserve, then the time elapsed till the second pump breaks down follows Gamma distribution.

Keeping in mind the importance of the Gamma distribution and the concept of weighted distributions, we introduce a weighted version of Gamma distribution with a particular weight function, called as Weighted Gamma (WG) distribution. Weighted Exponential (WE) distribution introduced by Gupta and Kundu[11],

Gamma and Exponential distributions can be obtained as special cases of WG distribution. This new distribution accommodates increasing and upside-down bathtub shaped hazard rate function and hence will have wider applicability in reliability and survival analysis. Our model can be interpreted as a hidden upper truncation model and also as a linear combination of two Gamma models.

The main objective of this piece of work is to study different properties of the WG distribution and check its superiority over some existing distributions viz WE, Generalized Exponential (GE), Weibull and Exponential. The maximum likelihood estimators (MLEs) of three unknown parameters which can't be obtained explicitly have been found as solutions of non-linear equations. The asymptotic distribution of MLEs is provided which can be used for testing of hypotheses and for construction of asymptotic confidence intervals. The paper is organized as follows.

In Section 2, the Weighted Gamma distribution is introduced and some interpretations provided. The expressions for the cumulative distribution function (cdf), the failure rate function, the moment generating function and moments till fourth order have been derived in Section 3. Section 4 provides the non-linear equations for finding MLEs, the elements of the Fisher's information matrix and the asymptotic distribution of MLEs. In Section 5, simulations are carried out for the validation of theory presented in Section 4. A real data set illustrates the importance of the new model. Concluding remarks are presented in Section 6.

2.   Weighted Gamma distribution

In this section, we define the Weighted Gamma (WG) distribution and provide some interpretations.

The random variable (r.v.) X follows Weighted Gamma distribution with scale parameter λ and shape parameters α and β if the probability density function (pdf) of X is given by

$$(2.1) \qquad f_X\left(x;\alpha,\beta,\lambda\right) = k\frac{(1-e^{-\alpha\lambda x})\lambda^{\beta}x^{\beta-1}e^{-\lambda x}}{\Gamma(\beta)} \ , x > 0, \alpha, \beta, \lambda > 0,$$

where $k^{-1} = 1 - \left(\frac{1}{1+\alpha}\right)^{\beta}$.

(2.1) is obtained from (1.1) by taking $f(x)$ to be the density function of Gamma with λ as scale parameter and β as shape parameter and $w(x) = 1 - e^{-\alpha\lambda x}$ since $E[w(X)] = k^{-1}$ is non-zero and finite.

If X is a random variable with pdf given in (2.1), we use the notation $X \sim WG(\alpha, \beta, \lambda)$.

Remark 1.

- (2.1) is the weighted version of the Gamma pdf with weight function

$$w(x) = 1 - e^{-\alpha\lambda x}, \quad \alpha, \ \lambda > 0.$$

- $\beta = 1$ gives the Weighted Exponential (WE) distribution introduced by Gupta and Kundu [11].

- As $\alpha \to \infty$, the weight function tends to 1. Hence the pdf (2.1) approaches the Gamma pdf with scale parameter λ and shape parameter β.

- For $\alpha \to \infty$ and $\beta = 1$, (2.1) reduces to the pdf of Exponential distribution with parameter λ.

- The pdf of the WG can be written as a linear combination of two Gamma pdfs as

$$(2.2) \qquad f_X(x) = k\left[\text{pdf of } G(x; \beta, \lambda) - \frac{1}{(1+\alpha)^\beta} \text{ pdf of } G(x; \beta, \lambda(1+\alpha))\right].$$

where $G(x; a, b) = \dfrac{b^a \int_0^x e^{-bt} t^{a-1} dt}{\Gamma(a)}$ is the cumulative distribution function of Gamma distribution with shape parameter a and scale parameter b.

Some observations about the new model are discussed below:

1. If for $i = 1$ and 2, $X_i's$ are independent random variables with pdfs $f_{X_i}(.)$ and cdfs $F_{X_i}(.)$, then given $\alpha X_1 > X_2$ for $\alpha > 0$, the pdf of the new r.v. $X = X_1$ can be written as

$$(2.3) \qquad f_X(x) = \frac{1}{P(\alpha X_1 > X_2)} f_{X_1}(x) F_{X_2}(\alpha x), \quad x > 0.$$

If $X_1 \sim G(\beta, \lambda)$ and $X_2 \sim Exp(\lambda)$, then (2.1) can be obtained from (2.3).
This means that Azzalini's results about skew-normal distribution (Azzalini [4]) can be extended to two independent and non-normal random variables.

2. Hidden truncation models arise when the random variable X is observed given that Y exceeds a certain level, say α, where X and Y follow Bivariate Normal distribution (Arnold *et al.* [1]). Such models use lower truncation.
Using upper truncation, our model can be obtained as a hidden truncation model as in case of skew-normal distribution (Arnold and Beaver [2]).

For details on models using lower or upper truncations, one can refer to Arnold [3].

We let X and Y to be two dependent random variables with joint pdf written as

$$(2.4) \qquad f_{X,Y}(x,y) = \frac{\beta \lambda^{\beta+1} x^{\beta+1-1} e^{-\lambda x} e^{-\lambda xy}}{\Gamma(\beta+1)} \; for \; \beta, \lambda > 0,$$

and consider the distribution of X when $Y \leq \alpha$.

