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ALGEBRAIC STRUCTURE FOR THE CROSSING OF BALANCED AND STAIR NESTED DESIGNS

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Abstract

Stair nesting allows us to work with fewer observations than the most usual form of nesting, the balanced nesting. In the case of stair nesting the amount of information for the different factors is more evenly distributed. This new design leads to greater economy, because we can work with fewer observations. In this work we present the algebraic structure of the cross of balanced nested and stair nested designs, using binary operations on commutative Jordan algebras. This new cross requires fewer observations than the usual cross balanced nested designs and it is easy to carry out inference.

Keywords: balanced nested designs, stair nested designs, crossing, commutative Jordan algebras, variance components, inference.

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1. Introduction

In the most usual form of nesting, the balanced nesting, the number of treatments will be $\prod_{k=1}^{u} a\,(k)$, where $a\,(k)$ is the number of levels for each factor. If we have $a\,(1)$ levels for the first factor, each of these levels nest $a\,(2)$ levels of the second factor and so on. In balanced nesting we are forced to divide repeatedly the plots and we have few degrees of freedom for the first levels, see for instance Khuri *et al.* (1998).

In stair nesting we have $a^{\bullet}\,(1)$ "active" levels for the first factor, combined with a single level of all the other factors. Then a new single level for the first factor, combined with $a^{\bullet}\,(2)$ new "active" levels of the second factor, combined with a single level of all the other factors and so on. The number of treatments will then be $\sum_{k=1}^{u} a^{\bullet}\,(k) = [a\,(1) - (u-1)] + \sum_{k=2}^{u} a\,(k)$ instead of $\prod_{k=1}^{u} a\,(k)$ which we would have with balanced nesting. Thus this model leads to a greater economy, because we can work with fewer observations, and the amount of information for the different factors is more evenly distributed, see Fernandes *et al.* (2010, 2012). Moreover it is easy to carry out inference since this new design is very similar to the usual cross of balanced nested designs, that is already well studied.

In our approach we use binary operations on commutative Jordan algebras, CJA, to study models obtained from simpler ones, through crossing and nesting. In the study of the algebraic structure of balanced nesting we use the restricted Kronecker product of CJA and in the study of the algebraic structure of stair nesting we use the cartesian product of CJA. To study models obtained by crossing we use the Kronecker product of CJA. In Section 2 we present results on CJA and these binary operations. In Section 3 we present the algebraic structure for the cross of balanced nested and stair nested designs. In Section 4 we show how to carry out inference for these models obtained by crossing. In Section 5 we present an application to compare both studies, the cross of balanced nesting and the cross of balanced nesting and stair nesting.

2. Commutative Jordan algebras

Jordan algebras were introduced by Jordan *et al.* (1934) while presenting a new formulation of Quantum Mechanics. Later on Jacobson (1968) published a book where he gave a comprehensive account of the structure and representation theory of Jordan algebras. It covers foundation material, structure theory and representation theory for Jordan algebras. Later they were rediscovered by Seely (1970a, 1970b, 1971) who used these structures in linear statistical inference. Seely (1970a, 1970b) called these structures quadratic spaces since they were linear spaces constituted by symmetric matrices that commute and containing the squares of their matrices, but for priority sake we name them as CJA.

Every commutative Jordan algebra $\mathcal{A}$ has one and only one basis, the principal basis $\mathrm{pb}(\mathcal{A})$, constituted by pairwise orthogonal orthogonal projection matrices, POOPM, see Seely (1971). If the sum of the matrices in $\mathrm{pb}(\mathcal{A})$ is the identity matrix, we say that $\mathcal{A}$ is complete.

Since the matrices in $\mathrm{pb}(\mathcal{A})$ are idempotent and pairwise orthogonal, any projection matrix belonging to $\mathcal{A}$ is idempotent, so it will be the sum of all or part of the matrices in $\mathrm{pb}(\mathcal{A})$. The rank of an orthogonal projection matrix will be the sum of the ranks of those matrices in $\mathrm{pb}(\mathcal{A})$ whose sum is that matrix. Thus an orthogonal projection matrix with rank 1 will belong to $\mathrm{pb}(\mathcal{A})$ whenever it belongs to $\mathcal{A}$. With $\mathbf{1}^n$ the vector with n components equal to 1 and $\mathbf{J}_n = \mathbf{1}^n(\mathbf{1}^n)'$, $\frac{1}{n}\mathbf{J}_n$ will be an orthogonal projection matrix with rank 1, belonging to $\mathrm{pb}(\mathcal{A})$ whenever it belongs to $\mathcal{A}$, where $\mathbf{B}'$ is the transpose matrix of $\mathbf{B}$. We say that a CJA of $n \times n$ matrices is regular when $\frac{1}{n}\mathbf{J}_n$ belongs to $\mathcal{A}$.

If $\mathrm{pb}(\mathcal{A})$ is constituted by matrices $\mathbf{Q}_1, \ldots, \mathbf{Q}_u$ and the row vectors of $\mathbf{A}_j$ constitute an orthogonal basis for the range space $\mathrm{R}(\mathbf{Q}_j)$ of $\mathbf{Q}_j$, $j = 1, \ldots, u$, we put $\mathrm{pb}(\mathcal{A})^{\frac{1}{2}} = \{\mathbf{A}_1, \ldots, \mathbf{A}_u\}$. We then have, for $j = 1, \ldots, u$,

$$\begin{cases} \mathbf{A}_j\mathbf{A}_j' = \mathbf{I}_{g_j} \\ \\ \mathbf{A}_j'\mathbf{A}_j = \mathbf{Q}_j \end{cases},$$

with $g_j = \mathrm{rank}(\mathbf{Q}_j)$, and $\mathbf{I}_s$ the $s \times s$ identity matrix. Moreover since the $\mathbf{Q}_1, \ldots, \mathbf{Q}_u$ are pairwise orthogonal matrices we will have $\mathbf{A}_j\mathbf{A}_{j'}' = \mathbf{0}_{g_j \times g_{j'}}$, with $j \neq j'$, where $\mathbf{0}_{r\times s}$ is the $r \times s$ null matrix.

