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BINOMIAL ARMA COUNT SERIES FROM RENEWAL PROCESSES

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Abstract

This paper describes a new method for generating stationary integer-valued time series from renewal processes. We prove that if the lifetime distribution of renewal processes is nonlattice and the probability generating function is rational, then the generated time series satisfy causal and invertible ARMA type stochastic difference equations. The result provides an easy method for generating integer-valued time series with ARMA type autocovariance functions. Examples of generating binomial ARMA$(p, p-1)$ series from lifetime distributions with constant hazard rates after lag p are given as an illustration.

Keywords: integer-valued time series, stochastic difference equations, autoregressive moving average, renewal process, lifetime distribution, probability generating function, palindromic polynomial, constant hazard rate.

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1. Introduction

Integer-valued time series have a broad range of applications including demographic studies, business planning and risk management. Among models developed for them integer-valued autoregressive (INAR) ones appear most frequently in the literature, see McKenzie (2003) for a review. However, their applicability is limited by their autocorrellation functions always being non-negative. More recent approaches include random coefficient processes of Zhang *et al.* (2007), and negative binomial model of Davis and Wu (2009).

We pursue a different method of generating time series by superposing independent integer-valued renewal processes, which unlike INAR models can induce negative autocorrelation functions. The method was originally proposed by Blight (1989) and developed by Cui and Lund (2009) to generate a variety of time series, Markov and long memory, with binomial and other marginals. Following Cui and Lund (2009) we choose renewal processes to be stationary from the very beginning to make the generated count process stationary. As Blight noticed, its autocovariance generating function can be easily expressed in terms of the lifetime distribution. In a couple of examples he computed it had the structure of the autocovariance of an autoregressive moving average (ARMA) count series, and he seemed to beleive this to be the case whenever the generating function is rational. The question reduces to a non-trivial factorization of the numerator of the generating function, which Blight performed explicitly in his examples. The main purpose of this paper is to prove that the resulting count series is always ARMA if a lifetime distribution is nonlattice and has rational probability generating function, see Theorem 1. Our proof involves palindromic polynomials and some subtle properties of probability characteristic functions. As an illustration, we use lifetime distributions with constant hazard rates after lag p to generate binomial ARMA$(p, p-1)$ count series and study their properties. For $p > 2$ explicit ARMA factorization is not available.

The paper is organized as follows. Section 2 recalls the construction of renewal count processes. In Section 3 we review the definition of ARMA processes and state our main result, Theorem 1, on generation of integer-valued binomial ARMA time series. The proof is given in Section 4. In Section 5 we apply our theorem to generate binomial ARMA$(p, p-1)$ time series from renewal processes with constant hazard rates after lag p, and show that the former possess the p'th order Markov property.

2. Renewal count processes

This section gives a brief review of renewal processes, see Feller (1968) and Ross (1995) for a thorough treatment. Let L be a nonnegative random variable, called lifetime, taking values in $\{1, 2, \ldots\}$ with $P(L = n) = f_n$ and $0 < f_1 < 1$. Let $L_0, L_1, L_2, \ldots$ be independent nonnegative integer-valued random variables with $L_1, L_2, \ldots$ having the same distribution as L. We allow L_0 to have a distribution other than L. Then a renewal is said to happen at time n if $L_0+L_1+\cdots+L_k = n$ for some $k \geq 0$. If L_0 has unit mass at 0, i.e. $L_0 \equiv 0$, the process is called non-delayed or pure, otherwise it is called delayed.

For a non-delayed process let u_n be the probability that a renewal occurs at time n, then u_n satisfies $u_0 = 1$ and $u_n = \Sigma_{j=0}^{n-1} u_j f_{n-j}$, $n \geq 1$. For a delayed process let ν_n be the probability of a renewal at time n, then $\nu_0 = b_0$, $\nu_n = \sum_{k=0}^{n} b_k u_{n-k}$ for $n \geq 1$, where $b_n = P(L_0 = n)$. When L is nonlattice, has finite mean, and $b_n = \mu^{-1} P(L > n)$, i.e. L_0 has the so-called equilibrium or first derived distribution of L, the delayed process is stationary with $\nu_n \equiv \mu^{-1}$ (Ross, 1995).

