STABLE HYPOTHESIS FOR MIXED MODELS WITH BALANCED CROSS-NESTING

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Abstract

Stable hypothesis are hypothesis that may refer either for the fixed part or for the random part of the model. We will consider such hypothesis for models with balanced cross-nesting. Generalized F tests will be derived. It will be shown how to use Monte-Carlo methods to evaluate p-values for those tests.

Keywords: generalized F tests; mixed balanced models; cross-nesting.

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1. Introduction

We will derive generalized F tests for stable hypothesis on models with balanced cross-nesting. Such hypothesis are named since they are hypothesis for sets of factors all of which may or may not have fixed effects. Thus these hypothesis refer either to the fixed part or to the random part of the model.

To formulate stable hypothesis we will consider the algebraic structure of the model. As we shall see these hypothesis are hypothesis of nullity for parameters. When the hypothesis refer to the fixed [random] part of the model the parameter is proportional to a non-centrality parameter [equals a variance components].

Generalized F tests were introduced by Michalsky & Zmyslony (see [1] and [2]). The statistics of these tests are the quocients of the positive by the negative part of quadratic unbiased estimators of parameters whose nullity is being tested.

Superscripts will be used to indicate the number of components of vectors, all s components of $\underline{1}^s$ will be 1. As for matrices, T_s will be the s order identity matrix and I_s will be obtained deleting the first row, equal to $\frac{1}{\sqrt{s}}\underline{1}^{s\top}$, of an $s \times s$ orthogonal matrix P_s. Kronecker matrix product will be indicated by $\otimes$.

We write $\underline{Z}^w \sim \mathcal{N}(\underline{\eta}^w, C)$ when $\underline{Z}^w$ is normal with mean vector $E(\underline{Z}^w) = \underline{\eta}^w$, variance-covariance matrix C and $k \sim \gamma\chi_g^2$ [$\sim \gamma\chi_{g,\delta}^2$] when k is the product by γ of a chi-square with g degrees of freedom [and non-centrality parameter δ].

2. Models

We assume that there are L groups of $u_1, .., u_L$ factors. For convenience we take $c_l(-1) = 0, c_l(0) = 1$ and write $a_l(1)$ for the number of levels of the first factor in the l-th group, $l = 1, ..., L$. If $u_l > 1$, each of the $a_l(1)$ levels of the first factor in the l-th group nests $a_l(2)$ levels of the second factor and so on, treating interactions between fixed and random effects factors as random. When $u_l = 1$ there is no nesting for the l-th group of factors. Then the h-th factor in the l-th group will have $c_l(h) = \prod_{k=1}^h a_l(k)$ levels each one for one combinations of levels of the first h factors in the group, $h = 1, ..., u_l$, $l = 1, ..., L$. Each of these levels nests $b_l(h) = c_l(u_l)/c_l(h)$ level combinations for the remaining factors in the l-th group $h = 1, ..., u_l$, $l = 1, ..., L$. Taking r replicates for each of the $w = \prod_{l=1}^L c_l(u_l)$ level combinations for all factors we will have $n = wr$ observations. Let $m(\underline{a}^L, \underline{b}^L)$ be the number of $a_l < b_l, l = 1, ..., L$. Non null vectors of

$$(1) \qquad \Gamma = \left\{ \underline{h}^L; h_l = 0, ..., u_l, l = 1, ..., L \right\}$$

label single factors when $m(\underline{0}^L, \underline{h}^L) = 1$, or sets of factors belonging to distinct groups when $m(\underline{0}^L, \underline{h}^L) > 1$. For instance if the non null components of $\underline{h}^L$ are $h_2 = 1$ and $h_3 = 2$ the first factor in the second group and the second factor in the third group will be labeled. For each of the $c(\underline{h}^L) = \prod_{l=1}^{L} c_l(h_l)$ level combinations for the factor labeled by $\underline{h}^L$, $b_l(h) = n/c(\underline{h}^L) = r \prod_{l=1}^{L} b_l(h_l)$ observations are taken.

