DMPS Page

Discussiones Mathematicae
Probability and Statistics 32 (2012) 87–99
doi:10.7151/dmps.1146

AN APPROACH TO DISTRIBUTION OF THE PRODUCT OF TWO NORMAL VARIABLES

Antonio Seijas-Macías

University of A Corunha – Spain
e-mail: antonio.smacias@udc.es

and

Amílcar Oliveira

Unversidade Aberta – Portugal
Center of Statistics and Applications – University of Lisbon
e-mail: aoliveira@uab.pt

Abstract

The distribution of product of two normally distributed variables come from the first part of the XX Century. First works about this issue were [1] and [2] showed that under certain conditions the product could be considered as a normally distributed.

A more recent approach is [3] that studied approximation to density function of the product using three methods: numerical integration, Monte Carlo simulation and analytical approximation to the result using the normal distribution. They showed as the inverse variation coefficient $\frac{\mu}{\sigma}$ increases, the distribution of the product of two independent normal variables tends towards a normal distribution.

Our study is focused in Ware and Lad approaches. The objective was studying which factors have more influence in the presence of normality for the product of two independent normal variables. We have considered two factors: the inverse of the variation coefficient value $\frac{\mu}{\sigma}$ and the combined ratio (product of the two means divided by standard deviation): $\frac{\mu_1\mu_2}{\sigma}$ for two normal variables with the same variance.

Our results showed that for low values of the inverse of the variation coefficient (less than 1) normal distribution is not a good approximation for the product. Another one, influence of the combined ratio value is less than influence of the inverse of coefficients of variation value.

Keywords: product of normally distributed variables, inverse coefficient of variation, numerical integration, Monte Carlo simulation, combined ratio.

2010 Mathematics Subject Classification: 62E17, 62E10.

1. Introduction

This work is focused to study distribution of the product of two uncorrelated normal variables. The distribution of the product of normal variables is not, in general, a normally distributed variable. However, under some conditions, is showed that the distribution of the product can be approximated by means of a Normal distribution. This problem appeared linked to diverse studies in several fields: business, statistics, psychology and so on.

Previous work involving the distribution of the product of two Normally distributed variables has been undertaken by Craig [1] and Aroian [4]. Craig [1], was the first to determine the algebraic expression of the moment-generating function of the product, but he could not determine the distribution of the product. Conclusions standing out that the distribution of XY is a function of the coefficient of correlation of both variables and of two parameters that are proportional to the inverse of the coefficient of variation of each variable. Aroian [2] advanced in the investigations of Craig and proved that when the inverse of the coefficients of variation are big, the function of density of $Z = XY$ approximates to a Normal curve and, under certain conditions, the product approaches the standardized Pearson type III distribution.

These works are relatively old, but there are not at all well-known among mathematicians. Until 2003, when the introduction of computer and numerical and symbolic calculus were extend there are new advances in this problem. In 2003 Ware and Lad [3] published an article where restart the problem of the probability of the product of two Normally distributed variables. They compare three different methods: a numerical method approximation, which involves implementing a numerical integration procedure on MATLAB, a Monte Carlo construction and an approximation to the analytic result using the Normal distribution. They presented new graphics to understand the shape of the distribution of $Z = XY$.

Our work begins with the study of the works mentioned, where find a group of theoretical and practical results of interest for solving this problem. In Section 2 we consider a theoretical and historical analysis of the problem considered. Section 3 is focused on the analysis of the distribution of the product of normal variables using different approaches to the problem. The fourth section considers the degree of influence of the value of the inverse of the coefficient of variation on the character of normality of the distribution of the product and the combined ratio. In our work, calculations and simulations were implemented on MATHEMATICA. Finally, in Section 5 conclusions are presented.