For a stationary renewal process define the following sequence of Bernoulli random variables: $X_t = 1$ if a renewal occurs at time t, otherwise $X_t = 0$. It can be shown that X_t is strictly stationary with

$$\gamma(h) = \operatorname{cov}(X_t, X_{t+h}) = \frac{1}{\mu}\left(u_h - \frac{1}{\mu}\right).$$

Many types of integer-valued time series with different marginal distributions can be generated by the above model. If we superposition (Cox and Smith, 1954) M independent and identical Bernoulli sequences $X_{i,t}$, $i = 1, 2, \ldots, M$ and define $Y_t = \sum_{i=1}^{M} X_{i,t}$ for $t \geq 0$, then Y_t is strictly stationary with binomial marginal distribution. The autocovariance of Y_t is

$$\operatorname{cov}(Y_t, Y_{t+h}) = \frac{M}{\mu}\left(u_h - \frac{1}{\mu}\right).$$

If L has a constant hazard rate after lag 1 then Y_t is Markov. Long memory binomial series can also be generated by taking L with finite mean but an infinite second moment (see Cui and Lund, 2009, for details).

3. ARMA processes

A stationary process X_t is called ARMA(p, q) process if for every t

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q},$$

where Z_t is a white noise process with variance σ^2. It is convenient to describe ARMA(p,q) processes using autocovariance generating functions. In general, if $\gamma(h)$ is the autocovariance function of a stationary process then its autocovariance generating function is defined by

$$G(z) = \sum_{h=-\infty}^{\infty} \gamma(h) z^h.$$

For an ARMA(p,q) process, the classic result shows that

$$G(z) = \sigma^2 \frac{\theta(z)\theta(z^{-1})}{\phi(z)\phi(z^{-1})}, \tag{1}$$

where $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ are called the autoregressive characteristic polynomial and the moving average characteristic polynomial respectively. It can be shown that a stationary process is ARMA(p,q) if its autocovariance generating function can be written in the form (1), where both $\phi(z)$ and $\theta(z)$ have all their roots outside the unit circle (see Priestley, 1981).

Now let Y_t be the integer-valued time series with binomial marginal distributions defined in the last section. The probability generating function of lifetime L is defined to be

$$F(z) := \sum_{n=1}^{\infty} f_n z^n.$$

As shown by Blight (1989), the autocovariance generating function of Y_t is given by

$$G(z) = \frac{M}{\mu} \frac{1 - F(z)F(\frac{1}{z})}{[1 - F(z)][1 - F(\frac{1}{z})]},$$

where M is the number of independent and identical renewal processes and μ is the mean of L. If $F(z)$ is rational, i.e. $F(z) = P(z)/Q(z)$ with $P(z)$ and $Q(z)$ polynomials, then

$$G(z) = \frac{M}{\mu} \frac{Q(z)Q(\frac{1}{z}) - P(z)P(\frac{1}{z})}{[Q(z) - P(z)][Q(\frac{1}{z}) - P(\frac{1}{z})]}. \tag{2}$$

Recall that a discrete probability distribution $P(L = n) = f_n$, $n \in Z$ is called lattice if it is supported on a sublattice of integers, i.e. there exists a $d > 1$ such that $\sum_{k=0}^{\infty} P(L = kd) = 1$. We will show that if L is nonlattice and has a rational probability generating function, then (2) can always be factorized as in (1). More precisely, the following is true.