Given $\mathbf{M}$, a regular matrix belonging to $\mathcal{A}$, we have $\mathbf{M} = \sum_{j=1}^{u} m_j\mathbf{Q}_j$, with $\mathbf{Q}_1, \ldots, \mathbf{Q}_u$ the matrices in $\mathrm{pb}(\mathcal{A})$. Since $\mathbf{Q}_1, \ldots, \mathbf{Q}_u$ are POOPM we will have

$$\mathbf{M}^{-1} = \sum_{j=1}^{u} m_j^{-1}\mathbf{Q}_j.$$

Moreover, for $j = 1, \ldots, u$, if $\mathbf{Q}_j = \mathbf{A}_j'\mathbf{A}_j$ we will have $\mathbf{M} = \sum_{j=1}^{u} m_j\mathbf{A}_j'\mathbf{A}_j$, so the row vectors of $\mathbf{A}_j$ will be the eigenvectors of $\mathbf{M}$ associated to the eigenvalues m_j with multiplicity $g_j = \mathrm{rank}(\mathbf{A}_j) = \mathrm{rank}(\mathbf{Q}_j)$ and so

$$\det(\mathbf{M}) = \prod_{j=1}^{u} m_j^{g_j}.$$

Three interesting binary operations have been defined in CJA, see Fernandes *et al.* (2010) and Fonseca *et al.* (2006). In our approach is of greater interest to know the principal basis of the CJA.

The first operation is the Kronecker product represented by $\otimes$, and is useful to study models obtained by crossing the treatments of two former models. Given the families of matrices $\mathcal{V}_1$ and $\mathcal{V}_2$, $\mathcal{V}_1 \otimes \mathcal{V}_2$ will be the family of matrices $\mathbf{G}_1 \otimes \mathbf{G}_2$ with $\mathbf{G}_d \in \mathcal{V}_d$, $d = 1, 2$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are CJA the principal basis of $\mathcal{A}_1 \otimes \mathcal{A}_2$ will be

$$\mathrm{pb}\left(\mathcal{A}_1 \otimes \mathcal{A}_2\right) = \mathrm{pb}\left(\mathcal{A}_1\right) \otimes \mathrm{pb}\left(\mathcal{A}_2\right).$$

These results can be generalized to more than two CJA.

The second operation is the restricted Kronecker product represented by $*$, and is useful to study models obtained by nesting all the treatments of one of these models within each treatment of the other. We use this operation in the study of models obtained by usual nesting. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two CJA of $n_1 \times n_1$ and $n_2 \times n_2$ matrices, respectively. If $\mathrm{pb}\left(\mathcal{A}_l\right) = \{\mathbf{Q}_{l,0}, \ldots, \mathbf{Q}_{l,u_l}\}$, $l = 1, 2$, the principal basis of $\mathcal{A}_1 * \mathcal{A}_2$ will be

$$\begin{aligned}\mathrm{pb}\left(\mathcal{A}_1 * \mathcal{A}_2\right) = &\left\{\mathbf{Q}_{1,0} \otimes \mathbf{Q}_{2,0}, \ldots, \mathbf{Q}_{1,u_1} \otimes \mathbf{Q}_{2,0}\right\} \cup \\ &\cup \left\{\sum_{i=0}^{u_1} \mathbf{Q}_{1,i} \otimes \mathbf{Q}_{2,1}, \ldots, \sum_{i=0}^{u_1} \mathbf{Q}_{1,i} \otimes \mathbf{Q}_{2,u_2}\right\}.\end{aligned}$$

These results can be generalized to more than two CJA.

The third operation is the cartesian product represented by $\times$, and we use this operation in the study of models obtained by stair nesting. Let $D\left(\mathbf{B}_1, \ldots, \mathbf{B}_u\right)$ be the block-wise diagonal matrix with principal blocks $\mathbf{B}_1, \ldots, \mathbf{B}_u$. Given the CJA $\mathcal{A}_1, \ldots, \mathcal{A}_u$, their cartesian product, $\times_{h=1}^{u} \mathcal{A}_h$, will be the set of the diagonal block-wise matrix $D\left(\mathbf{M}_1, \ldots, \mathbf{M}_u\right)$ with $\mathbf{M}_h \in \mathcal{A}_h$, $h = 1, \ldots, u$.

Let $\mathcal{A}_h$ be commutative Jordan algebra constituted by $a_h \times a_h$ matrices with principal basis $\mathbf{Q}_h = \{\mathbf{Q}_{h,1}, \ldots, \mathbf{Q}_{h,v_h}\}$. Then the principal basis of the cartesian product of commutative Jordan algebras, $\times_{h=1}^{u} \mathcal{A}_h$, will be $\bigcup_{h=1}^{u} \mathbf{Q}_{a,h}$, where $\mathbf{Q}_{a,h}$ is the family of the diagonal block-wise matrices $D\left(\mathbf{B}_1, \ldots, \mathbf{B}_u\right)$, with $\mathbf{B}_{h'} = \mathbf{0}_{a_{h'} \times a_{h'}}$, if $h' \neq h$, and $\mathbf{B}_h \in \mathbf{Q}_h$, with $h = 1, \ldots, u$.

3. Model

In this design we consider L groups with $u_1, \ldots, u_L$ factors in each group, respectively, and r replicates. In the first s^{th} groups we have balanced nesting and in the other groups we have stair nesting. In each group in which we have balanced nesting we have $a_i(1), \ldots, a_i(u_i)$, with $i = 1, \ldots, s$, levels for each factor and in each group in which we have stair nesting we have $a_j(1), \ldots, a_j(u_j)$, with $j = s+1, \ldots, L$, "active" levels for each factor. So we have $n_i = \prod_{k=1}^{u_i} a_i(k)$, with $i = 1, \ldots, s$, treatments in the i^{th} group and $n_j = \sum_{k=1}^{u_j} a_j(k)$, with $j = s+1, \ldots, L$, treatments in the j^{th} group. Finally, we have $n = (n_1 \times \ldots n_{s-1} \times n_s \times n_{s+1} \times \ldots \times n_L) \times r$ observations.