Using (2.4), we get

$$(2.5) \qquad P(X = x, Y \leq \alpha) = \frac{\lambda^\beta x^{\beta-1} e^{-\lambda x} [1 - e^{-\alpha \lambda x}]}{\Gamma(\beta)};$$

$$(2.6) \qquad P(Y \leq \alpha) = 1 - (1+\alpha)^{-\beta} = k^{-1}.$$

Hence

$$f_{X|Y<\alpha}(x) = \frac{k\lambda^\beta x^{\beta-1} e^{-\lambda x} [1 - e^{-\alpha \lambda x}]}{\Gamma(\beta)},$$

which is the pdf of WG as given in (2.1).

Hence our model is a type of hidden truncation model using upper truncation. The plot of the joint density function given by (2.4) for $\beta = 0.5$ and $\lambda = 1$ is shown in Figure 1.
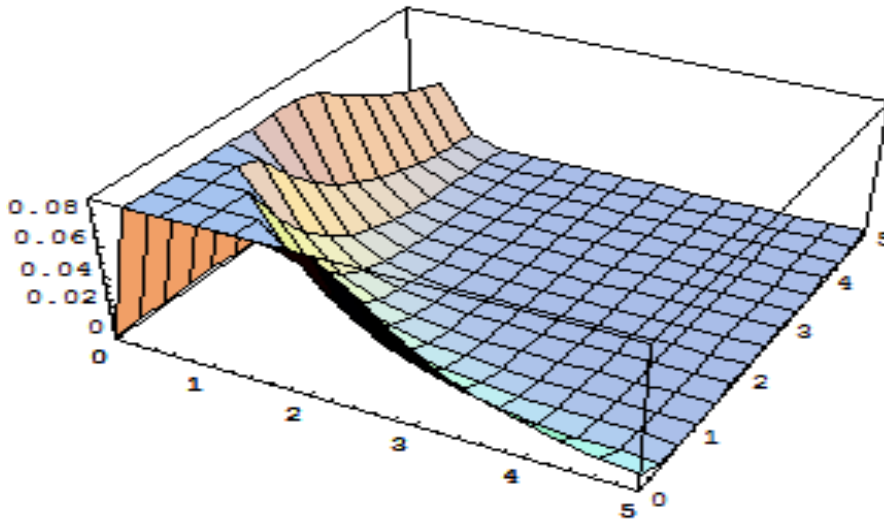

Figure 1. Plot of the joint density function of X and Y.

For sake of simplicity, we assume that $\lambda = 1$. The pdf of the WG distribution then transforms to

$$(2.7) \qquad f_X(x) = \frac{k[1 - e^{-\alpha x}]x^{\beta - 1}e^{-x}}{\Gamma(\beta)}.$$

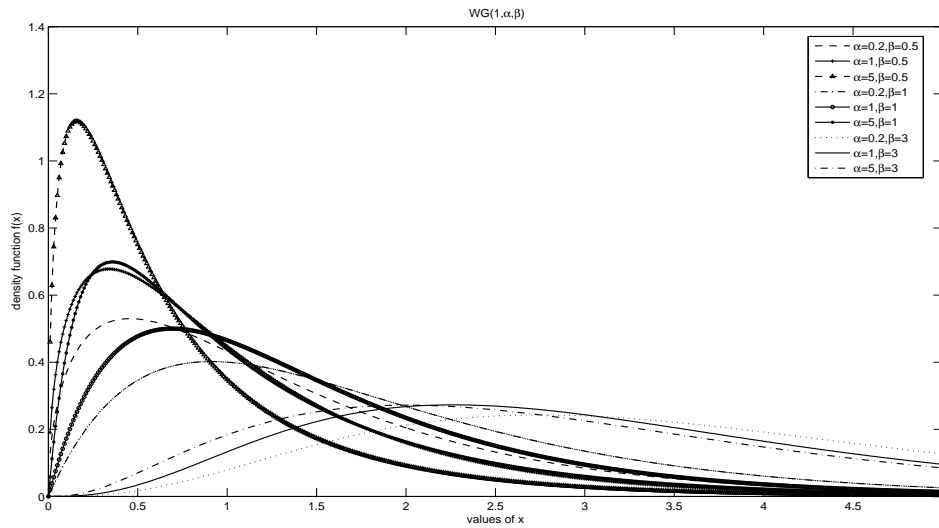The plots of this density function for different values of α and β are shown in Figure 2.

Figure 2. Plots of the density function of WG with $\lambda = 1$.

From the plots in Figure 2, it is observed that

- $f(x)$ approaches 0 as x tends to 0 or infinity;

- the WG distribution is positively skewed for all α and β;

- with increasing β, the density function $f(x)$ approaches symmetry.

3.   PROPERTIES OF WEIGHTED GAMMA DISTRIBUTION

3.1.   Distributional properties

Using (2.2), the moment generating function of the WG distribution can be written as

$$M_X(t) = k\left[\text{mgf of } G(x; \beta, \lambda)\text{at t} \quad - \frac{1}{(1+\alpha)^\beta} \text{ mgf of } G(x; \beta, \lambda(1+\alpha))\text{at t}\right.$$

$$= k\left[\frac{1}{(1-\frac{t}{\lambda})^\beta} - \frac{1}{(1+\alpha)^\beta}\frac{1}{(1-\frac{t}{\lambda(1+\alpha)})^\beta}\right]\text{for } t < \lambda.$$

The nth order moment of the WG distribution is obtained as

$$E(X^n) = k\left[\int_0^\infty x^n \text{ pdf of } G(x; \beta, \lambda)dx\right.$$

$$\left. - \frac{1}{(1+\alpha)^\beta}\int_0^\infty x^n \text{ pdf of } G(x; \beta, \lambda(1+\alpha))dx\right]$$

$$= k\frac{\Gamma(n+\beta)}{\lambda^n\Gamma(\beta)}\left[1 - \frac{1}{(1+\alpha)^{n+\beta}}\right].$$