Theorem 1. *Let L be a nonlattice distribution with a rational probability generating function $F(z) = P(z)/Q(z)$, written in lowest terms, and variance σ_L^2. Then it represents a causal and invertible ARMA process. Moreover, its autocovariance generating function can be factorized as* $G(z) = \frac{kM}{\mu} \frac{\theta(z)\theta(\frac{1}{z})}{\phi(z)\phi(\frac{1}{z})}$ *with* $k = \frac{\sigma_L^2 Q^2(1)}{\theta^2(1) Q^2(0)}$*, where $\phi(z)$ and $\theta(z)$ have all their zeros outside the unit circle, and no common zeros.*

Formula for k given in Blight (1989) has a missing factor. We prove Theorem 1 in the next section.

4. ARMA FACTORIZATION

In this section we prove our main result, Theorem 1. First, recall a result on nonlattice distributions, which is crucial to factorizing (2). Substituting $z = e^{it}$ into the probability generating function $F(z)$ we get exactly the characteristic function $\chi(t) = F(e^{it})$ of the lifetime distribution L. Of course, any characteristic function has $\chi(0) = 1$, which corresponds to $F(1) = 1$. But it turns out that for nonlattice distributions $|\chi(t)| \neq 1$ on $(0, 2\pi)$. In other words, for nonlattice lifetime distributions $F(z) \neq 1$ on the unit circle except at $z = 1$. The following Lemma also shows that in equation (2) $Q(z)-P(z)$ and $Q(z)Q(1/z)-P(z)P(1/z)$ have no common zeros on the unit circle except at $z = 1$.

Lemma 2. *Let f_n, $n \in \mathrm{Z}$ be a nonlattice distribution and $F(z)$ be its probability generating function. Assume that $F(z)$ is rational and $F(z) = P(z)/Q(z)$ in lowest terms, i.e. $P(z)$ and $Q(z)$ are polynomials with no common factors. Then $1-F(z)$ and $1-F(z)F(1/z)$ have only one zero on the unit circle, namely $z = 1$, and all other zeros are outside the unit circle. Moreover, $z = 1$ is the only common zero of $1 - F(z)$ and $1 - F(z)F(1/z)$, as well as of $Q(z) - P(z)$ and $Q(z)Q(1/z) - P(z)P(1/z)$.*

Proof. It is proved in Gnedenko and Kolmogorov (1968) that $|\chi(t)| < 1$ on $(0, 2\pi)$ except when $t = 0$ if f_n is nonlattice. This means that $|F(z)| < 1$ for $|z| = 1$ and $z \neq 1$. Hence, on the unit circle if $z \neq 1$, then $|1 - F(z)| \geq |1 - |F(z)|| > 0$, and $1 - F(z)F(1/z) = 1 - |\chi(t)|^2 > 0$. Consequently, $1 - F(z)$ and $1 - F(z)F(1/z)$ have no zeros on the unit circle except at $z = 1$. Since $F(z)F(1/z) = |P(z)|^2/|Q(z)|^2$, we see that $Q(z)Q(1/z) - P(z)P(1/z) > 0$ on the unit circle for $z \neq 1$, which means $Q(z)Q(1/z) - P(z)P(1/z)$ also has only $z = 1$ as a zero on the unit circle.

By the maximum modulus principle from complex analysis, $|F(z)| < 1$ for all $|z| < 1$. Thus $|1 - F(z)| \geq |1 - |F(z)|| > 0$ for all $|z| < 1$. We conclude that except for $z = 1$ all zeros of $1 - F(z)$ are outside the unit circle. Suppose z^* is a common zero of $1 - F(z)$ and $1 - F(z)F(1/z)$ and $z^* \neq 1$. Then $F(z^*) = 1$ and $F(1/z^*) = 1$. By the above, z^* can not be on the unit circle so z^* or $1/z^*$ is inside of it. But this contradicts $|F(z)| < 1$ for $|z| < 1$.