If we consider the intervenient factors, the parameters will be:

- the general mean value;
- the effects of the different factors;
- the interactions between levels of factors in distinct groups.

These parameters will be associated to the $\mathbf{h} = (h_1, \ldots, h_L)$ vectors of

$$\Gamma = \{\mathbf{h} : 0 \leq h_l \leq u_l, l = 1, \ldots, L\}.$$

If all components of $\mathbf{h}$ are null, the vector corresponds to general mean value; if $\mathbf{h}$ has only one non-null component, the vector corresponds to the effects of the factor indicated by that component; otherwise $\mathbf{h}$ will be the vector associated to the interactions between the factors indicated by the non-null components of $\mathbf{h}$.

Now we will present the structure for the s groups with balanced nesting. For the i^{th} group, with $i = 1, \ldots, s$, the first factor will have $a_i(1)$ levels. If $u_i > 1$, each level of the first factor nests $a_i(2)$ levels of the second factor and so on. So we will have $c_i(k) = \prod_{m=1}^{k} a_i(m)$ level combinations for the k first factors, with $k = 1, \ldots, u_i$. Each of these combinations nest $b_i(k) = \frac{c_i(u_i)}{c_i(k)}$, with $k = 1, \ldots, u_i$, level combinations of the following factors. Finally we have $n_i = c_i(u_i)$ level combinations.

The model can be written in its canonical form as

$$\mathbf{y}_i = \sum_{k=0}^{u_i} \mathbf{A}_i(k)' \widetilde{\eta}_i(k),$$

where vectors $\widetilde{\eta}_i(k) = \mathbf{A}_i(k)\mathbf{y}_i$ correspond to the effects of the factors. Matrices $\mathbf{A}_i(k)$ are defined as

$$\mathbf{A}_i(0) = \bigotimes_{k=1}^{u_i} \frac{1}{\sqrt{a_i(k)}} \left(\mathbf{1}^{a_i(k)}\right)'$$

and

$$\mathbf{A}_i(k) = \left[\bigotimes_{m=0}^{k-1} \mathbf{I}_{a_i(m)}\right] \otimes \mathbf{T}_{a_i(k)} \otimes \left[\bigotimes_{m=k+1}^{u_i} \frac{1}{\sqrt{a_i(m)}} \left(\mathbf{1}^{a_i(m)}\right)'\right],$$

where $i = 1, \ldots, s$, $a_i(0) = 1$, $k = 1, \ldots, u_i$, $\bigotimes_{m=p+q}^{p} \mathbf{P}_m = \mathbf{I}_1$, $q > 0$ and $\mathbf{T}_1 = \mathbf{I}_1$. Matrix $\mathbf{T}_v$ is obtained deleting the first row equal to $\frac{1}{\sqrt{v}}(\mathbf{1}^v)'$ of a $v \times v$ orthogonal matrix and $\mathbf{T}_v'\mathbf{T}_v = \mathbf{I}_v - \frac{1}{v}\mathbf{J}_v$. We thus have

$$g_i(k) = \operatorname{rank}[\mathbf{A}_i(k)] = [a_i(k) - 1] \prod_{m=0}^{k-1} a_i(m).$$

We assume that the observations vector $\mathbf{y}_i$ has mean vector

$$\mu = \frac{1}{\sqrt{n_i}} \left(1^{n_i}\right)' \mu$$

and variance-covariance matrix

$$\mathbf{V}_i = \sum_{k=1}^{u_i} \gamma_i(k) \, \mathbf{Q}_i(k) \, ,$$

where $\mathbf{Q}_i(k) = \mathbf{A}_i(k)' \, \mathbf{A}_i(k)$ and $\gamma_i(k) = \sum_{l=k}^{u_i} b_i(l) \, \sigma_i^2(l)$, with $k = 1, \ldots, u_i$.

The principal basis of the CJA associated to the balanced nesting,

$$\mathrm{pb}\left[*_{k=1}^{u_i} \mathcal{A}\left(a_i(k)\right) \right] ,$$

is constituted by matrices $\mathbf{Q}_i(k)$, with $i = 1, \ldots, s$.

Now we will present the structure for the $L - s$ remaining groups with stair nesting. For the j^{th} group, with $j = s+1, \ldots, L$, we have $a_j(1)$ "active" levels for the first factor, combined with a single level of all other factors; then a new single level for the first factor, combined with $a_j(2)$ new "active" levels of the second factor, combined with a single level of all other factors; and so on. So we will have $c_j(k) = (u_j - k) + \sum_{m=1}^{k} a_j(m)$ level combinations for the k first factors, with $k = 1, \ldots, u_j$.

We assume that the observations vector $\mathbf{y}_j$ is normal with mean vector μ and variance-covariance matrix $\mathbf{V}_j$. For $z = 1, 2$ and $k = 1, \ldots, u_j$, the

$$\widetilde{\eta}_{j,z}(k) = \mathbf{A}_{j,z}(k) \, \mathbf{y}_j$$

will be normal with mean vectors $\eta_{j,z}(k) = \mathbf{A}_{j,z}(k) \, \mu_j$ and variance-covariance matrices $\gamma_{j,z}(k) \, \mathbf{I}_{g_{j,z}(k)}$.