2. Distribution of the product of two variables

Let X and Y be two continuous random variables, where $F_X(x), F_Y(y), f_X(x)$, $f_Y(y)$ are the respective Cumulative Distribution Function (CDF) and Probability Density Function (PDF). We consider a bivariate distribution of the two variables:

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y), \tag{1}$$

and PDF will be

$$f_{X,Y}(x,y) = \frac{\partial F_{X,Y}(x,y)}{\partial x \partial y}, \tag{2}$$

and CDF of a distribution of two variables

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)dudv. \tag{3}$$

Marginal density functions are:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \tag{4}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx. \tag{5}$$

Along this work we study only the product of two independent variables then, relationship between marginal distribution and joint distribution is:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y). \tag{6}$$

The density function of Y to X when two variable are continuous and the expected value are:

$$f_Y(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \tag{7}$$

$$E(Y \mid x) = \int_{-\infty}^{\infty} y f_Y(y \mid x)dy. \tag{8}$$

The expected value is a random variable and following is verified for X and Y independent random variables:

$$E(X \mid Y) = E(X) \tag{9}$$

$$E(XY) = E(X)E(Y). \tag{10}$$

Let $Z = XY$ a continuous random variable, product of two independent continuous random variables X and Y. Distribution function of Z is

$$F_Z(z) = \int_{\{(x,y):xy\leq z\}} f_{XY}(xy)dxdy, \tag{11}$$

where $\{(x,y) : xy \leq z\} = \{-\infty < x \leq 0, \frac{z}{x} \leq y < \infty\} \cup \{0 \leq x < \infty, -\infty < y \leq \frac{z}{x}\}$. Then

$$F_Z(z) = \int_{-\infty}^{0}\int_{\frac{z}{x}}^{\infty} f_{XY}(x,y)dydx + \int_{0}^{\infty}\int_{-\infty}^{\frac{z}{x}} f_{XY}(x,y)dydx. \tag{12}$$

Differentiating with respect to z we can get the density function of Z:

$$f_Z(z) = \int_{-\infty}^{0}\left(\frac{-1}{x}\right) f_{XY}\left(x,\frac{z}{x}\right)dx + \int_{0}^{\infty}\left(\frac{1}{x}\right) f_{XY}\left(x,\frac{z}{x}\right)dx. \tag{13}$$

In particular, if X and Y are non negative random variables we have following expressions for distribution and density functions:

$$F_Z(z) = \int_{0}^{\infty}\int_{0}^{\frac{z}{x}} f_{XY}(x,y)dydx, \tag{14}$$

$$f_Z(z) = \int_{0}^{\infty} f_{XY}\left(x,\frac{z}{x}\right)dx, \tag{15}$$

$$z \geq 0. \tag{16}$$

Then, from a theoretical point of view, the problem of calculating the distribution function and density function of a product of two variables is solved from the joint distribution of a two bivariate variable distribution function. But when we consider different types of distributions we have several problems to get the distribution of the product.

2.1. Product of two independent normal variables

Let (X, Y) a bivariate Normal distribution with independent variables and parameters: $\mu_x, \mu_y, \sigma_x, \sigma_y$ and ρ with:

$$-\infty < \mu_x, \mu_y < \infty, \tag{17}$$

$$\sigma_x > 0, \sigma_y > 0. \tag{18}$$

The joint density function is:

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma y} \exp\left\{-\frac{1}{2}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right)\right\} \tag{19}$$

with $-\infty < x, y < \infty$. Marginal density functions are:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right), \tag{20}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left(-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right). \tag{21}$$

Let $Z = XY$ a product of two normally distributed random variables, we consider the distribution of the random variable Z.

First approaches to this question are considered in [5], authors conclusions is that distribution function of a product of two independent normal variables is proportional to a Bessel function of the second kind of a purely imaginary argument of zero order. From this work, Craig [1] consider two independent normal variables and studies the distribution of $Z = \frac{XY}{\sigma_x\sigma_y}$, he gets this function as a subtraction of integrals:

$$F_Z(z) = \frac{e - \frac{r_1^2+r_2^2}{2}}{2\pi} \left(A(x) - B(x)\right), \tag{22}$$

$$A(x) = \int_0^\infty \left\{\left(-\frac{x^2}{2} - r_1 x - r_2 \frac{z}{x} + \frac{z^2}{2x^2}\right)\right\} \frac{dx}{x}, \tag{23}$$

$$B(x) = \int_{-\infty}^0 \left\{\left(-\frac{x^2}{2} - r_1 x - r_2 \frac{z}{x} + \frac{z^2}{2x^2}\right)\right\} \frac{dx}{x}, \tag{24}$$

with $r_1 = \frac{\mu_x}{\sigma_x}$ and $r_2 = \frac{\mu_y}{\sigma_y}$. This expression doesn't admit an analytical expression. For purposes of numerical computation this result can be expanded in an infinite Laurent series and Bessel functions; however, for large values of parameters r_1 and r_2 convergence of the series is very slow, even for values of r_i as 2.