For $\lambda = 1$,

$$M_X(t) = k\left[\frac{1}{(1-t)^\beta} - \frac{1}{(1+\alpha-t)^\beta}\right], \qquad \text{for } t < 1$$

and

(3.1) $$E(X^n) = \frac{k\Gamma(n+\beta)}{\Gamma(\beta)}\left[1 - \frac{1}{(1+\alpha)^{n+\beta}}\right].$$

Using (3.1), the first four moments are

$$E(X) = \frac{\beta}{1+\alpha}\left[\frac{(1+\alpha)^{\beta+1}-1}{(1+\alpha)^\beta - 1}\right];$$

$$E(X^2) = \frac{\beta(\beta+1)}{(1+\alpha)^2}\left[\frac{(1+\alpha)^{\beta+2}-1}{(1+\alpha)^\beta - 1}\right];$$

$$E(X^3) = \frac{\beta(\beta+1)(\beta+2)}{(1+\alpha)^3}\left[\frac{(1+\alpha)^{\beta+3}-1}{(1+\alpha)^\beta - 1}\right];$$

$$E(X^4) = \frac{\beta(\beta+1)(\beta+2)(\beta+3)}{(1+\alpha)^4}\left[\frac{(1+\alpha)^{\beta+4}-1}{(1+\alpha)^\beta - 1}\right].$$

Hence

$$V(X) =$$

$$= \frac{\beta}{(1+\alpha)^2 \left\{(1+\alpha)^\beta - 1\right\}} \left[(\beta+1)\left\{(1+\alpha)^{\beta+2} - 1\right\} - \beta\frac{\left\{(1+\alpha)^{\beta+1} - 1\right\}^2}{(1+\alpha)^\beta - 1}\right].$$

and coefficient of skewness=

$$\sqrt{\frac{\left[(\beta+1)(\beta+2)\left\{(1+\alpha)^{\beta+3}-1\right\}\left\{(1+\alpha)^\beta-1\right\}^2 - 3\beta(\beta+1)\left\{(1+\alpha)^{\beta+1}-1\right\}\left\{(1+\alpha)^{\beta+2}-1\right\}\left\{(1+\alpha)^\beta-1\right\}+2\beta^2\left\{(1+\alpha)^{\beta+1}-1\right\}^3\right]^2}{\beta\left[(\beta+1)\left\{(1+\alpha)^{\beta+2}-1\right\}\left\{(1+\alpha)^\beta-1\right\}-\beta\left\{(1+\alpha)^{\beta+1}-1\right\}^2\right]^3}}.$$

Coefficient of variation is written as

$$CV = \frac{\sqrt{V(X)}}{E(X)}$$

$$= \sqrt{\frac{(\beta+1)\left\{(1+\alpha)^{\beta+2} - 1\right\}\left\{(1+\alpha)^\beta - 1\right\} - \beta\left\{(1+\alpha)^{\beta+1} - 1\right\}^2}{\beta\left\{(1+\alpha)^{\beta+1} - 1\right\}^2}}.$$

For $\beta = 1$, we get

$$E(X) = 1 + \frac{1}{1+\alpha};$$

$$V(X) = 1 + \frac{1}{(1+\alpha)^2};$$

coefficient of skewness $= \sqrt{\frac{4\left\{(1+\alpha)^3+1\right\}^2}{\left\{(\alpha+1)^2+1\right\}^3}}$

and

$$CV = \sqrt{1 - \frac{2(1+\alpha)}{(\alpha+2)^2}} = \frac{\sqrt{\alpha^2+2}}{(\alpha+2)}.$$

The above expressions are same as those given by Gupta and Kundu [11] for Weighted Exponential (WE) distribution.

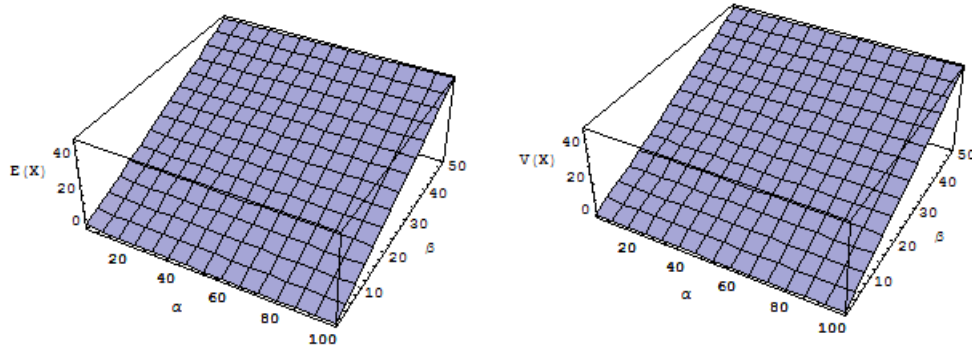Figures 3–5 display the plots for mean, variance, coefficient of skewness and CV.