Since there are no interactions between factors in the same groups we can use the $\underline{h}^L \in \Gamma$ to index the model parameters. Firstly $c(\underline{0}^L) = \prod_{l=1}^{L} c_l(0) = 1$ and so we can represent the general mean value μ by $\underline{\beta}(\underline{0}^L)^s$. Nextly, if $m(\underline{0}^L, \underline{h}^L) = 1[> 1]$, the components of $\underline{\beta}(\underline{h}^L)^{c(\underline{h}^L)}$ will be the effects of the factor [interactions between the factors] labeled by $\underline{h}^L$.

If all factors labeled by $\underline{h}^L$ have fixed effects, $\underline{\beta}(\underline{h}^L)^{c(\underline{h}^L)}$ will be a fixed vector while in the second case it will be random vector with null mean vector. Taking

$$(2) \qquad X(\underline{h}^L) = \left[\bigotimes_{l=1}^{L} \left(I_{c_l(h_l)} \bigotimes 1^{b_l(h_l)}\right)\right] \bigotimes 1^r, \underline{h}^L \in \Gamma,$$

we have, (see [1]), for the observations vector $\underline{Y}^n$ the model

$$\underline{Y}^n = \sum_{\underline{h}^L \in \Gamma} X\left(\underline{h}^L\right)\underline{\beta}\left(\underline{h}^L\right)^{c(\underline{h}^L)} + \underline{e}^n$$

$$(3)$$

$$= \underline{\mu}^n + \sum_{\underline{h}^L \in \Gamma_r} X\left(\underline{h}^L\right)\underline{\beta}\left(\underline{h}^L\right)^{c(\underline{h}^L)} + \underline{e}^n,$$

where $\underline{e}^n$ is an normal error vector with null mean vector and vectors $\underline{\beta}(\underline{h}^L)^{c(\underline{h}^L)}$, $\underline{h}^L \in \Gamma_f$ and $\underline{e}^n$ are independent. Γ_r is the set of $\underline{h}^L \in \Gamma$ that label one or more random effects. Now

$$(4) \qquad \underline{\mu}^n = E\left(\underline{Y}^n\right) = \sum_{\underline{h}^L \in \Gamma_f} X\left(\underline{h}^L\right)\underline{\beta}(\underline{h}^L)^{c(\underline{h}^L)}$$

with $\underline{h}^L \in \Gamma_f = \Gamma - \underline{h}^L \in \Gamma_r$, will constitute the fixed part of the model, the remaining terms in the model will constitute it's random part. Stable hypothesis will be hypothesis on parameters $\theta(\underline{h}_0^L)$ that may have a common formulation when $\underline{h}_0^L \in \Gamma_f$ and when $\underline{h}_0^L \in \Gamma_r$.

3. Stable hypothesis

We write $\underline{a}^L < \underline{b}^L$ when $\underline{a}^L \leq \underline{b}^L, l = 1, ..., L$ and for at least one l_0, $a_{l_0} < b_{l_0}$. With

$$(5) \qquad D\left(\underline{h}_0^L\right) = \left\{ \underline{k}^L : \underline{h}_0^L < \underline{k}^L; \underline{k}^L \in \Gamma \right\}$$

let us assume that:

(a)
$$\underline{\beta}\left(\underline{k}^L\right)^{c\left(\underline{k}^L\right)} \sim \mathcal{N}\left(0^{c\left(\underline{k}^L\right)}, \sigma^2\left(\underline{k}^L\right) I_{c\left(\underline{k}^L\right)}\right),$$

$$\underline{k}^L \in D\left(\underline{h}_0^L\right); \ D\left(\underline{h}_0^L\right) \subset \Gamma_r;$$

(b)
$$\underline{e}^n \sim \mathcal{N}\left(\underline{0}^n, \sigma^2 I_n\right);$$

while

(c)
$$\underline{\beta}\left(\underline{h}_0^L\right)^{c\left(\underline{h}_0^L\right)} \quad \text{may be fixed or have distribution}$$

$$\mathcal{N}\left(\underline{0}^{c\left(\underline{h}_0^L\right)}, \sigma^2\left(\underline{h}_0^L\right) I_{c\left(\underline{h}_0^L\right)}\right).$$

