

RECURRENCE RELATIONS FOR CONDITIONAL
MOMENT GENERATING FUNCTIONS OF
ORDER STATISTICS AND RECORD VALUES

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Abstract

In this paper, recurrence relations for conditional moment generating functions and conditional moments of order statistics and record values based on random samples drawn from members of a class of doubly truncated distributions \mathfrak{S}_d are obtained.

Keywords: order statistics, record values, conditional moment generating functions, conditional moments, recurrence relations.

2000 Mathematics Subject Classification: Primary 62E10, Secondary 62G30.

1. INTRODUCTION

Order statistics and records are used in a variety of disciplines and have extensively appeared in statistical literature. Many authors have investigated either of the topics or both, among others: Sarhan and Greenberg (1962), Reiss (1989), Arnold, Balakrishnan and Nagaraja (1992, 1998), Ahsanullah (1995) and Ahsanullah and Nevzorov (2001).

In this paper, recurrence relations of conditional moment generating functions and conditional moments of powers of order statistics and records based on random samples drawn from a population whose distribution is a member of a general class of distributions, denoted by \mathfrak{S}_d , are obtained.

Suppose that a random variable X having an absolutely continuous distribution function (*df*), considered by AL-Hussaini and Osman (1997), AL-Hussaini (1999) and AL-Hussaini and Ahmad (2003a, 2003b), is given by

$$F(x) \equiv F_X(x; \theta) = 1 - e^{-\lambda(x; \theta)} \equiv 1 - e^{-\lambda(x)}, \quad x > 0,$$

and the probability density function (*pdf*), given by

$$f(x) = \lambda'(x) e^{-\lambda(x)}, \quad x > 0,$$

where $\lambda(x) \equiv \lambda(x; \theta)$ is a nonnegative, monotone increasing and differentiable function of x such that $\lambda(x) \rightarrow 0$ as $x \rightarrow 0^+$ and $\lambda(x) \rightarrow \infty$ as $x \rightarrow \infty$, $\lambda'(x)$ is the derivative of $\lambda(x)$ with respect to x and the parameter θ (may be a vector) belongs to some parameter space.

We shall write the class \mathfrak{S} of distributions as

$$(1.1) \quad \mathfrak{S} = \left\{ F : F(x) = 1 - e^{-\lambda(x)}, \quad x > 0 \right\}.$$

A doubly truncated *pdf* on $[P_1, Q_1]$, denoted by $f_d(x)$, is given by

$$(1.2) \quad f_d(x) = A_d \lambda'(x) e^{-\lambda(x)}, \quad P_1 \leq x \leq Q_1, \quad (P_1 \geq 0, Q_1 \leq \infty),$$

where

$$(1.3) \quad A_d = 1 / \left[e^{-\lambda(P_1)} - e^{-\lambda(Q_1)} \right].$$

The corresponding doubly truncated *df* and the survival function (*sf*) are given, respectively, for $0 \leq P_1 \leq x \leq Q_1 \leq \infty$, by

$$(1.4) \quad F_d(x) = A_d \left[e^{-\lambda(P_1)} - e^{-\lambda(x)} \right]$$

and

$$(1.5) \quad \bar{F}_d(x) = Q_2 + \frac{f_d(x)}{\lambda'(x)},$$

where $\bar{F}_d(\cdot) = 1 - F_d(\cdot)$, $F_d(\cdot)$ is given by (1.4) and

$$(1.6) \quad Q_2 = -A_d e^{-\lambda(Q_1)} = e^{-\lambda(Q_1)} / [e^{-\lambda(Q_1)} - e^{-\lambda(P_1)}].$$

Notice that $\bar{F}_d(P_1) = 1$ and $\bar{F}_d(Q_1) = 0$.