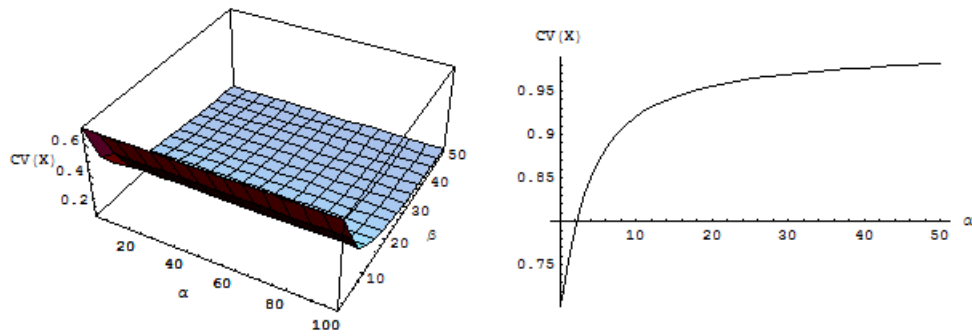Figure 3. Plots for mean and variance of WG.

Figure 4. Plots of coefficient of skewness for WG for general β and for $\beta = 1$.
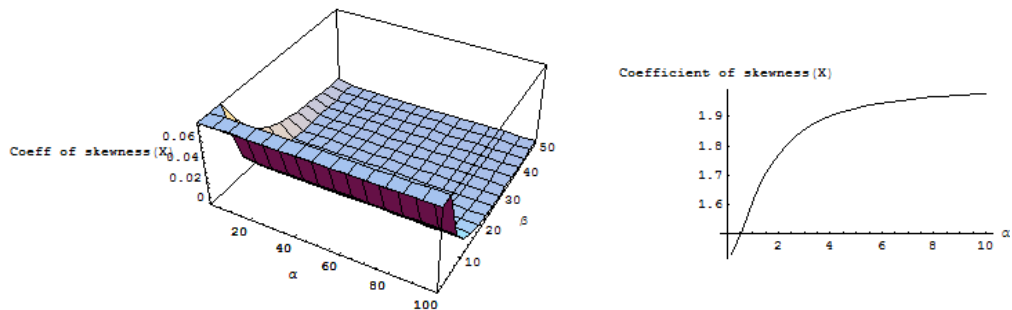

Figure 5. Plots of coefficient of variation (C.V.) for general β and for $\beta = 1$.

Figures 3–5 help us to conclude the following:

- for all values of α, E(X) and V(X) are increasing functions of β,

- coefficient of skewness

  – increases from $\sqrt{2}$ to 2 with an increase in α for $\beta= 1$;

  – decreases in β for $\beta>1$ and all α;

  – is zero when β approaches infinity,

- coefficient of variation is

  – increasing from $1/\sqrt{2}$ to 1 in α for $\beta=1$ which is true for WE distribution;

  – decreasing in β for $\beta > 1$ and all α.

For Weighted Gamma distribution, the mode does not exist in a closed form and can be found by solving the equation

$$\alpha x = \log \left[\frac{x(\alpha + 1) - (\beta - 1)}{x - (\beta - 1)} \right]$$

which is equivalent to $e^{\alpha x} = \dfrac{x(\alpha + 1) - (\beta - 1)}{x - (\beta - 1)} = 1 + \dfrac{\alpha x}{x - (\beta - 1)}.$

Remark 2.

- If we use the approximation $e^{\alpha x} \approx 1 + \alpha x$ for small x, then $x = \beta$ is the mode.

- For $\beta = 1$, we get $x = \frac{\log(1+\alpha)}{\alpha}$, which is the mode of WE (Gupta and Kundu [11]).

3.2. Ageing properties

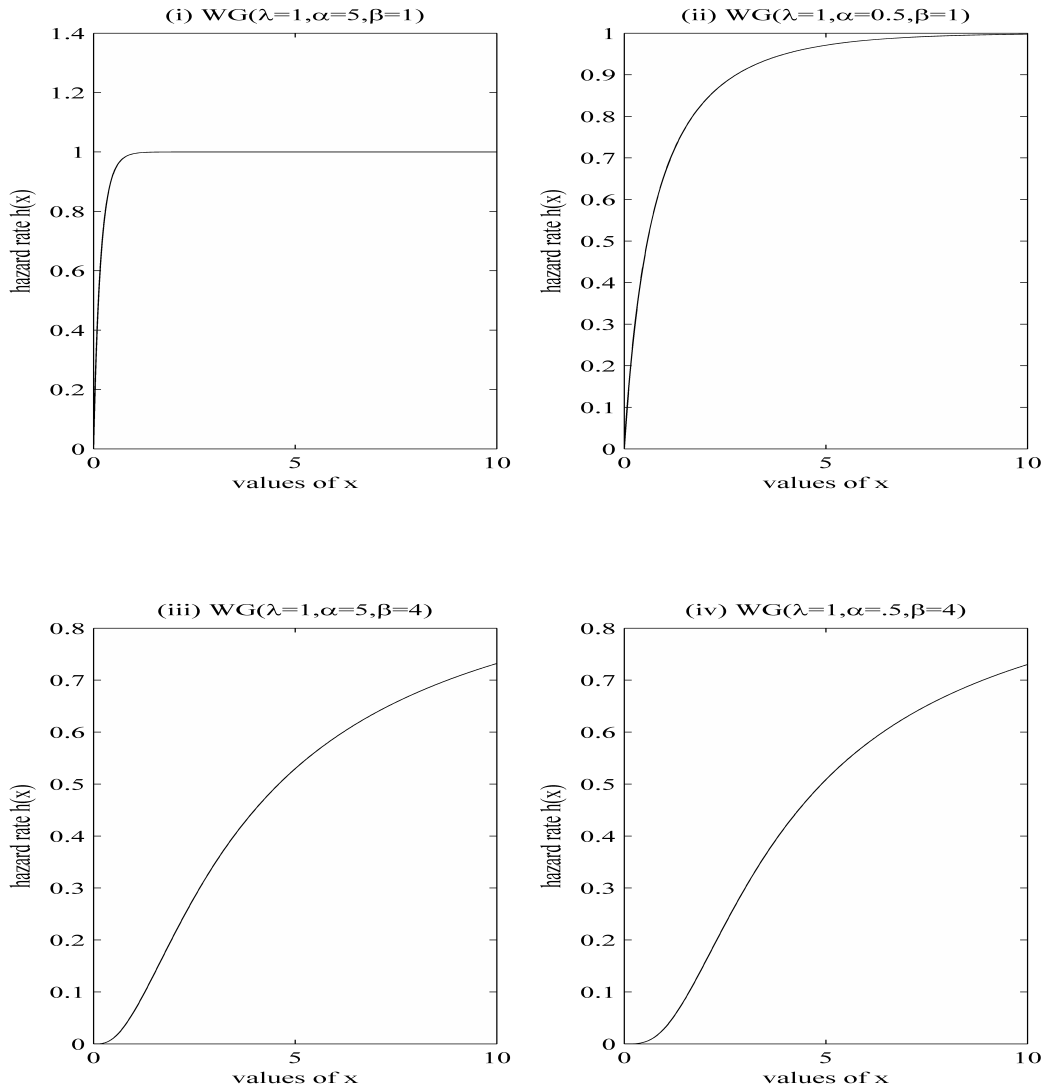For $\lambda = 1$, the cumulative distribution function (cdf) of X can be written using (2.2) and is given by