The random vectors considered above are assumed to be independent. These assumptions only refer to part of the vectors in the model so that they are less stringent then the usual ones, (see [1]). We point out that (c) covers both cases: $\underline{h}_0^L \in \Gamma_f$ and $\underline{h}_0^L \in \Gamma_r$. Taking

$$
(6) \quad
\begin{cases}
Q(\underline{h}^L) = \left\{ \bigotimes_{l=1}^{L} \left[I_{c_l}(h_l - 1) \bigotimes T_{a_l(h_l)} \bigotimes \left(\frac{1}{\sqrt{b_l(h_l)}} \mathbf{1}^{b_l(h_l)\top} \right) \right] \right\} \bigotimes \\[4mm]
\qquad\qquad \bigotimes \left(\frac{1}{\sqrt{r}} \mathbf{1}^{r\top} \right), \underline{h}^L \in \Gamma \\[6mm]
g(\underline{h}^L) = \left(\prod_{l=1}^{L} c_l(h_l) - c_l(h_l - 1) \right), \underline{h}^L \in \Gamma
\end{cases}
$$

we get, (see [1]),

$$
(7) \quad
\begin{cases}
S\left(\underline{k}^L\right) = \|Q\left(\underline{k}^L\right)\underline{Y}^n\|^2 \sim \gamma\left(\underline{k}^L\right)\chi^2_{g(\underline{k}^L)}, \underline{k}^L \in D\left(\underline{h}_0^L\right) \\[4mm]
S\left(\underline{h}_0^L\right) = \|Q\left(\underline{h}_0^L\right)\underline{Y}^n\|^2 \sim \gamma\left(\underline{h}_0^L\right)\chi^2_{g(\underline{h}_0^L),\delta(\underline{h}_0^L)}, \underline{h}_0^L \in \Gamma
\end{cases}
$$

with

$$
(8) \quad
\begin{cases}
\gamma\left(\underline{h}^L\right) = \sigma^2 + \sum_{\underline{h}'^L:\underline{h}^L \leq \underline{h}'^L} b\left(\underline{h}'^L\right)\sigma^2\left(\underline{h}'^L\right), \underline{h}^L \in \Gamma \\[4mm]
\delta\left(\underline{h}_0^L\right) = \frac{1}{\gamma\left(\underline{h}_0^L\right)} \|\underline{\eta}_0^{g(\underline{h}_0^L)}\|^2, \underline{h}_0^L \in \Gamma, \underline{h}_0^L \in \Gamma_f
\end{cases}
$$

where

$$
(9) \quad \underline{\eta}_0^{g(\underline{h}_0^L)} = E\left(Q\left(\underline{h}_0^L\right)X\left(\underline{h}_0^L\right)\beta\left(\underline{h}_0^L\right)^{c(\underline{h}_0^L)} \right)
$$

while $Q(\underline{h}_0^L)$ is defined in expression (6).

Thus we get

Proposition 1. *Hypothesis $H_0(\underline{h}_0^L) : \xi(\underline{h}_0^L) = 0$ with*

$$\xi\left(\underline{h}_0^L\right) = \sigma^2\left(\underline{h}_0^L\right) + \frac{\|\underline{\eta}_0^{g(\underline{h}_0^L)}\|^2}{g\left(\underline{h}_0^L\right)}$$

is stable.

Proof. It suffices to point out that, when $\underline{h}_0^L \in \Gamma_f, \sigma^2(\underline{h}_0^L) = 0$ and so

$$\xi\left(\underline{h}_0^L\right) = \frac{\|\underline{\eta}_0^{g(\underline{h}_0^L)}\|^2}{g\left(\underline{h}_0^L\right)},$$

while, when $\underline{h}_0^L \in \Gamma_r, \underline{\eta}_0^{g(\underline{h}_0^L)} = \underline{0}^{g(\underline{h}_0^L)}$ and so $\xi(\underline{h}_0^L) = \sigma^2(\underline{h}_0^L)$.   ∎

4. Tests

Let $\ominus(\underline{h}^L)^+$ $[\ominus(\underline{h}^L)^-]$ be the sets of the $\underline{k}^L \in \Gamma$ with $m(\underline{h}^L, \underline{k}^L)$ is even [odd] such that

(10) $$h_{0,l} \leq k_l \leq min\{h_{0,l}; u_l\}, l = 1, ..., L.$$

then, (see [1]),

(11) $$\sigma^2\left(\underline{h}_0^L\right) = \frac{1}{b(\underline{h}_0^L)}\left(\sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^+} \gamma\left(\underline{k}^L\right) - \sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^-} \gamma\left(\underline{k}^L\right)\right).$$