We shall write \mathfrak{S}_d to denote the doubly truncated class. So that, for $P_1 \leq x \leq Q_1$, $P_1 \geq 0$, $Q_1 \leq \infty$,

$$(1.7) \quad \mathfrak{S}_d = \left\{ F_d : F_d(x) = \frac{[e^{-\lambda(P_1)} - e^{-\lambda(x)}]}{[e^{-\lambda(P_1)} - e^{-\lambda(Q_1)}]} \right\}.$$

Special cases of the doubly truncated class \mathfrak{S}_d are the non-truncated, left and right truncated classes, denoted by \mathfrak{S} , \mathfrak{S}_L and \mathfrak{S}_R , where the non-truncated class \mathfrak{S} is given by (1.1), the left truncated class is given by

$$(1.8) \quad \mathfrak{S}_L = \left\{ F_L : F_L(x) = 1 - e^{-[\lambda(x) - \lambda(P_1)]}, \quad x \geq P_1, \quad P_1 > 0 \right\},$$

in which case, it is only required for $\lambda(x)$ to satisfy the condition $\lambda(x) \rightarrow \infty$ as $x \rightarrow \infty$. The right truncated class \mathfrak{S}_R is given by

$$(1.9) \quad \mathfrak{S}_R = \left\{ F_R : F_R(x) = \frac{[1 - e^{-\lambda(x)}]}{[1 - e^{-\lambda(Q_1)}]}, \quad 0 \leq x \leq Q_1, \quad Q_1 < \infty \right\},$$

in which case, it is only required for $\lambda(x)$ to satisfy the condition $\lambda(x) \rightarrow 0$ as $x \rightarrow 0^+$.

AL-Hussaini, Ahmad and El-Boghdady (2004a, 2004b) have obtained recurrence relations of multivariate moment generating functions of powers of order statistics and records, respectively, based on random samples drawn from a population whose distribution is a member of the doubly truncated class of distributions \mathfrak{S}_d .

Members of \mathfrak{S}_d include important distributions, used in areas as life testing and other areas of statistics as well, such as the doubly truncated distributions of each of the Weibull, Compound Weibull, Pareto, power function, Gompertz and compound Gompertz distributions. Recurrence relations obtained in this paper are applied to such members as illustrative examples.

2. RECURRENCE RELATIONS FOR CONDITIONAL MOMENT GENERATING FUNCTION OF ORDER STATISTICS

Suppose that X_1, \dots, X_n are independently identically distributed random variables as a random variable X having a *df* $F_d(x)$, $x \in [P_1, Q_1]$. Let $X_{1:n} < \dots < X_{n:n}$ be the order statistics of X_1, \dots, X_n . For integers r, s such that $1 \leq r < s \leq n$, the conditional density function of $X_{s:n}$ given $X_{r:n}$ is known to be given by

$$(2.1) \quad f_{X_{s:n}|X_{r:n}}(y | x) = A_1 \left[\bar{F}_d(x) - \bar{F}_d(y) \right]^{s-r-1} \left[\bar{F}_d(y) \right]^{n-s} f_d(y),$$

$$P_1 \leq x < y \leq Q_1,$$

where

$$(2.2) \quad A_1 = (n-r)! / \left[(s-r-1)! (n-s)! \{ \bar{F}_d(x) \}^{n-r} \right].$$

(See, for example David 1981).

The following theorem gives recurrence relations for the conditional moment generating function or conditional moments of order statistics.

Theorem 1. *The necessary and sufficient condition for a random variable X to be distributed as (1.4), is that, for integers r, s and a such that $1 \leq r < s \leq n$ and $a \geq 1$,*

$$\begin{aligned}
 & M_{X_{s:n}|X_{r:n}}(t | x) - M_{X_{s-1:n}|X_{r:n}}(t | x) \\
 (2.3) \quad &= \frac{a t}{n - s + 1} E \left[\frac{X_{s:n}^{a-1} e^{tX_{s:n}^a}}{\lambda'(X_{s:n})} \mid X_{r:n} = x \right] + \frac{(n - r) Q_2}{(n - s + 1) \bar{F}_d(x)} \\
 & \quad \left[M_{X_{s:n-1}|X_{r:n-1}}(t | x) - M_{X_{s-1:n-1}|X_{r:n-1}}(t | x) \right],
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & E \left[X_{s:n}^a \mid X_{r:n} = x \right] - E \left[X_{s-1:n}^a \mid X_{r:n} = x \right] \\
 (2.4) \quad &= \frac{a}{n - s + 1} E \left[\frac{X_{s:n}^{a-1}}{\lambda'(X_{s:n})} \mid X_{r:n} = x \right] + \frac{(n - r) Q_2}{(n - s + 1) \bar{F}_d(x)} \\
 & \quad \left\{ E \left[X_{s:n-1}^a \mid X_{r:n-1} = x \right] - E \left[X_{s-1:n-1}^a \mid X_{r:n-1} = x \right] \right\}.
 \end{aligned}$$