Since $P(z)$ and $Q(z)$ have no common factors $1 - F(z)$ and $Q(z) - P(z)$ have the same zeros. From the above we conclude that $Q(z) - P(z)$ have all zeros outside the unit circle except for $z = 1$. Analogously, $Q(z)Q(1/z) - P(z)P(1/z)$ and $1 - F(z)F(1/z)$ have the same zeros. We conclude that $Q(z) - P(z)$ and $Q(z)Q(1/z) - P(z)P(1/z)$ have no common zeros except for $z = 1$. ■

Next we investigate the behavior of $1 - F(z)$ and $1 - F(z)F(1/z)$ near their common zero $z = 1$. Applying Taylor series expansion to $F(z)$ around $z = 1$ one gets $F(z) = 1 + a(z-1) + b(z-1)^2 + o((z-1)^2)$. Also, expanding $1/z$ around $z = 1$ we have

$$\frac{1}{z} = \frac{1}{1 + (z-1)} = 1 - (z-1) + (z-1)^2 + o((z-1)^2).$$

Since $z = 1$ is a fixed point of $1/z$ we can compose the Taylor expansions:

$$\begin{aligned} F\left(\tfrac{1}{z}\right) &= 1 + a(\tfrac{1}{z} - 1) + b(\tfrac{1}{z} - 1)^2 + o\left((\tfrac{1}{z} - 1)^2\right) \\ &= 1 + a\left[-(z-1) + (z-1)^2\right] + b(z-1)^2 + o((z-1)^2) \\ &= 1 - a(z-1) + (a+b)(z-1)^2 + o((z-1)^2). \end{aligned}$$

This yields $F(z)F\left(\frac{1}{z}\right) = 1 + (a + 2b - a^2)(z-1)^2 + o((z-1)^2)$. Thus, $1 - F(z)F(1/z)$ has a double zero at $z = 1$ unless $a + 2b - a^2 = 0$. But $a = F'(1) = E[L]$ is the first moment of lifetime, and $2b = F''(1) = E[L^2] - E[L]$. Therefore, $Var[L] = a + 2b - a^2$ is the variance of L. For notation, let $\sigma_L^2 = a + 2b - a^2$, then it is easy to verify that

$$F(z)F\left(\frac{1}{z}\right) = 1 + \sigma_L^2 (z-1)^2 + o((z-1)^2). \tag{3}$$

We now factorize equation (2) in the form (1). Recall that we assume $F(z) = P(z)/Q(z)$ in lowest terms. Since $F(1) = 1$ the difference $Q(z) - P(z)$ from the denominator of (2) has a zero at $z = 1$. By Lemma 2, $Q(z) - P(z)$ can be factorized as

$$Q(z) - P(z) = (1 - z)Q(0)\phi(z),$$

where the polynomial $\phi(z)$ has all zeros outside the unit circle. We factored out $Q(0)$ to make the constant term of $\phi(z)$ equal to 1 and $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$

for some integer p and constants ϕ_i. After dividing out common factors the denominator of (2) takes the desired form (see (1)):

$$\frac{[Q(z)-P(z)][Q(1/z)-P(1/z)]}{(1-z)(1-1/z)Q(0)^2}=\phi(z)\phi(1/z). \tag{4}$$

It remains to factorize the numerator. Here are two simple but important observations concerning $Q(z)Q(1/z)-P(z)P(1/z)$. If a is a zero then $1/a$ is also a zero, and if a is a complex zero then $\overline{a}$ is also a zero since $P(z)$ and $Q(z)$ have real coefficients. Therefore, zeros of $Q(z)Q(1/z)-P(z)P(1/z)$ come in quartets unless some of a, $\overline{a}$, $1/a$, $1/\overline{a}$ coincide. The latter occurs in two cases. If $a=\overline{a}$ then a is real and the quartet reduces to a real pair a, $1/a$; if a is complex and on the unit circle the quartet reduces to a complex conjugate pair a, $\overline{a}$.

In fact, $Q(z)Q(1/z)-P(z)P(1/z)$ is closely related to palindromic polynomials in which coefficients read the same from left to right as from right to left. Namely, it becomes a palindromic polynomial after being multiplied by the highest power of z. Zeros of real palindromic polynomials also generically come in quartets a, $\overline{a}$, $1/a$, and $1/\overline{a}$.