In 1947 Aroian [2] shows that the probability function of Z approaches a normal curve and the Type III function and the Gram-Charlier Type a series are excellent approximations. He proofs that Z is assymtoptically normal if r_1 or r_2 (or both of them) tends to infinity.

Later, in 1978, Aroian [6] shows that if $r1 = r2 = r$, then when $r \to \infty$ the standard distribution of Z approaches to a standardized Type III Pearson distribution.

Last contribution to this problem is in [3]. Let $Z = XY$ and consider the conditional distribution of $Z \mid (Y = y)$ and the distribution of Y: $Z \mid Y = y \sim N(y\mu_x, y^2\sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$. Thus we can calculate the joint density of Z and Y:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{Z|Y}(z \mid y) f_Y(y) \\ &= \frac{1}{2\pi \mid y \mid \sigma_x\sigma_y} \exp\left\{ -\frac{1}{2\sigma_x^2}\left(\frac{z}{y} - \mu_x\right)^2 - \frac{1}{2\sigma_y^2}(y-\mu_y)^2 \right\}. \end{aligned} \tag{25}$$

Then, the density of the product $f_Z(z)$ will be the integral of $f(z \mid y)f(y)$ with respect to y.

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi \mid y \mid \sigma_x\sigma_y} \exp\left\{ -\frac{1}{2\sigma_x^2}\left(\frac{z}{y} - \mu_x\right)^2 - \frac{1}{2\sigma_y^2}(y-\mu_y)^2 \right\} dy. \tag{26}$$

There's no analytical solution to this integral. In order to solve them we can take a numerical integration procedure. We use an adaptive recursive Newton Cotes 8 panel rule that is implemented in the mathematical software package: Mathematica (version 8.1). A numeric value of the $\int f(z,y)dy$ is obtained for an array of points in the domain of Z. These points are all extremely close to one another.We consider the density as uniform on these intervals.

3. APPROXIMATION THE DISTRIBUTION OF THE PRODUCT OF TWO NORMAL VARIABLES

We consider two independent normal distributed variables $X \sim N(\mu_x, \sigma_x)$ and $Y \sim N(\mu_y, \sigma_y)$ and we study different values of the parameters. In order to calculate the density function of the product of variables $f_Z(z)$, we have to obtain the marginal density function with respect to Z and the joint density function $f_{ZY}(z,y)$, then we have to calculate the integral:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZY}(z,y)dy. \tag{27}$$

The solution to this integral requires using a numerical integration method. We have used Newton-Cotes 8 panel method (see [7]). To calculate values we have used the mathematical software Mathematica (v.8.1), using NIntegrate function for the variable y and the variable z. Options used were: Method: "Global-Adaptive", Method: "NewtonCotesRule", "Points":8, MaxRecursion:100, Exclusions:(0,0).

An alternative method to approximate $f(z)$ is by calculating the first two moments of Z, and then finding a distribution whose parameters match the moments of Z. We shall derive the moment-generating function for Z, and show that Z can be approximated by a normal curve under certain conditions. The moment-generating function was studied for Craig (1936) and Aroian (1947) (see [1] and [2]).

We only consider the case for two independent uncorrelated variables $\rho = 0$. The moment-generating function of $Z = XY$ can be written as:

$$M_Z(t) = \frac{\exp\left\{\frac{t\mu_x\mu_y + \frac{1}{2}(\mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2)t^2}{1 - t^2\sigma_x^2\sigma_y^2}\right\}}{\sqrt{1 - t^2\sigma_x^2\sigma_y^2}}. \tag{28}$$

We define the variables: $\delta_x = \frac{\mu_x}{\sigma_x}$ and $\delta_y = \frac{\mu_y}{\sigma_y}$, then the moment-generating function can be written as:

$$M_Z(t) = \frac{\exp\left\{\frac{t\mu_x\mu_y + (t\delta_y^2\mu_x\mu_y + \delta_x^2(2\delta_y^2 + t\mu_x\mu_y))}{2\delta_x^2\delta_y^2 - 2t^2\mu_x^2\mu_y^2}\right\}}{\sqrt{1 - \frac{t^2\mu_x^2\mu_y^2}{\delta_x^2\delta_y^2}}}. \tag{29}$$

We study the limit of the moment-generating function when δ tends to increase, in this case the moment-generating function tends to:

$$\exp\left\{t\mu_x\mu_y + \frac{1}{2}\left(\mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2\right)t^2\right\}. \tag{30}$$

Although the product of two normal distributed variables is not, usually, normally distributed; the limit of the moment-generating function of the product is normally distributed. Then the product of two variables $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ tends to the distribution $N(\mu_x\mu_y, \mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2)$ as the values for δ_x and δ_y increase.