$$(3.2) \qquad F(x) = \left[\frac{(1 + \alpha)^{\beta}}{(1 + \alpha)^{\beta} - 1} \right] \left[G(x; \beta, 1) - \frac{1}{(1 + \alpha)^{\beta}} G(x; \beta, (1 + \alpha)) \right],$$

Using (2.1) and (3.2), the corresponding failure rate function is given by

$$(3.3) \quad h\left(x\right) = \frac{f(x)}{\overline{F}(x)} = \frac{(1 - e^{-\alpha x})x^{\beta-1}e^{-x}}{\Gamma(\beta)[\{1 - G\left(x;\beta,1\right)\} - \frac{1}{(1+\alpha)^{\beta}}\{1 - G\left(x;\beta,(1+\alpha)\right)\}]},$$

where $\overline{F}(x)$ is the survival function of X.

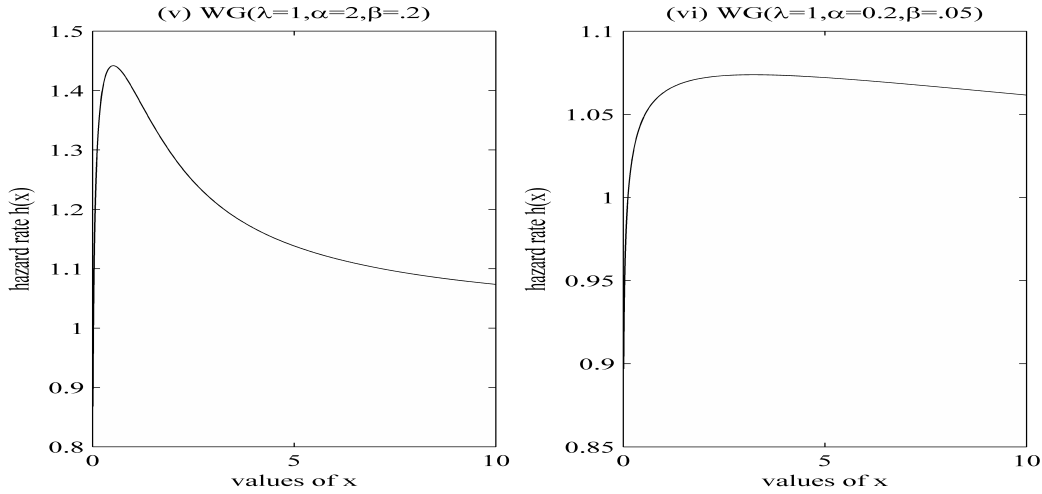The plots of failure rate function for different values of shape parameters are given in Figure 6.

Figure 6. Plots of the failure rate function.

It is evident from Figure 6 that for any α and

(a) for $\beta \geq 1$, $h(x)$ is increasing implying F is IFR;

(b) for $\beta < 1$, $h(x)$ has an upside-down bathtub shape giving that F is UBT.

This can also be proved mathematically.

Theorem 1. *The Weighted Gamma* (WG) *distribution has*

(a) *increasing failure rate* (IFR) *if $\beta \geq 1$;*

(b) *upside-down bathtub* (UBT) *failure rate if $\beta < 1$.*

Proof. For proving this, we use the theorem of Glaser [8].
    Using (2.1), we write

$$\eta'(x) = -\frac{d^2}{dx^2}\log f(x)$$

$$= \frac{\alpha^2 e^{-\alpha x}}{(1 - e^{-\alpha x})^2} + \frac{\beta - 1}{x^2}.$$

(a) If $\beta \geq 1$, then $\eta'(x) > 0$ for all $x > 0$ which implies that WG is IFR.

(b) $\eta'(x) = 0$ if

$$\frac{\alpha^2 e^{-\alpha x}}{(1 - e^{-\alpha x})^2} = \frac{1 - \beta}{x^2}$$

that is, if

$$\frac{(e^{\alpha x} + e^{-\alpha x} - 2)}{\alpha^2} = \frac{x^2}{1 - \beta}.$$

Using $e^{\alpha x} \cong 1 + \alpha x + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!} + \frac{(\alpha x)^4}{4!}$, we get that $\eta'(x) = 0$ for $x = \frac{\sqrt{12\beta/(1-\beta)}}{\alpha} = x_0$ when $\beta < 1$.