Lemma 3. *For a nonlattice lifetime distribution with rational generating function $F(z)=P(z)/Q(z)$, written in lowest terms, there exist a real polynomial $\theta(z)$ with all zeros outside the unit circle, and a constant c such that*

$$Q(z)Q(1/z)-P(z)P(1/z)=c\,(1-z)(1-1/z)\theta(z)\theta(1/z),$$

where $\theta(z)=1+\theta_1 z+\theta_2 z^2+\cdots+\theta_q z^q$ for some integer q and real constants θ_i.

Proof. Since $P(z)$ and $Q(z)$ have no common factors, $1-F(z)F(1/z)$ and $Q(z)Q(1/z)-P(z)P(1/z)$ have identical zeros. It follows from (3) that $z=1$ is a double zero of the former and therefore of the latter. In other words, $(1-z)(1-1/z)$ can be factored from $Q(z)Q(1/z)-P(z)P(1/z)$. Lemma 2 tells us that $Q(z)Q(1/z)-P(z)P(1/z)$ has no other zeros on the unit circle. Therefore, the remaining factors come in quartets

$$\left(1-\frac{1}{a_j}z\right),\left(1-\frac{1}{\overline{a_j}}z\right),\left(1-\frac{1}{a_j}\frac{1}{z}\right),\left(1-\frac{1}{\overline{a_j}}\frac{1}{z}\right),$$

with a_j complex and $|a_j|>1$ or pairs

$$\left(1-\frac{1}{a_k}z\right)\left(1-\frac{1}{a_k}\frac{1}{z}\right)$$

with a_k real and $|a_k| > 1$. Define $\theta(z)$ to be the product of all factors $(1-\frac{1}{a_j}z)(1-\frac{1}{\overline{a_j}}z)$ in the first case, and all factors $(1-\frac{1}{a_k}z)$ in the second case. It is clear that $\theta(z)$ has real coefficients since

$$\left(1-\frac{1}{a_j}z\right)\left(1-\frac{1}{\overline{a_j}}z\right) = 1-\left(\frac{1}{a_j}+\frac{1}{\overline{a_j}}\right)z+\frac{1}{|a_j|^2}z^2.$$

■

Now we are in a position to prove the main theorem.

***Proof of Theorem* 1.** Dividing the numerator and the denominator of equation (2) by $(1-z)(1-1/z)Q(0)^2$ we get (4) as the new denominator. For the numerator we apply Lemma 3 to get a real polynomial θ satisfying

$$k\,\theta(z)\theta\left(\frac{1}{z}\right) = \frac{Q(z)Q(\frac{1}{z})-P(z)P(\frac{1}{z})}{(1-z)(1-\frac{1}{z})Q^2(0)}, \tag{5}$$

where k is selected to make $\theta(z)$ have unit constant term. To compute k we divide both sides of (5) by $Q(z)Q(1/z)$ and get

$$k\,\frac{\theta(z)\theta(\frac{1}{z})}{Q(z)Q(\frac{1}{z})} = \frac{1-F(z)F\left(\frac{1}{z}\right)}{(1-z)(1-\frac{1}{z})Q^2(0)}.$$

Set $z \to 1$ on both sides. The lefthand side becomes simply $k\,\theta^2(1)/Q^2(1)$. The righthand side is seen from (3) to approach $\sigma_L^2/Q^2(0)$. Solving for k yields the desired formula. Since $k > 0$ our Y_t is an ARMA time series. By Lemmas 2 and 3, $\phi(z)$ and $\theta(z)$ have all zeros outside the unit circle and no common zeros. It follows that the corresponding ARMA process is causal and invertible (Brockwell and Davis, 1991, Ch.3). ■

5. BINOMIAL ARMA$(p, p-1)$ TIME SERIES

In this section we show how to generate some binomial ARMA$(p, p-1)$ time series using Theorem 1. We use distributions with constant hazard rates after lag p as lifetimes. We also discuss Markov properties of the generated series.