The associated moments of the product of variables will be:

$$E(Z) = \mu_x \mu_y, \tag{31}$$

$$V(Z) = \mu_y^2 \sigma_x^2 + (\mu_x^2 + \sigma_x^2)\sigma_y^2 = (1 + \delta_x^2 + \delta_y^2)\sigma_x^2 \sigma_y^2, \tag{32}$$

$$\alpha_3(Z) = \frac{6\delta_x \delta_y \sigma_x^3 \sigma_y^3}{((1 + \delta_x^2 + \delta_y^2)\sigma_x^2 \sigma_y^2)^(3/2)}. \tag{33}$$

Skewness depends on δ values. When $\delta \to \infty$ then skewness tends to zero. Skenewss is large for small δ values. Solving for maximum and minimum values, we get that skewness will be maximal for $\sigma = \mu$ and will be minimal for $\sigma = -\mu$. Larger values for skewness means more shape differences of the Normal approximations for density product.

In Ware and Lad (2003) (see [3]) authors consider different approaches to two product of two normal variables. Case 1: $\mu_x = 1, \mu_y = 0.5, \sigma_x = \sigma_y = 1$ and Case 2: $\mu_x = 5, \mu_y = 2, \sigma_x = \sigma_y = 1$. First case has small values for δ but second case has large values $\delta > 1$. Authors consider three types of approximation: numerical integration, Monte-Carlo simulation and via a Normal distribution. Results show that in both cases the mean and the variance calculate from $f(z)$ are very similar, but for Case 1, the shape of the Normal approximation is very different to the shape of the other two approximations. In both Cases the skewness of the numerical integration and Monte Carlo simulation approximations is very close to the exact known skewness of the product distribution.

Conclusion for Ware and Lad is that as μ_x and μ_y increase relative to σ^2 the approximations of $f(z)$ obtained via numerical integration will become increasingly similar to $N(\mu_x \mu_y, \sigma^2(\mu_x^2 + \mu_y^2 + \sigma^2))$ density.

4. Inverse of coefficients of variation and quasi-normality

To calculate the effect of the inverse of variation coefficient (δ) over the normality of the product of two normal distributed variables, we have investigated the consequences of different values, basically, through the value of parameters (mean, variance and skewness) and we have observed the shape of distributions. We have calculated the values of the product of normal variables using the numerical integration methods.

Let X and Y be two independent and non-correlated normal variables with parameters: $\mu_x, \sigma_x^2, \alpha_3(x), \delta_x = \frac{\mu_x}{\sigma_x}$.

There several cases.

4.1. Same variance

We have considered two normal distributed variables with the same variance: $\sigma_x^2 = \sigma_y^2 = 1$. For this case, then $\delta_x = \mu_x$ and $\delta_y = \mu_y$. Then we have Table 1.

Table 1. Two normal variables - Same Variance

Parameters	Type	Mean	Variance	Skewness
$\mu_x = 1,\ \mu_y = 5$	Numerical Integration	4.99355	26.9609	0.208871
$\mu_x = 1,\ \mu_y = 5$	Moments	5	27	0.2138333
$\mu_x = .1,\ \mu_y = .5$	Numerical Integration	0.495	2.25001	0.8886
$\mu_x = .1,\ \mu_y = .5$	Moments	0.5	2.25	0.88889
$\mu_x = 1,\ \mu_y = .05$	Numerical Integration	0.045511	1.99672	0.128552
$\mu_x = 1,\ \mu_y = .05$	Moments	0.05	2.0025	0.105867

Normal approach is a good approximation only for the first case, where values of mean are large (bigger than 1), but the other cases with two small values for mean or only one small case, produces very skewness distribution and a bad normal approximation. In the three cases numerical integration produces very good estimations for parameters.