It is also seen that $\eta'(x) > 0$ for $x < \frac{\sqrt{12\beta/(1-\beta)}}{\alpha} = x_0$; $\eta'(x) < 0$ for $x > \frac{\sqrt{12\beta/(1-\beta)}}{\alpha} = x_0$. Using Theorem by Glaser [8], it can be concluded that WG is UBT when $\beta < 1$. ∎

3.3. Stochastic ordering results

In the next theorem, the relationship between a Gamma and a Weighted Gamma random variable in terms of likelihood ratio, failure rate, stochastic and mean residual life orderings is established.

Theorem 2. *If $X \sim G(\beta, \lambda)$ and $X^w \sim WG(\alpha, \beta, \lambda)$, then*

(a) $X \overset{LR}{\leq} X^w$,

(b) $X \overset{FR}{\leq} X^w$,

(c) $X \overset{ST}{\leq} X^w$,

(d) $X \overset{MRL}{\leq} X^w$.

Proof. (a) The result follows since

$$\frac{f_X^w(x; \alpha, \beta, \lambda)}{f_X(x; \beta, \lambda)} = k(1 - e^{-\alpha \lambda x}) \text{ is an increasing function of } x.$$

(b), (c) and (d) follow because of the chain of implications given in (1.2). ∎

4. ESTIMATION AND INFERENCE

In this section, we derive the equations for finding the maximum likelihood estimators (MLEs) of parameters.

Suppose X follows WG distribution and let $\boldsymbol{\theta} = (\alpha, \beta, \lambda)^T$ be the parameter vector. The log likelihood based on the observed sample $(x_1, x_2, \ldots, x_n)$ is

$$l = l(\alpha, \beta, \lambda)$$

$$= n\left\{\log(1+\alpha)^\beta - \log\left\{(1+\alpha)^\beta - 1\right\}\right\} + \sum_{i=1}^{n}\log\left(1 - e^{-\alpha\lambda x_i}\right) + n\beta\log\lambda$$

$$+ (\beta-1)\sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}x_i - n\log\{\Gamma(\beta)\}.$$

The components of the score vector $\mathbf{U} = \left(\frac{\partial l}{\partial\alpha}, \frac{\partial l}{\partial\beta}, \frac{\partial l}{\partial\lambda}\right)^T$ are given by

$$(4.1) \quad \frac{\partial l}{\partial\alpha} = -n\beta(1+\alpha)^{\beta-1}\left\{\frac{1}{\left\{(1+\alpha)^\beta - 1\right\}(1+\alpha)^\beta}\right\} + \lambda\sum_{i=1}^{n}\frac{x_i e^{-\alpha\lambda x_i}}{1 - e^{-\alpha\lambda x_i}},$$

$$(4.2)$$

$$\frac{\partial l}{\partial\beta} = n\log(1+\alpha) - \frac{n(1+\alpha)^\beta\log(1+\alpha)}{\left\{(1+\alpha)^\beta - 1\right\}} + n\log\lambda + \sum_{i=1}^{n}\log x_i - n\psi(\beta),$$

$$(4.3) \qquad \frac{\partial l}{\partial\lambda} = \alpha\sum_{i=1}^{n}\frac{x_i e^{-\alpha\lambda x_i}}{1 - e^{-\alpha\lambda x_i}} + \frac{n\beta}{\lambda} - \sum_{i=1}^{n}x_i,$$

where $\psi(.)$ denotes the digamma function, the logarithmic derivative of the gamma function. The MLE $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is obtained numerically by equating (4.1)–(4.3) to zeros and solving for α, β and λ. When the parameters are unknown, the asymptotic distribution of $\sqrt{n}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)$ is $N_3(0, \boldsymbol{K}(\boldsymbol{\theta})^{-1})$.

For interval estimation and testing of hypotheses for the parameters in $\boldsymbol{\theta}$, the $3x3$ information matrix can be obtained as

$$\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{\theta}) = \begin{bmatrix} K_{\lambda,\lambda} & K_{\lambda,\alpha} & K_{\lambda,\beta} \\ K_{\lambda,\alpha} & K_{\alpha,\alpha} & K_{\alpha,\beta} \\ K_{\lambda,\beta} & K_{\alpha,\beta} & K_{\beta,\beta} \end{bmatrix}.$$

For $u\,(a,b) = \int_0^1 \left\{ -\log(1-x) \right\}^{b+1}(1-x)^a x^{-1} dx$, the elements of $\boldsymbol{K}$ are derived below:

$$K_{\lambda,\lambda} = -E\left[\frac{\partial^2 l}{\partial \lambda^2}\right]$$

$$= \frac{\beta}{\lambda^2} + \left\{ \frac{(1+\alpha)^\beta}{\left\{(1+\alpha)^\beta - 1\right\}(\Gamma(\beta))\alpha^\beta \lambda^2} \right\} u\left(\frac{1}{\alpha}, \beta\right);$$

$$K_{\lambda,\alpha} = -E\left[\frac{\partial^2 l}{\partial \lambda \partial \alpha}\right]$$

$$= \frac{1}{\lambda\left\{(1+\alpha)^\beta - 1\right\}} \left[\frac{1}{1+\alpha} - \frac{(1+\alpha)^\beta}{\beta \alpha^{\beta+1}} u\left(\frac{1}{\alpha}, \beta\right)\right];$$

$$K_{\lambda,\beta} = -E\left[\frac{\partial^2 l}{\partial \lambda \partial \beta}\right]$$