If a lifetime distribution has a constant hazard rate after lag 2, the probability mass function is $P(L=n) = f_3 r^{n-3}$ with $0 < f_3$, $r < 1$ for $n \geq 3$. It is clearly nonlattice. It can also be shown that the hazard rate is $h_k = P(L=k|L \geq k) = (1-r)$ for $k \geq 3$. The probability generating function of L is

$$F(z) = \frac{z[f_1+(f_2-f_1r)z+(f_3-f_2r)z^2]}{1-rz}.$$

From the last section we know that $Q(z) = 1 - rz$ and $P(z) = z[f_1 + (f_2 - f_1 r)z + (f_3 - f_2 r)z^2]$. Plugging $z = 1/r$ into $P(z)$ we get $P(1/r) = f_3/r^2 \neq 0$. Since $1/r$ is the only zero of $Q(z)$ polynomials $Q(z)$ and $P(z)$ have no common factors.

To factorize the covariance generating function we first compute

$$Q(z) - P(z) = 1 - (r + f_1)z - (f_2 - f_1 r)z^2 - (f_3 - f_2 r)z^3 = (1 - z)(1 - \phi_1 z - \phi_2 z^2),$$

with $\phi_1 = r + f_1 - 1$, $\phi_2 = f_2 r - f_3$. The numerator of (2), $Q(z)Q(z^{-1}) - P(z)P(z^{-1})$, has a factor $(1 - z)(1 - z^{-1})$. Besides a double zero at $z = 1$ there exists another pair of zeros, a_1 and a_1^{-1}. Since $Q(z)Q(z^{-1}) - P(z)P(z^{-1}) = (1 - z)(1 - z^{-1})(\pi_0 z + \pi_1 + \pi_0 z^{-1})$, where $\pi_0 = f_1(f_3 - f_2 r)$, $\pi_1 = f_1 f_2 (1 - r)^2 + f_1 f_3 (2 - r) + r(1 - f_1^2 - f_2^2) + f_2 f_3$, one can solve for a_1 from $\pi_0 z + \pi_1 + \pi_0 z^{-1} = 0$ and get

$$a_1 = \frac{-\pi_1 - \sqrt{\pi_1^2 - 4\pi_0^2}}{2\pi_0}.$$

Letting $\theta = -a_1^{-1}$ one has as in Lemma 3

$$Q(z)Q(z^{-1}) - P(z)P(z^{-1}) = k(1 - z)\left(1 - \frac{1}{z}\right)(1 + \theta z)\left(1 + \theta z^{-1}\right), \tag{6}$$

where k can be found from the formula in Theorem 1, or by comparing the constant terms on both sides of (6). This yields

$$k = \frac{(1 - f_1^2 - f_2^2 - f_3^2) + r^2(1 - f_1^2 - f_2^2) + 2f_1 f_2 r + 2f_2 f_3 r}{2 + 2\theta^2 - 2\theta}.$$

The autocovariance generating function of Y_t is

$$G(z) = \frac{Mk}{\mu} \frac{(1 + \theta z)(1 + \theta z^{-1})}{(1 - \phi_1 z - \phi_2 z^2)(1 - \phi_1 z^{-1} - \phi_2 z^{-2})}.$$

An AR$(2, 1)$ type stochastic difference equation for Y_t is now readily written.