For $k = 1, \ldots, u_j$, matrices $\mathbf{A}_{j,z}(k)$ are defined as

$$\begin{cases} \mathbf{A}_{j,1}(k) = D\left(\mathbf{C}_{1,1}(k), \ldots, \mathbf{C}_{1,u_j}(k)\right) \\ \mathbf{A}_{j,2}(k) = D\left(\mathbf{C}_{2,1}(k), \ldots, \mathbf{C}_{2,u_j}(k)\right) \end{cases} ,$$

with

$$\begin{cases} \mathbf{C}_{1,k^*}(k) = \mathbf{C}_{2,k^*}(k) = \left[\mathbf{0}^{a_j}(k^*)\right]' , \quad k \neq k^* \\ \mathbf{C}_{1,k}(k) = \frac{1}{\sqrt{a_j(k)}} \left[\mathbf{1}^{a_j(k)}\right]' \\ \mathbf{C}_{2,k}(k) = \mathbf{T}_{a_j(k)} \, . \end{cases}$$

We thus have

$$\begin{cases} g_{j,1}(k) = \operatorname{rank}\left[\mathbf{A}_{j,1}(k)\right] = 1 \\ g_{j,2}(k) = \operatorname{rank}\left[\mathbf{A}_{j,2}(k)\right] = a_j(k) - 1\,, \end{cases}$$

for $k = 1, \ldots, u_j$.

Moreover the observations vector $\mathbf{y}_j$ has mean vector $\mu_j = \mathbf{1}^{n_j}\mu$ and variance-covariance matrix

$$\mathbf{V}_j = \sum_{k=1}^{u_j}\sum_{z=1}^{2}\gamma_{j,z}(k)\,\mathbf{Q}_{j,z}(k)\,,$$

where

$$\begin{cases} \gamma_{j,1}(h) = \displaystyle\sum_{m=1}^{k-1} a_j(m)\,\sigma_j^2(m) + \sum_{m=k}^{u_j}\sigma_j^2(m) \\ \gamma_{j,2}(k) = \displaystyle\sum_{m=h}^{u_j}\sigma_j^2(m) \end{cases}$$

and $\mathbf{Q}_{j,z}(k) = \mathbf{A}_{j,z}(k)'\,\mathbf{A}_{j,z}(k)$, with $z = 1, 2$.

The principal basis of the CJA associated to the stair nesting,

$$\mathrm{pb}\left[\times_{k=1}^{u_j}\mathcal{A}(a_j(k))\right],$$

is constituted by matrices $\mathbf{Q}_{j,z}(k)$, with $j = s+1, \ldots, L$ and $z = 1, 2$.

When we cross these L groups we obtain the model in its canonical form as

$$\mathbf{y} = \sum_{\mathbf{h}\in\Gamma}\mathbf{A}(\mathbf{h})'\,\eta(\mathbf{h}) + \mathbf{A}^{\perp}\mathbf{e},$$

with

$$\mathbf{A}(\mathbf{h}) = \mathbf{A}(1,\mathbf{h}) \otimes \mathbf{A}(2,\mathbf{h}) \otimes \frac{1}{\sqrt{r}}\left(\mathbf{1}^{r}\right)'.$$

Matrices $\mathbf{A}(\mathbf{h})$ are obtained as follows:

- if all components of $\mathbf{h}$ are null, for the first s components we have

  $$\mathbf{A}(1,\mathbf{h}) = \bigotimes_{i=1}^{s}\mathbf{A}_i(0)$$

  and for the $L - s$ remaining components we have

  $$\mathbf{A}(2,\mathbf{h}) = \bigotimes_{j=s+1}^{L}\left(\sum_{m=1}^{u_j}\mathbf{A}_{j,1}(m)\right).$$

- if in the first s components of $\mathbf{h}$ we have a single non-null component and the $L-s$ remaining components are null we put

$$\mathbf{A}(1,\mathbf{h}) = \left(\bigotimes_{i=1}^{l-1} \mathbf{A}_i(0)\right) \otimes \mathbf{A}_l(k) \otimes \left(\bigotimes_{i=l+1}^{s} \mathbf{A}_i(0)\right),$$

with $k = 1, \ldots, u_l$, and

$$\mathbf{A}(2,\mathbf{h}) = \bigotimes_{j=s+1}^{L} \left(\sum_{m=1}^{u_j} \mathbf{A}_{j,1}(m)\right).$$

- if the first s components of $\mathbf{h}$ are null component and in the $L-s$ remaining components we have a single non-null component we put

$$\mathbf{A}(1,\mathbf{h}) = \bigotimes_{i=1}^{s} \mathbf{A}_i(0)$$

and

$$\mathbf{A}(2,\mathbf{h}) = \left[\bigotimes_{j=s+1}^{f} \left(\sum_{m=1}^{u_j} \mathbf{A}_{j,1}(m)\right)\right] \otimes \mathbf{A}_{f,2}(t) \otimes \left[\bigotimes_{j=f+1}^{L} \left(\sum_{m=1}^{u_j} \mathbf{A}_{j,1}(m)\right)\right],$$

with $t = 1, \ldots, u_f$.

- if in the first s components of $\mathbf{h}$ we have a single non-null component and in the $L-s$ remaining components we have too a single non-null component we put

$$\mathbf{A}(1,\mathbf{h}) = \left(\bigotimes_{i=1}^{l-1} \mathbf{A}_i(0)\right) \otimes \mathbf{A}_l(k) \otimes \left(\bigotimes_{i=l+1}^{s} \mathbf{A}_i(0)\right),$$

with $k = 1, \ldots, u_l$, and

$$\mathbf{A}(2,\mathbf{h}) = \left[\bigotimes_{j=s+1}^{f} \left(\sum_{m=1}^{u_j} \mathbf{A}_{j,1}(m)\right)\right] \otimes \mathbf{A}_{f,2}(t) \otimes \left[\bigotimes_{j=f+1}^{L} \left(\sum_{m=1}^{u_j} \mathbf{A}_{j,1}(m)\right)\right],$$

with $t = 1, \ldots, u_f$.

If we have more than two non-null components in $\mathbf{h}$, we obtain matrices $\mathbf{A}(\mathbf{h})$ proceeding in the same way as we did for the interaction between two factors. Matrix $\mathbf{A}^{\perp}$ is defined as

$$\mathbf{A}^{\perp} = \mathbf{I}_n \otimes \mathbf{T}_r.$$

The principal basis of the CJA associated to the cross of balanced nesting and stair nesting,

$$\mathrm{pb}\left[\left[\bigotimes_{i=1}^{s}\left(*_{k=1}^{u_i}\mathcal{A}\left(a_i\left(k\right)\right)\right)\right]\otimes\left[\bigotimes_{j=s+1}^{L}\left(\times_{k=1}^{u_j}\mathcal{A}\left(a_j\left(k\right)\right)\right)\right]\right]$$

is constituted by matrices $\mathbf{Q}\left(\mathbf{h}\right)=\mathbf{A}\left(\mathbf{h}\right)'\mathbf{A}\left(\mathbf{h}\right)$, $\mathbf{h}\in\Gamma$.