It is assumed that all of the moment generating functions and conditional moments involved exist.

Proof.

$$\begin{aligned}
 & M_{X_{s:n}|X_{r:n}}(t | x) \\
 &= E \left[e^{tX_{s:n}^a} \mid X_{r:n} = x \right] \\
 (2.5) \quad &= \int_x^{Q_1} e^{ty^a} f_{X_{s:n}|X_{r:n}}(y | x) dy \\
 &= A_1 \int_x^{Q_1} e^{ty^a} \left[\bar{F}_d(x) - \bar{F}_d(y) \right]^{s-r-1} \left[\bar{F}_d(y) \right]^{n-s} f_d(y) dy \\
 &= -A_2 \int_x^{Q_1} e^{ty^a} \left[\bar{F}_d(x) - \bar{F}_d(y) \right]^{s-r-1} d \left[\bar{F}_d(y) \right]^{n-s+1},
 \end{aligned}$$

where

$$(2.6) \quad A_2 = \frac{A_1}{n-s+1} = \frac{(n-r)!}{(n-s+1)!(s-r-1)! [\bar{F}_d(x)]^{n-r}}.$$

Integrating by parts, we then have

$$(2.7) \quad \begin{aligned} & M_{X_{s:n}|X_{r:n}}^a(t|x) \\ &= A_2 \int_x^{Q_1} [\bar{F}_d(y)]^{n-s+1} \left\{ a t y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} \right. \\ & \quad \left. + e^{ty^a} (s-r-1) [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-2} f_d(y) \right\} dy \\ &= a t A_2 \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s+1} dy \\ & \quad + A_3 \int_x^{Q_1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-2} [\bar{F}_d(y)]^{n-s+1} f_d(y) dy, \end{aligned}$$

where

$$A_3 = (s-r-1) A_2 = \frac{(n-r)!}{(n-s+1)!(s-r-2)! [\bar{F}_d(x)]^{n-r}}.$$

The second term in (2.7) is the same as (2.5) when s is replaced by $s-1$. Therefore, (2.7) can be written as

$$(2.8) \quad \begin{aligned} & M_{X_{s:n}|X_{r:n}}^a(t|x) - M_{X_{s-1:n}|X_{r:n}}^a(t|x) \\ &= a t A_2 \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s+1} dy. \end{aligned}$$

By using (1.5), we can write

$$[\bar{F}_d(y)]^{n-s+1} = [\bar{F}_d(y)]^{n-s} [\bar{F}_d(y)] = [\bar{F}_d(y)]^{n-s} \left[Q_2 + \frac{f_d(y)}{\lambda'(y)} \right].$$

By substituting in (2.8) with A_2 being written in terms of A_1 as in (2.6), we have

$$\begin{aligned}
 & M_{X_{s:n}^a | X_{r:n}}(t | x) - M_{X_{s-1:n}^a | X_{r:n}}(t | x) \\
 &= \frac{a t A_1}{n - s + 1} \int_x^{Q_1} \frac{y^{a-1} e^{ty^a}}{\lambda'(y)} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s} f_d(y) dy \\
 (2.9) \quad &+ \frac{a t Q_2 A_1}{n - s + 1} \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s} dy \\
 &= \frac{a t}{n - s + 1} E \left[\frac{X_{s:n}^{a-1} e^{tX_{s:n}^a}}{\lambda'(X_{s:n})} \mid X_{r:n} = x \right] \\
 &+ \frac{a t Q_2 A_1}{n - s + 1} \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s} dy.
 \end{aligned}$$