4.2. Same mean

We have considered two normal distributed variables with the same mean: $\mu_x = \mu_y = 1$. For this case, then $\delta_x = \frac{1}{\sigma_x}$ and $\delta_y = \frac{1}{\sigma_y}$. Then we have Table 2.

Table 2. Two normal variables - Same Mean

Parameters	Type	Mean	Variance	Skewness
$\sigma_x = .1,\ \sigma_y = 10$	Numerical Integration	0.992487	100.719	0.0.0029
$\sigma_x = .1,\ \sigma_y = 10$	Moments	1	101.01	0.0059
$\sigma_x = 2,\ \sigma_y = 5$	Numerical Integration	0.902385	113.895	0.238112
$\sigma_x = 2,\ \sigma_y = 5$	Moments	1	129	0.409512
$\sigma_x = .5,\ \sigma_y = .2$	Numerical Integration	0.995	0.300	0.365133
$\sigma_x = .5,\ \sigma_y = .2$	Moments	1	0.3	0.35148

Normal approach is a good approximation only for the first case and third cases, where some of the values of variance is small (less than 1), but the other case with two large values for sigma produces very skewness distribution and a bad normal approximation, in addition this case presents a bad result for numerical integration.

4.3. Different mean and different variance

We have considered two normal distributed variables with the different mean: μ_x and μ_y and different variance: σ_x^2 and σ_y^2. For this case, then $\delta_x = \frac{\mu_x}{\sigma_x}$ and $\delta_y = \frac{\mu_y}{\sigma_y}$. Then we have Table 3.

Table 3. Two normal variables - Different Mean and Different Variance

Parameters	Type	Mean	Variance	Skewness
$\mu_x = .5, \sigma_x = .5,\ \mu_y = 1, \sigma_y = 1$	N.i.	0.494997	0.749974	1.15423
$\delta_x = \delta_y = 1$	Moments	0.5	0.75	1.1547
$\mu_x = 2, \sigma_x = 2,\ \mu_y = 5, \sigma_y = 5$	N.i.	8.89135	231.247	0.660361
$\delta_x = \delta_y = 1$	Moments	10	300	1.1547
$\mu_x = .5, \sigma_x = 2,\ \mu_y = 1, \sigma_y = .5$	N.I.	0.497226	5.05385	0.262004
$\delta_x = 0.25, \delta_y = 2$	Moments	0.5	5.0625	0.263374
$\mu_x = 2, \sigma_x = .5,\ \mu_y = .1, \sigma_y = 1$	N.I.	0.194999	4.25247	0.03418
$\delta_x = 4, \delta_y = 0.1$	Moments	0.2	4.2525	0.0342101
$\mu_x = 1, \sigma_x = 10,\ \mu_y = .1, \sigma_y = 2$	N.I.	0.0527474	230.179	0.0055336
$\delta_x = 0.1, \delta_y = 0.05$	Moments	0.1	405	0.0294462
$\mu_x = 5, \sigma_x = 2,\ \mu_y = 2, \sigma_y = 1$	N.I.	9.99318	44.925	0.787368
$\delta_x = 2.5, \delta_y = 2$	Moments	10	45	0.795046

Normal approach is a good approximation only for the case number 4. Rest of the cases produces different shapes or large skewness distributions. Case 4 presents one normal distributed variable with a large mean and small variance and the opposite for the other one. At this situation one of the δ has a very large value (4).

Case number 2 and Case number 5 produces very bad approximations for numerical integration. Both cases has small values for δ, that is ≤ 1, but at the second case parameters are large values, and at the fourth case they are small values. Rest of the cases, numerical integration produces very good approximations.

Numerical integration approximation is very sensitive to the presence of small values for δ, that is ≤ 1. The bigger δ the best approximation for numerical integration.

4.4. Combined ratio and normality

We consider the influence over the Normal approximation for the product of two independent non correlated Normal variables of the combined ratio. Let X and Y be two normal variables with mean μ_x and μ_y, respectively, and with the same variance $\sigma_x = \sigma_y = \sigma$. Combined Ratio is defined as:

$$\frac{\mu_x \mu_y}{\sigma^2}. \tag{34}$$

This ratio is used to study the joint influence of the two inverse coefficients of variation, that is, same magnitude for both variables or influence is independent for each one.