$$= -\frac{1}{\lambda};$$

$$K_{\alpha,\alpha} = -E\left[\frac{\partial^2 l}{\partial \alpha^2}\right]$$

$$= \frac{\beta}{(1+\alpha)^2} - \frac{\beta\left\{(1+\alpha)^{2\beta-2} + (\beta-1)(1+\alpha)^{\beta-2}\right\}}{\left\{(1+\alpha)^\beta - 1\right\}^2}$$

$$+ \left\{ \frac{(1+\alpha)^\beta}{\left\{(1+\alpha)^\beta - 1\right\}(\Gamma(\beta))\alpha^{\beta+2}} \right\} u\left(\frac{1}{\alpha}, \beta\right);$$

$$K_{\alpha,\beta} = -E\left[\frac{\partial^2 l}{\partial \alpha \partial \beta}\right]$$

$$= \frac{(1+\alpha)^{\beta-1}\left\{(1+\alpha)^\beta - \beta\log(1+\alpha) - 1\right\}}{\left\{(1+\alpha)^\beta - 1\right\}^2} - \frac{1}{(1+\alpha)};$$

$$K_{\beta,\beta} = -E\left[\frac{\partial^2 l}{\partial \beta^2}\right]$$

$$= -\frac{\left\{\log(1+\alpha)\right\}^2(1+\alpha)^\beta}{\left\{(1+\alpha)^\beta - 1\right\}^2}.$$

If we wish to compare the WE and WG distributions using the likelihood ratio (LR) test, the null hypothesis $H_0 : \beta = 1$ is to be tested against H_1: $\beta \neq 1$. For this, LR statistic=-2 (log likelihood under H_0 - log likelihood under H_1).

In the next section, the Monte Carlo Simulations are carried out to check the validity of the theoretical results reported in this section.

We also compare WG distribution with Weighted Exponential (WE), Generalized Exponential (GE) (Gupta and Kundu [10]), Weibull and Exponential distributions using a real data set.

5.   Simulations and application

5.1.   Estimation through simulation

Simulations are carried out by generating observations from Weighted Gamma (WG) distribution with parametric values $\alpha = 5$, $\beta = 5$ and $\lambda = 2$, using Acceptance-Rejection procedure. The estimates of the parameters are found using (4.1)–(4.3) and equating these to zeros. The considered sample sizes are n= 50, 75, 100, 200 and 300 and the number of repetitions is 10,000. The results have been found using the BFGS quasi-newton method in R package.

The mean estimates and the corresponding Root Mean Square Errors (RMSEs) are reported in Table 1.
The values in Table 1 substantiate the theoretical results reported in Section 4. It is observed that as n becomes large, the estimates of parameters get closer to the initial parametric values.

5.2.   Real life illustration

We consider a real data set to estimate the parameters and establish the superiority of WG over WE, GE, Weibull and Exponential distributions. The data set is given by Birnbaum and Saunders [6] on the fatigue life of 6061-T6 aluminium coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. With maximum stress per cycle 31,000 psi, there are 101 observations given as
70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120,120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 131,132, 132, 132, 133, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 201, 212.
The plot for the Total Time on Test (TTT) for the above data set is shown in Figure 7.

Table 1. Mean Estimates of parameters and Root Mean Squared Errors.

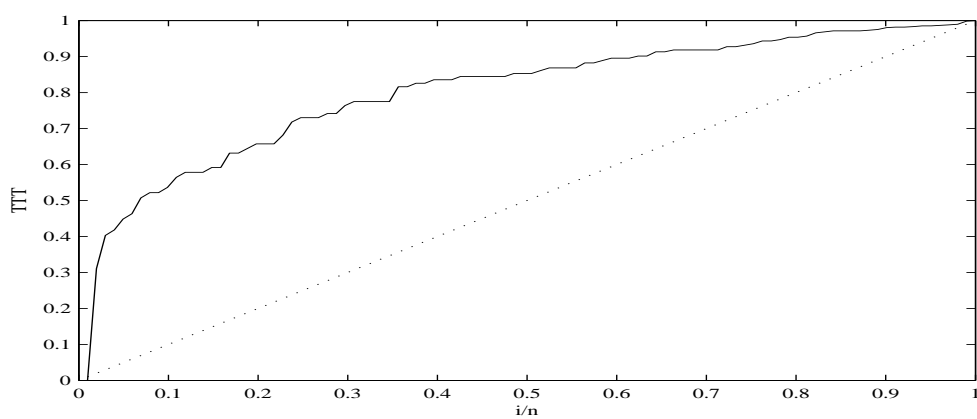| n	parameter	mean estimate	RMSE
50	α	3.99	3.01
	β	5.18	1.18
	λ	2.09	0.49
75	α	3.99	3.16
	β	5.07	0.91
	λ	2.05	0.38
100	α	4.15	3.24
	β	5.03	0.78
	λ	2.03	0.32
200	α	4.28	3.05
	β	4.96	0.55
	λ	1.99	0.22
300	α	4.67	2.39
	β	4.92	0.45
	λ	1.99	0.18

Figure 7. Total Time on Test (TTT) plot.