4. Inference

We consider that the model has random effects and we assume that vector $\mathbf{y}$ is normal with mean vector μ and variance-covariance matrix

$$\mathbf{V}=\sum_{\mathbf{h}\in\Gamma}\gamma\left(\mathbf{h}\right)\mathbf{Q}\left(\mathbf{h}\right)+\sigma^2\mathbf{Q}^{\perp},$$

with

$$\begin{cases}\mathbf{Q}\left(\mathbf{h}\right)=\mathbf{A}\left(\mathbf{h}\right)'\mathbf{A}\left(\mathbf{h}\right)\\ \mathbf{Q}^{\perp}=\left(\mathbf{A}^{\perp}\right)'\mathbf{A}^{\perp}.\end{cases}$$

We put $\mathbf{y}\sim\mathcal{N}\left(\mu,\mathbf{V}\right)$. So we have

$$\begin{cases}\widetilde{\eta}\left(\mathbf{h}\right)=\mathbf{A}\left(\mathbf{h}\right)\mathbf{y}\\ \widetilde{\eta}^{\perp}=\mathbf{A}^{\perp}\mathbf{y},\end{cases}$$

where

$$\begin{cases}\widetilde{\eta}\left(\mathbf{h}\right)\sim\mathcal{N}\left[\mathbf{0},\gamma\left(\mathbf{h}\right)\mathbf{I}_{g(\mathbf{h})}\right]\\ \widetilde{\eta}^{\perp}\sim\mathcal{N}\left[\mathbf{0},\sigma^2\mathbf{I}_g\right],\end{cases}$$

with $g\left(\mathbf{h}\right)=\mathrm{rank}\left[\mathbf{A}\left(\mathbf{h}\right)\right]$, $g=\mathrm{rank}\left[\mathbf{A}^{\perp}\right]$ and

$$\gamma\left(\mathbf{h}\right)=\sigma^2+\sum_{\mathbf{k}:\mathbf{h}\leq\mathbf{k}}\sigma^2\left(\mathbf{k}\right),\quad \mathbf{k}\in\Gamma.$$

We have the sums of squares

$$S\left(\mathbf{h}\right)=\left\|\mathbf{A}\left(\mathbf{h}\right)\mathbf{y}\right\|^2\sim\gamma\left(\mathbf{h}\right)\chi^2_{g(\mathbf{h})},\quad \mathbf{h}\in\Gamma$$

and

$$S = \left\| \mathbf{A}^{\perp} \mathbf{y} \right\|^2 \sim \sigma^2 \chi_g^2.$$

Thus we have the unbiased estimators

$$\widetilde{\gamma}\left(\mathbf{h}\right) = \frac{S\left(\mathbf{h}\right)}{g\left(\mathbf{h}\right)}$$

and

$$\widetilde{\sigma}^2 = \frac{S}{g}.$$

Then we can obtain the unbiased estimators for the variance components

$$\widetilde{\sigma}^2\left(\mathbf{k}\right) = \sum_{\mathbf{h}\in\Theta(\mathbf{k})} \left(-1\right)^{n(\mathbf{k},\mathbf{h})} \widetilde{\gamma}\left(\mathbf{h}\right),$$

with $n\left(\mathbf{k},\mathbf{h}\right)$ the number of components of $\mathbf{k}$ lesser than homologous components of $\mathbf{h}$ and

$$\Theta\left(\mathbf{k}\right) = \left\{\mathbf{h} : k_l \leq h_l \leq \min\left\{u_l; k_l + 1\right\}, \quad l = 1, 2\right\}.$$

Furthermore we have

$$\begin{cases} \det\left[\mathbf{V}\right] = \prod_{\mathbf{h}\in\Gamma} \left[\gamma\left(\mathbf{h}\right)\right]^{g(\mathbf{h})} \left(\sigma^2\right)^{g^{\perp}} \\ \mathbf{V}^{-1} = \sum_{\mathbf{h}\in\Gamma} \gamma^{-1}\left(\mathbf{h}\right) \mathbf{Q}\left(\mathbf{h}\right) + \frac{1}{\sigma^2}\mathbf{Q}^{\perp}. \end{cases}$$

The normal density of $\mathbf{y}$ will be

$$n\left(\mathbf{y}\right) = \frac{\exp\left[-\frac{1}{2}\left(\frac{\|\mathbf{A}(\mathbf{0})\mathbf{y} - \mathbf{A}(\mathbf{0})\mu\|^2}{\gamma(\mathbf{0})} + \sum_{\mathbf{h}\in\Gamma\setminus\{\mathbf{0}\}} \frac{S(\mathbf{h})}{\gamma(\mathbf{h})} + \frac{S}{g}\right)\right]}{\left(2\pi\right)^{\frac{n}{2}} \prod_{\mathbf{h}\in\Gamma} \left[\gamma\left(\mathbf{h}\right)\right]^{\frac{g(\mathbf{h})}{2}} \left(\sigma^2\right)^{\frac{g^{\perp}}{2}}}.$$

Using the factorization theorem we see that $\widetilde{\eta}\left(\mathbf{0}\right) = \mathbf{A}\left(\mathbf{0}\right)\mathbf{y}$, $S\left(\mathbf{h}\right)$, with $\mathbf{h} \in \Gamma \setminus \{\mathbf{0}\}$, and S are sufficient statistics. According to the Rao-Blackwell theorem estimators should, as we have previously shown, be function of the sufficient statistics.