Now, according to expression (3) in the previous section, we have the mean values

(12) $$\begin{cases} E\left(\dfrac{S(\underline{k}^L)}{(\underline{k}^L)}\right) = \gamma\left(\underline{k}^L\right), \underline{k}^L \in D\left(\underline{h}_0^L\right) \\[4mm] E\left(\dfrac{S(\underline{h}_0^L)}{g(\underline{h}_0^L)}\right) = \gamma\left(\underline{h}_0^L\right) + \dfrac{\|\underline{\eta}_0^{g(\underline{h}_0^L)}\|^2}{g(\underline{h}_0^L)}, \underline{h}_0^L \in \Gamma \end{cases}$$

Now we also have

Proposition 2. *The*

$$\widetilde{\xi}\left(\underline{h}_0^L\right)^+ = \sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^+} E\left(\frac{S(\underline{k}^L)}{g(\underline{k}^L)}\right)$$

and

$$\widetilde{\xi}\left(\underline{h}_0^L\right)^- = \sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^-} E\left(\frac{S(\underline{k}^L)}{g(\underline{k}^L)}\right)$$

are the positive and negative parts of an unbiased quadratic estimator for $\xi(\underline{h}_0^L)$.

Proof. According to (12) we have

$$\sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^+} E\left(\frac{S(\underline{k}^L)}{g(\underline{k}^L)}\right) - \sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^-} E\left(\frac{S(\underline{k}^L)}{g(\underline{k}^L)}\right)$$

$$= E\left(\frac{S(\underline{h}_0^L)}{g(\underline{h}_0^L)}\right) + \sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^+/\underline{h}_0^L} E\left(\frac{S(\underline{k}^L)}{g(\underline{k}^L)}\right) - \sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^-} E\left(\frac{S(\underline{k}^L)}{g(\underline{k}^L)}\right)$$

$$= \gamma\left(\underline{h}_0^L\right) + \frac{\|\underline{\eta}_0^{g(\underline{h}_0^L)}\|^2}{g(\underline{h}_0^L)} + \sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^+/\underline{h}_0^L} \gamma\left(\underline{k}^L\right) - \sum_{\underline{k}^L \in \ominus(\underline{h}_0^L)^-} \gamma\left(\underline{k}^L\right)$$

$$= \sigma^2\left(\underline{h}_0^L\right) + \frac{\|\underline{\eta}_0^{g(\underline{h}_0^L)}\|^2}{g(\underline{h}_0^L)} = \xi\left(\underline{h}_0^L\right).$$

which establishes the thesis. ∎

Thus we can use, (see [2] and [3]), $\dfrac{\widetilde{\xi}(\underline{h}_0^L)^+}{\widetilde{\xi}(\underline{h}_0^L)^-}$ as the statistic of a generalized F test for $H_0(\underline{h}_0^L)$. In $\ominus(\underline{h}_0^L)^+$ or $\ominus(\underline{h}_0^L)^-$ there are, (see [1]), $2^{m(\underline{h}_0^L, \underline{u}^L)-1}$ vectors. The $S(\underline{k}^L)$ and $g(\underline{k}^L)$ with $\underline{k}^L \in \ominus(\underline{h}_0^L)^+$ or $\ominus(\underline{h}_0^L)^-$ from 1 to a or from $a+1$ to $2a$, respectively. The test statistic may written as

$$(13) \qquad \mathcal{F}\left(\underline{h}_0^L\right) = \frac{\displaystyle\sum_{j=1}^{a} \frac{S_j}{g_j}}{\displaystyle\sum_{j=a+1}^{2a} \frac{S_j}{g_j}} \,.$$

Let us now generate N sets $U_{1,i}, ..., U_{2a,i}, i = 1, ..., N$ of independent chi-squares with $g_1, ..., g_{2a}$ degrees of freedom. If N^0 is the number of the

$$(14) \qquad Z_i = \frac{\displaystyle\sum_{j=1}^{a} \frac{S_j}{U_{j,i}}}{\displaystyle\sum_{j=a+1}^{2a} \frac{S_j}{U_{j,i}}}, i = 1, ..., N$$

lesser then $\mathcal{F}(\underline{h}_0^L)$, $\dfrac{N^0}{N}$ may be used to evaluate the p-value of the test, (see [4]).

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