Since TTT plot is concave, hence the failure rate function is increasing implying that an IFR distribution models the data set. The plots of the empirical and fitted survival functions are displayed in Figure 8.
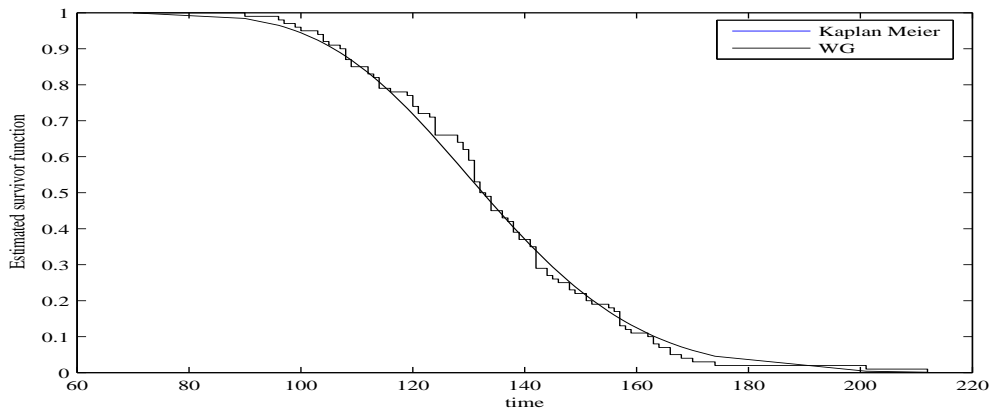

Figure 8. The empirical and fitted survival functions for the considered data set.

This figure depicts that WG distribution provides a good fit to the data set.

Estimation of parameters

The values of the parameters for Weighted Gamma (WG), Weighted Exponential (WE), Generalized Exponential (GE), Weibull and Exponential distributions, for the data set on the fatigue life, have been calculated and are reported in Table 2.

Table 2. Estimates of the parameters.

Distribution	MLE		
	λ	α	β
WG	.265	12.43	35.38
WE	.015	.001	1
GE	.039	119.52	-
Weibull	.0098	-	1.41
Exponential	.0074	-	1

Comparison of WG distribution with its submodels

(I) **Using Akaike Information Criterion and Bayesian Information Criterion:**

For comparing Weighted Gamma distribution with Weighted Exponential, Generalized Exponential, Weibull and Exponential distributions, we first use the concept of Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The best model is the one which has least values of AIC and BIC. For establishing its superiority, the calculated values of AIC and BIC are reported in Table 3.

Table 3. AIC and BIC for different models.

Distribution	AIC	BIC
WG	919.54	927.39
WE	1119.83	1125.06
GE	934.74	939.97
Weibull	2149.30	2154.53
Exponential	1192.99	1195.61

Since the values of AIC and BIC are lowest for WG, hence WG distribution provides the best fit to the data.

(II) **Through Histogram:**

The histogram for the above data set and estimated densities of WG, WE, GE, Weibull and Exponential distributions have been plotted in Figure 9.
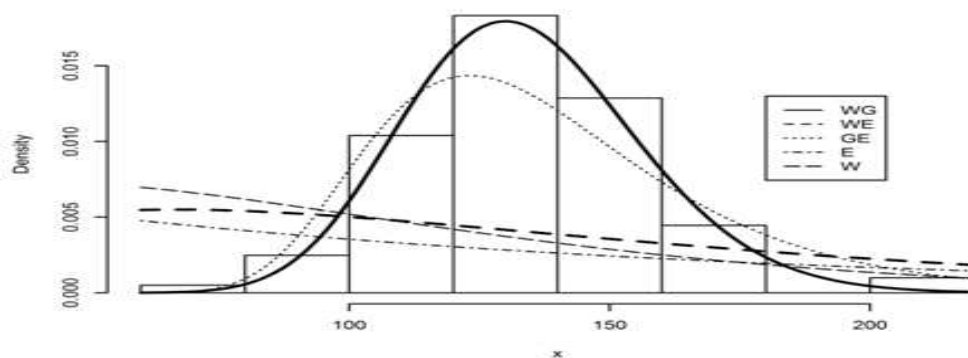

Figure 9. Plots for the histogram and estimated densities.

Figure 9 also shows that WG distribution fits the data in a better way as compared to the WE, GE, Weibull and Exponential distributions.

(III) **Using Kolmogorov-Smirnov Distances:**

For checking the best fitted distribution to the data set, KS distances between empirical and fitted survival functions are computed for WG, WE, GE, Weibull and Exponential distributions. These distances and the corresponding p-values are reported in Table 4.

Table 4. Kolmogorov-Smirnov (KS) distances and the corresponding p-values.

Distribution	Kolmogorov-Smirnov distances	p-value
WG	.0491	.5852
WE	.2503	.0800
GE	.1224	.1082
Weibull	.0995	.2469
Exponential	.2559	.0954

Since least KS distance and highest p-value correspond to WG distribution, hence it is concluded that WG provides the best fit against the rival distributions WE, GE, Weibull and Exponential distributions.

(IV) **Using Likelihood Ratio Test:**

We compare WG with WE distribution using the Likelihood Ratio test since WE is a special case of WG distribution. This amounts to testing H_0: $\beta = 1$ (WE) versus H_1: $\beta \neq 1$ (WG). The value of LR statistic is computed to be 202.284 with corresponding p-value as .00001. Hence H_0 is rejected and it is concluded that WG provides a better fit than WE distribution.

6. Conclusions

We introduce a new three-parameter distribution known as Weighted Gamma (WG) distribution which provides extension to Exponential, Gamma and Weighted Exponential distributions with a broader class of failure rate functions. It can be obtained as a hidden upper truncation model and also as a linear combination of two Gamma models. We study distributional and ageing properties of the WG distribution and check its superiority over some existing distributions viz WE, GE, Weibull and Exponential. The estimation of parameters is done by the method of maximum likelihood and the information matrix is derived. An application to a real data set shows that the fit of the WG distribution is superior to the fits using the WE, GE, Weibull and Exponential distributions.

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