5. Application

The problem of genetic homogeneity of the grapevine castes is of great practical interest. We will consider an experiment to test the genetic homogeneity. The grapevines are produced through cloning. Clones with a possible common ancestor constitute a caste. Although the castes should be genetically homogeneous, some farmers consider that this is not always true.

Now we will discuss the results of an experiment in which two groups of three clones, obtained in different regions for "Touriga Nacional" were cultivated jointly. In this experiment we use a model with four random factors, the location inside the field, the degree of humidity on the ground, the origin of the plants and the clone. We consider three replicates. The grapevines were planted in a rectangular grid and in each row of the grid a clone was planted. The production in kilos per grapevine are presented in Table 1.

In this work we intend to compare two studies, firstly the cross of balanced nesting, see for instance Fonseca *et al.* (2003, 2006), and secondly the cross balanced and stair nesting, that we present in this work.

Table 1. Production in kg per plant.

		L 1					L 2				
		H 1	H 2	H 3	H 4	H 5	H 1	H 2	H 3	H 4	H 5
		3,00	1,85	0,75	1,35	1,45	1,80	0,70	2,50	1,70	0,40
	C 1	2,85	1,75	0,90	1,40	1,40	1,85	0,75	2,55	1,75	0,50
		3,05	1,90	0,85	1,30	1,50	1,90	0,80	2,65	1,80	0,45
		1,00	1,10	1,00	1,60	1,50	1,60	1,75	0,50	1,35	1,10
O 1	C 2	1,15	1,20	1,05	1,65	1,55	1,50	1,70	0,55	1,30	1,15
		0,95	0,95	1,10	1,50	1,60	1,55	1,75	0,60	1,40	1,20
		1,10	1,50	1,80	1,45	1,25	0,85	0,65	0,55	0,90	0,90
	C 3	1,05	1,55	1,85	1,50	1,20	0,90	0,60	0,45	0,95	0,95
		1,10	1,60	1,70	1,55	1,35	0,95	0,55	0,60	1,00	1,00
		1,75	3,50	2,50	2,00	0,65	2,00	3,00	2,55	3,00	2,65
	C 1	1,80	3,45	2,55	2,10	0,70	2,05	3,05	2,50	3,05	2,60
		1,70	3,40	2,60	2,15	0,55	2,00	2,95	2,45	2,90	2,55
		1,10	1,05	0,50	1,05	1,25	1,20	1,35	1,20	0,30	2,50
O 2	C 2	1,15	1,00	0,60	1,00	1,30	1,25	1,30	1,30	0,40	2,55
		1,00	1,05	0,65	1,10	1,35	1,35	1,25	1,40	0,45	2,60
		1,05	1,25	2,00	1,50	2,10	1,00	2,70	2,15	2,10	2,70
	C 3	1,00	1,20	2,05	1,60	2,05	0,90	2,60	2,00	2,00	2,75
		1,10	1,35	2,20	1,65	2,00	1,05	2,55	2,10	2,05	2,80

We consider that when we have a cross of balanced nesting we will have $3 \times 5 \times 2 \times 3 = 90$ treatments. On one side we have the origin, O, with two levels, that nest the clone, C, with three levels. On the other side we have the localization on the field, L, with three levels, that nest the degree of humidity, H, with five levels. Finally we have three replicates for each combination of factors. The number of levels for the different factores are $a_1(1) = 2$ and $a_1(2) = 3$ levels for the first group, and $a_2(1) = 3$ and $a_2(2) = 5$ levels for the second group. We have $n_1 = 2 \times 3 = 6$ treatments in the first group and $n_2 = 3 \times 5 = 15$ treatments in the second group. So we have $n = n_1 \times n_2 \times r = 6 \times 15 \times 3 = 270$ observations.

The parameters of the model will be associated to the $\mathbf{h} = (h_1, h_2)$ vectors of $\Gamma = \{\mathbf{h} : 0 \le h_l \le u_l, \quad l = 1, 2\}$.

For the first group we have matrices $\mathbf{A}_1(h)$, with $h = 1, 2$, defined by

$$\begin{cases} \mathbf{A}_1(0) = \frac{1}{\sqrt{2}}\left[\mathbf{1}^2\right]' \otimes \frac{1}{\sqrt{3}}\left[\mathbf{1}^3\right]' \\ \mathbf{A}_1(1) = \mathbf{T}_2 \otimes \frac{1}{\sqrt{3}}\left[\mathbf{1}^3\right]' \\ \mathbf{A}_1(2) = \mathbf{I}_2 \otimes \mathbf{T}_3\,, \end{cases}$$

with

$$\mathbf{T}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

and

$$\mathbf{T}_3 = \begin{bmatrix} \frac{2\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

For the second group we have matrices $\mathbf{A}_2(h)$, with $h = 1, 2$, defined by

$$\begin{cases} \mathbf{A}_2(0) = \frac{1}{\sqrt{3}}\left[\mathbf{1}^3\right]' \otimes \frac{1}{\sqrt{5}}\left[\mathbf{1}^5\right]' \\ \mathbf{A}_2(1) = \mathbf{T}_3 \otimes \frac{1}{\sqrt{5}}\left[\mathbf{1}^5\right]' \\ \mathbf{A}_2(2) = \mathbf{I}_3 \otimes \mathbf{T}_5\,, \end{cases}$$

with

$$\mathbf{T}_5 = \begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{10} & -\frac{\sqrt{5}}{10} & -\frac{\sqrt{5}}{10} & -\frac{\sqrt{5}}{10} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

When we cross both groups we obtain the model in its canonical form as

$$\mathbf{y} = \sum_{\mathbf{h}\in\Gamma} \mathbf{A}(\mathbf{h})'\, \eta(\mathbf{h}) + \mathbf{A}^{\perp}\mathbf{e},$$

where matrices $\mathbf{A}(\mathbf{h})$ are defined as

$$\mathbf{A}(\mathbf{h}) = \mathbf{A}(h_1, h_2) = \mathbf{A}_1(h_1) \otimes \mathbf{A}_2(h_2)\,,$$

with $h_1 = 0, 1, 2$ and $h_2 = 0, 1, 2$. Matrix $\mathbf{A}^{\perp}$ is given by

$$\mathbf{A}^{\perp} = \mathbf{I}_{90} \otimes \mathbf{T}_3.$$

In Tables 2 and 3 we present the results for the cross of balanced nesting.