More generally, suppose L has a constant hazard rate after lag p. Then L has $P(L = n) = f_{p+1} r^{n-p-1}$ with $0 < f_{p+1}$, $r < 1$ for $n \geq p + 1$. The probability generating function of L can be represented by a ratio of two polynomials as follows

$$\begin{aligned} F(z) &= f_1 z + f_2 z^2 + \cdots + f_p z^p + \frac{f_{p+1} z^{p+1}}{1 - rz} \\ &= \frac{z[f_1 + (f_2 - f_1 r)z + \cdots + (f_{p+1} - f_p r)z^p]}{1 - rz}. \end{aligned}$$

As above, we conclude that $P(z)$ and $Q(z)$ have no common factors since $P(1/r) = f_{p+1}/r^p \neq 0$. By Theorem 1, $Q(z) - P(z)$ can be factorized as $(1 - z)(1 - \phi_1 z - \cdots - \phi_p z^p)$ and $Q(z^{-1}) - P(z^{-1})$ can be factorized as $(1 - z^{-1})(1 - \phi_1 z^{-1} - \cdots - \phi_p z^{-p})$ for some real constants $\phi_1, \ldots, \phi_p$. Explicit factorization of $Q(z)Q(z^{-1}) - P(z)P(z^{-1})$ is no longer possible but Theorem 1 still ensures that the stationary time series has ARMA$(p, p-1)$ structure.

Now we consider the Markov property for our binomial ARMA$(p, p-1)$ processes. For simplicity we only treat the case $p = 2$, but the proof is analogous, albeit more cumbersome, for general p. The trivariate binomial distribution mentioned below is discussed by Chandrasekar and Balakrishnan (2002).

Theorem 4. *Let* $Y_t = \sum_{i=1}^{M} X_{i,t}$*, where* $X_{i,t}$*,* $i = \{1, \ldots, M\}$ *are the underlying Bernoulli series. Then* Y_t *is a second-order Markov chain, i.e.* Y_t *is independent of* $\{Y_{t-3}, Y_{t-4}, \ldots, Y_0\}$*. The vector* (Y_t, Y_{t-1}, Y_{t-2}) *has the trivariate binomial distribution with the moment generating function*

$$
\begin{aligned}
&E\left[e^{Y_t s_1} e^{Y_{t-1} s_2} e^{Y_{t-2} s_3}\right] = \\
&= \left(q + \sum_{1\le i\le 3} p_i e^{s_i} + \sum_{1\le i\le 3}\sum_{1\le j\le 3} p_{ij} e^{s_i} e^{s_j} + p_{123} e^{s_1} e^{s_2} e^{s_3}\right)^M .
\end{aligned}
\tag{7}
$$

Proof. We start by computing the following probabilities for the underlying Bernoulli series $X_{i,t}$.

$$
\begin{aligned}
p_1 &:= P(X_{i,t} = 1, X_{i,t-1} = 0, X_{i,t-2} = 0) = \mu^{-1}(1 - f_1 - f_2);\\
p_{13} &:= P(X_{i,t} = 1, X_{i,t-1} = 0, X_{i,t-2} = 1) = \mu^{-1} f_2;\\
p_{12} &:= P(X_{i,t} = 1, X_{i,t-1} = 1, X_{i,t-2} = 0) = \mu^{-1} f_1(1 - f_1);\\
p_{123} &:= P(X_{i,t} = 1, X_{i,t-1} = 1, X_{i,t-2} = 1) = \mu^{-1} f_1 f_1;\\
p_3 &:= P(X_{i,t} = 0, X_{i,t-1} = 0, X_{i,t-2} = 1) = \mu^{-1}(1 - f_1 - f_2);\\
p_{23} &:= P(X_{i,t} = 0, X_{i,t-1} = 1, X_{i,t-2} = 1) = \mu^{-1} f_1(1 - f_1);\\
p_2 &:= P(X_{i,t} = 0, X_{i,t-1} = 1, X_{i,t-2} = 0) = \mu^{-1}(1 - f_1)^2;\\
q &:= P(X_{i,t} = 0, X_{i,t-1} = 0, X_{i,t-2} = 0) = 1 - \sum_{1\le i\le 3} p_i - \sum_{1\le i\le 3}\sum_{1\le j\le 3} p_{ij} - p_{123}.
\end{aligned}
$$

The conditional probabilities of $X_{i,t}$ can also be explicitly computed. In particular, we use that $\mu = 1 - f_1 + \frac{2-r-f_1-f_2}{1-r}$ to simplify $p_{1|0,0}$ and get the expression for $p_{0|0,0}$ from $p_{1|0,0} + p_{0|0,0} = 1$.