We have considered four cases: two cases with a large combined ratio (one of them with two large inverse variation coefficient and other one with one large and one small value) and two cases with small combined ratio (one of them with two small inverse variation coefficient and other one with one large and one small value). Results are in Table 4.

When combined ratio is small there are several important differences between numerical integration approximation and Normal distributed approximation. A large combined ratio produces a small skewness and small skewness is a guarantee for Normality. On the other side, a small combined ratio is related to a bad approximation for the value of the moments calculated for numerical integration approximation. The more important this effect the large value for inverse variation coefficient.

Table 4. Combined Ratio and Normality

	Value	Ratio = 200 $\mu_x = 1$ $\mu_y = 2$ $\sigma = 0.1$	Ratio = 0.5 $\mu_x = 1$ $\mu_y = 2$ $\sigma = 2$	Ratio = 0.0125 $\mu_x = 0.1$ $\mu_y = 0.5$ $\sigma = 2$	Ratio = 5 $\mu_x = 0.1$ $\mu_y = 0.5$ $\sigma = 0.1$
Numerical Integration	Mean	1.99496	1.93715	0.49019	0.495
	Variance	0.0501462	33.8743	20.9304	0.012608
	Skewness	0.09650	0.689244	0.482127	0.211902
Moments	Mean	2	2	0.5	0.5
	Variance	0.0501	36	21	0.0126
	Skewness	0.10701	0.88889	0.498784	0.212112

5. Conclusions

The approximation of the distribution of the product of normal variables is an ancient problem whose first resolutions trace back to the first-half of the 20th century. In general lines, we could assume that the product would have to follow a normal distribution; nevertheless, it does not seem to be this option the correct, in all the cases, but rather remains linked to the apparition of determinate conditions in the normal variables that compose the product.

Previous works of determinate authors have showed that it is possible to try estimate the function of density of the product by means of diverse methods. In this paper, we have centered our approximation using numerical integration by means of Newton-Cotes and we have doing several comparisons with normal approximation by means of the calculation of the parameters.

Our conclusions follow lines of investigation established and show that the normal approximation appears linked to the presence of large values in the inverse of the coefficients of variation (average divided by typical deviation) of the variables, the normal approximation results quite adapted for values of the upper coefficients to 1 in both variables. In the other side, when the value of some inverse of the coefficient of variation is lower to 1, the normal approximation does not coincide with the approximation by means of numerical integration.

A second conclusion shows that the sensitivity is greater regarding the inverse of coefficients of variation of the individual variables that to the presence of a high value in the ratio combined product of averages with regard to variance, for those cases where both variables possess the same variance.

The presence of normality can be accepted for values of the reverse of the coefficient of variation of both upper variables to the unit and, on the other hand, the influence of the designated ratio combined is inferior to the one of the inverse of coefficients of individual variation.

To the future remains one asks after to answer: Is there some critical value of the coefficient of asymmetry that justifies the normal approximation when the inverse coefficients of variation are large? A second line of investigation would be to improve the methods of integration so that the approximation by means of numerical integration show better results for values very small of the parameters of the variables or for values very big of the same.

References

[1] Cecil C. Craig, *On the frequency function of xy*, Annals of Mathematical Society **7** (1936) 1–15.

[2] L.A. Aroian, *The probability function of a product of two normal distributed variables*, Annals of Mathematical Statistics **18** (1947) 256–271.

[3] R. Ware and F. Lad, Approximating the Distribution for Sums of Product of Normal Variables. Research-Paper 2003–15. Department of Mathematics and Statistics (University of Canterbury – New Zealand, 2003).

[4] L.A. Aroian, V.S. Taneja and L.W. Cornwell, *Mathematical forms of the distribution of the product of two normal variables*, Communication in Statistics – Theory and Method **7** (1978) 164–172.

[5] J. Whisart and M.S.Bartlett, *The distribution of second order moment statistics in a normal system*, Proceedings of the Cambridge Philosophical Society **XXVIII** (1932) 455–459.

[6] L.A. Aroian, V.S. Taneja and L.W. Cornwell, *Mathematical forms of the distribution of the product of two normal variables*, Communications in Statistics. Theoretical Methods **A7 (2)** (1978) 165–172.

[7] S.C. Chapra and R.P. Canale, Numerical Methods for Engineers (McGraw-Hill: New York, 2010).

Received 29 October 2012