Table 2. Production in kg per plant.

		L 3				
		H 1	H 2	H 3	H 4	H 5
		1,05	1,50	1,15	0,85	1,15
	C 1	1,10	1,60	1,20	0,70	1,10
		1,15	1,45	1,10	0,95	1,05
		0,75	0,65	0,90	0,85	1,05
O 1	C 2	0,80	0,60	0,95	0,95	1,10
		0,95	0,75	1,10	1,00	1,15
		0,90	0,90	0,55	0,70	0,35
	C 3	0,95	0,95	0,75	0,80	0,45
		0,90	1,05	0,60	0,65	0,40
		1,60	3,05	0,25	1,65	2,65
	C 1	1,65	3,00	0,35	1,70	2,50
		1,60	2,95	0,40	1,75	2,70
		1,05	1,95	2,00	2,20	2,35
O 2	C 2	1,10	1,90	2,05	2,30	2,40
		1,15	2,00	2,15	2,15	2,45
		1,60	1,10	2,05	1,50	3,00
	C 3	1,65	1,15	2,20	1,60	2,90
		1,75	1,25	1,95	1,65	3,15

Table 3. Results for the cross of balanced nesting.

$\mathbf{h}$	$S(\mathbf{h})$	$g(\mathbf{h})$	$\widetilde{\gamma}(\mathbf{h})$
$(1,0)$	0,0073	1	0,0073
$(2,0)$	4,3498	4	1,0875
$(0,1)$	0,1075	2	0,0538
$(0,2)$	21,1211	12	1,7601
$(1,1)$	3,7048	2	1,8524
$(1,2)$	24,0568	12	2,0047
$(2,1)$	7,9815	8	0,9977
$(2,2)$	81,6798	48	1,7017

The variances for the estimators are given by

$$\operatorname{Var}\left[\widetilde{\gamma}(\mathbf{h})\right] = \operatorname{Var}\left[\frac{S(\mathbf{h})}{g(\mathbf{h})}\right] = \frac{2\gamma^2(\mathbf{h})}{g(\mathbf{h})}.$$

So $\widetilde{\mathrm{Var}}\left[\widetilde{\gamma}\left(\mathbf{h}\right)\right]=\frac{2\widetilde{\gamma}^2(\mathbf{h})}{g(\mathbf{h})}$ and we have

$$\begin{cases}\widetilde{\mathrm{Var}}\left[\widetilde{\gamma}\left(1,0\right)\right]=0,0001054\\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}\left(2,0\right)\right]=0,5912780\\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}\left(0,1\right)\right]=0,0028871\\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}\left(0,2\right)\right]=0,5163180\\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}\left(1,1\right)\right]=3,4313800\\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}\left(1,2\right)\right]=0,6698280\\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}\left(2,1\right)\right]=0,2488460\\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}\left(2,2\right)\right]=0,1206520\\ \widetilde{\mathrm{Var}}\left[\widetilde{\sigma}^2\right]=0,0000002\,.\end{cases}$$

Now we will study the cross between two groups in which, in the first group we have a balanced nested design, and in the second group we have a stair nested design. So we have $L=2$ groups with $u_1=2$ and $u_2=2$ factors in each group and $r=3$ replicates. In the first group we have balanced nesting with two factors, the origin, O, and the clone, C, and in the second group we have stair nesting with two factors, the localization in the field, L, and the degree of humidity, H. The number of levels for the different factors are $a_1\left(1\right)=2$ and $a_1\left(2\right)=3$ levels for the first group, and $a_2\left(1\right)=2$ and $a_2\left(2\right)=5$ "active" levels for the second group. We have $n_1=2\times 3=6$ treatments in the first group and $n_2=2+5=7$ treatments in the second group. So we have $n=n_1\times n_2\times r=6\times 7\times 3=126$ observations. The chosen observations are presented in Table 4.

Table 4. Results for the cross of balanced nesting.

S	g	$\widetilde{\sigma}^2$
0,8683	180	0,0048

The parameters of the model will be associated to the $\mathbf{h}=(h_1,h_2)$ vectors of $\Gamma=\{\mathbf{h}:0\le h_l\le u_l,\ l=1,2\}$.

For the first group we have matrices $\mathbf{A}_1\left(k\right)$, with $k=1,2$, defined by

$$\begin{cases}\mathbf{A}_1\left(0\right)=\frac{1}{\sqrt{2}}\left[\mathbf{1}^2\right]'\otimes\frac{1}{\sqrt{3}}\left[\mathbf{1}^3\right]'\\ \mathbf{A}_1\left(1\right)=\mathbf{T}_2\otimes\frac{1}{\sqrt{3}}\left[\mathbf{1}^3\right]'\\ \mathbf{A}_1\left(2\right)=\mathbf{I}_2\otimes\mathbf{T}_3\,.\end{cases}$$

For the second group we have matrices $\mathbf{A}_{2,z}(k)$, with $k = 1,2$ and $z = 1,2$, defined by

$$\begin{cases} \mathbf{A}_{2,1}(1) = D\left(\left[\frac{1}{\sqrt{2}}\mathbf{1}^2\right]', \left[\mathbf{0}^5\right]'\right) \\ \mathbf{A}_{2,2}(1) = D\left(\mathbf{T}_2, \left[\mathbf{0}^5\right]'\right) \\ \mathbf{A}_{2,1}(2) = D\left(\left[\mathbf{0}^2\right]', \left[\frac{1}{\sqrt{5}}\mathbf{1}^5\right]'\right) \\ \mathbf{A}_{2,2}(2) = D\left(\left[\mathbf{0}^2\right]', \mathbf{T}_5\right) . \end{cases}$$