$$
\begin{aligned}
p_{1|0,0} &:= P(X_{i,t}=1|X_{i,t-1}=0, X_{i,t-2}=0) = 1-r;\\
p_{1|0,1} &:= P(X_{i,t}=1|X_{i,t-1}=0, X_{i,t-2}=1) = f_2/(1-f_1);\\
p_{1|1,0} &:= P(X_{i,t}=1|X_{i,t-1}=1, X_{i,t-2}=0) = f_1;\\
p_{1|1,1} &:= P(X_{i,t}=1|X_{i,t-1}=1, X_{i,t-2}=1) = f_1;\\
p_{0|0,1} &:= P(X_{i,t}=0|X_{i,t-1}=0, X_{i,t-2}=1) = (1-f_1-f_2)/(1-f_1);\\
p_{0|1,1} &:= P(X_{i,t}=0|X_{i,t-1}=1, X_{i,t-2}=1) = (1-f_1);\\
p_{0|1,0} &:= P(X_{i,t}=0|X_{i,t-1}=1, X_{i,t-2}=0) = (1-f_1);\\
p_{0|0,0} &:= P(X_{i,t}=0|X_{i,t-1}=0, X_{i,t-2}=0)\\
&= \left[1-\sum_{1\le i\le 3} p_i - \sum_{1\le i\le 3}\sum_{1\le j\le 3} p_{ij} - p_{123}\right] / \left[1-2\mu^{-1}+f_1\mu^{-1}\right] = r. \qquad (8)
\end{aligned}
$$

After some algebra one also finds that the probabilities conditioned on $X_{i,t-1}$, $\dots, X_{i,0}$ are the same as above,

$$P(X_{i,t}|X_{i,t-1}, X_{i,t-2}) = P(X_{i,t}|X_{i,t-1}, X_{i,t-2}, X_{i,t-3}, \dots, X_{i,0}),$$

i.e. the underlying Bernoulli series $X_{i,t}$ is a second-order Markov chain.

Next, we need to find $P(Y_t|Y_{t-1}, \dots, Y_0)$. To this end, let ϵ_j, $j = 1, \dots, M$, be a zero-one vector with three components, and $\epsilon_j(i)$ denote its i'th component, $i = 1, \dots, 3$. Define a set of ϵ_j's by

$$
\begin{aligned}
&A_{Y_t|Y_{t-1},Y_{t-2}} =\\
&= \left\{ \Lambda = (\epsilon_1, \dots, \epsilon_M) \,\middle|\, \sum_{j=1}^{M} \epsilon_j(1) = Y_t, \sum_{j=1}^{M} \epsilon_j(2) = Y_{t-1}, \sum_{j=1}^{M} \epsilon_j(3) = Y_{t-2} \right\}.
\end{aligned}
$$

By independence and the Markov property of the underlying Bernoulli series $X_{i,t}$, we have

$$P(Y_t|Y_{t-1}, \dots, Y_0) = \sum_{\Lambda \in A_{Y_t|Y_{t-1},Y_{t-2}}} \Pi_{j=1}^{M} p_{\epsilon_j(1)|\epsilon_j(2),\epsilon_j(3)}, \qquad (9)$$

where $p_{\epsilon_j(1)|\epsilon_j(2),\epsilon_j(3)}$ can be calculated from (8). Since (9) is not affected by $\{Y_{t-3}, \dots, Y_0\}$, we conclude that $P(Y_t|Y_{t-1}, \dots, Y_0) = P(Y_t|Y_{t-1}, Y_{t-2})$, so $\{Y_t\}$ is a second-order Markov chain. The formula for the moment generating function $E[e^{Y_t s_1} e^{Y_{t-1} s_2} e^{Y_{t-2} s_3}]$ follows from (8) by a straightforward computation. ■

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