When we cross the both groups we obtain the model in its canonical form as

$$\mathbf{y} = \sum_{\mathbf{h}\in\Gamma} \mathbf{A}(\mathbf{h})' \eta(\mathbf{h}) + \mathbf{A}^{\perp}\mathbf{e},$$

where matrices $\mathbf{A}(\mathbf{h})$ are defined as

$$\begin{cases} \mathbf{A}(0,0) = \mathbf{A}_1(0,1) + \mathbf{A}_1(0,2) \\ \mathbf{A}(i,0) = \mathbf{A}_1(i,1) + \mathbf{A}_1(i,2) \quad i = 1,2 \\ \mathbf{A}(0,l) = \mathbf{A}_2(0,l) \quad l = 1,2 \\ \mathbf{A}(i,1) = \mathbf{A}_2(i,1) \quad i = 1,2 \\ \mathbf{A}(i,2) = \mathbf{A}_2(i,2) \quad i = 1,2 \, , \end{cases}$$

with matrices

$$\mathbf{A}_z(k_1, k_2) = \mathbf{A}_1(k_1) \otimes \mathbf{A}_{2,z}(k_2) \otimes \frac{1}{\sqrt{3}}\left(\mathbf{1}^3\right)',$$

for $z = 1,2$, $k_1 = 0,1,2$ and $k_2 = 1,2$. Matrix $\mathbf{A}^{\perp}$ is given by

$$\mathbf{A}^{\perp} = \mathbf{I}_{42} \otimes \mathbf{T}_3.$$

In Tables 5 and 6 we present the results for the cross of balanced nesting and stair nesting.

The variances of these estimators are given by

$$\text{Var}\left[\widetilde{\gamma}(\mathbf{h})\right] = \text{Var}\left[\frac{S(\mathbf{h})}{g(\mathbf{h})}\right] = \frac{2\gamma^2(\mathbf{h})}{g(\mathbf{h})}.$$

Table 5. Production in kg per plant.

		L 1					L 2				
		H 1	H 2	H 3	H 4	H 5	H 1	H 2	H 3	H 4	H 5
		3,00	1,85	0,75	1,35	1,45		0,70			
	C 1	2,85	1,75	0,90	1,40	1,40		0,75			
		3,05	1,90	0,85	1,30	1,50		0,80			
		1,00	1,10	1,00	1,60	1,50		1,75			
O 1	C 2	1,15	1,20	1,05	1,65	1,55		1,70			
		0,95	0,95	1,10	1,50	1,60		1,75			
		1,10	1,50	1,80	1,45	1,25		0,65			
	C 3	1,05	1,55	1,85	1,50	1,20		0,60			
		1,10	1,60	1,70	1,55	1,35		0,55			
		1,75	3,50	2,50	2,00	0,65		3,00			
	C 1	1,80	3,45	2,55	2,10	0,70		3,05			
		1,70	3,40	2,60	2,15	0,55		2,95			
		1,10	1,05	0,50	1,05	1,25		1,35			
O 2	C 2	1,15	1,00	0,60	1,00	1,30		1,30			
		1,00	1,05	0,65	1,10	1,35		1,25			
		1,05	1,25	2,00	1,50	2,10		2,70			
	C 3	1,00	1,20	2,05	1,60	2,05		2,60			
		1,10	1,35	2,20	1,65	2,00		2,55			

Table 6. Production in kg per plant.

		L 3				
		H 1	H 2	H 3	H 4	H 5
		1,05				
	C 1	1,10				
		1,15				
		0,75				
O 1	C 2	0,80				
		0,95				
		0,90				
	C 3	0,95				
		0,90				
		1,60				
	C 1	1,65				
		1,60				
		1,05				
O 2	C 2	1,10				
		1,15				
		1,60				
	C 3	1,65				
		1,75				

Table 7. Results for the cross of balanced nesting and stair nesting.

$\mathbf{h}$	$S(\mathbf{h})$	$g(\mathbf{h})$	$\widetilde{\gamma}(\mathbf{h})$
$(1,0)$	0,3821	2	0,1910
$(2,0)$	3,4081	8	0,4260
$(0,1)$	1,5625	1	1,5625
$(0,2)$	0,7618	4	0,1905
$(1,1)$	0,3025	1	0,3025
$(1,2)$	10,8458	4	2,7115
$(2,1)$	4,0741	4	1,0185
$(2,2)$	29,5411	16	1,8463

Table 8. Results for the cross of balanced nesting and stair nesting.

S	g	$\widetilde{\sigma}^2$
0,3733	84	0,0044

So $\widetilde{\mathrm{Var}}\left[\widetilde{\gamma}(\mathbf{h})\right] = \frac{2\widetilde{\gamma}^2(\mathbf{h})}{g(\mathbf{h})}$. For this case we have

$$\begin{cases} \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}(1,0)\right] = 0,0364916 \\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}(2,0)\right] = 0,0453705 \\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}(0,1)\right] = 4,8828100 \\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}(0,2)\right] = 0,0181372 \\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}(1,1)\right] = 0,1830120 \\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}(1,2)\right] = 3,6760000 \\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}(2,1)\right] = 0,5186850 \\ \widetilde{\mathrm{Var}}\left[\widetilde{\gamma}(2,2)\right] = 0,4261120 \\ \widetilde{\mathrm{Var}}\left[\widetilde{\sigma}^2\right] = 0,0000005\,. \end{cases}$$

We obtained similar results for both studies. The unbiased estimators for the canonical variance components are, in general, similar and the variances of these estimators are generally small in both studies. This similarity is even more interesting since when we cross balanced and stair nesting, only 126 observations are required, less that the 270 ones required for cross of balanced nesting. This is in fact a big advantage when we compare both studies since the cross of balanced nesting and stair nesting will allow experiments that will become cheaper, due to the fewer number of observations involved, or with the same resources we produce more experiments. Since in a practical experiment, the implementation cost is, many times, a decisive factor, the cross balanced and stair nesting will be a strong alternative to the cross of balanced nesting. Moreover, for the cross of balanced and stair nesting it is easy to carry out inference because it is very similar to the cross of balanced nesting, that is well studied.

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