By replacing n by $n - 1$, in (2.8), we obtain

$$\begin{aligned}
 & \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s} dy \\
 (2.10) \quad &= \frac{(n - r)}{a t A_1 \bar{F}_d(x)} \left[M_{X_{s:n-1}^a | X_{r:n-1}}(t | x) - M_{X_{s-1:n-1}^a | X_{r:n-1}}(t | x) \right].
 \end{aligned}$$

Notice, from (2.6) and (2.2), that if n is replaced by $n - 1$ in (2.6), then $A_2 = A_1 \bar{F}_d(x) / (n - r)$ where A_1 is given by (2.2).

Substituting in (2.9), we obtain (2.3).

On the other hand, if (2.3) is satisfied, its left hand side is then given, from (2.8), by

$$(2.11) \quad a t A_2 \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s+1} dy.$$

The right hand side of (2.3) is given, from definition and the use of (2.10) and (2.9), by

$$\begin{aligned}
 & \frac{a t A_1}{n-s+1} \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s} \left[\frac{f_d(y)}{\lambda'(y)} \right] dy \\
 (2.12) \quad & + \frac{a t Q_2 A_1}{n-s+1} \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s} dy \\
 & = a t A_2 \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s} \left[Q_2 + \frac{f_d(y)}{\lambda'(y)} \right] dy.
 \end{aligned}$$

By equating (2.11) and (2.12), we obtain

$$0 = \int_x^{Q_1} y^{a-1} e^{ty^a} [\bar{F}_d(x) - \bar{F}_d(y)]^{s-r-1} [\bar{F}_d(y)]^{n-s} \left[\bar{F}_d(y) - Q_2 - \frac{f_d(y)}{\lambda'(y)} \right] dy.$$

By applying the extension of Müntz-Szás theorem [see, Hwang and Lin (1984)], it follows that

$$\bar{F}_d(y) = Q_2 + \frac{f_d(y)}{\lambda'(y)}.$$

By differentiating both sides of (2.3) and then setting $t = 0$, the recurrence relation (2.4) of conditional moments of order statistics is obtained.

2.1. Left, right and nontruncated cases

Special conditional doubly truncated cases are the conditional left, right and nontruncated distributions. Recurrence relations of moment generating functions and product moments of order statistics corresponding to each one of such cases characterize its members.

Corollary 1. *The necessary and sufficient condition for a random variable X to be distributed as a member of the left truncated class (1.8) is that, for integers r, s and a such that $1 \leq r < s \leq n$ and $a \geq 1$,*

$$\begin{aligned}
 & M_{X_{s:n}^a | X_{r:n}}(t | x) - M_{X_{s-1:n}^a | X_{r:n}}(t | x) \\
 (2.13) \quad &= \frac{a t}{n - s + 1} E \left[\frac{X_{s:n}^{a-1} e^{t X_{s:n}^a}}{\lambda'(X_{s:n})} \mid X_{r:n} = x \right],
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & E \left[X_{s:n}^a \mid X_{r:n} = x \right] - E \left[X_{s-1:n}^a \mid X_{r:n} = x \right] \\
 (2.14) \quad &= \frac{a}{n - s + 1} E \left[\frac{X_{s:n}^{a-1}}{\lambda'(X_{s:n})} \mid X_{r:n} = x \right].
 \end{aligned}$$

Corollary 2. *The necessary and sufficient condition for a random variable X to be distributed as a member of the right truncated class (1.9) is that, for integers r, s and a such that $1 \leq r < s \leq n$ and $a \geq 1$,*

$$\begin{aligned}
 & M_{X_{s:n}^a | X_{r:n}}(t | x) - M_{X_{s-1:n}^a | X_{r:n}}(t | x) = \frac{a t}{n - s + 1} \\
 (2.15) \quad & E \left[\frac{X_{s:n}^{a-1} e^{t X_{s:n}^a}}{\lambda'(X_{s:n})} \mid X_{r:n} = x \right] + \frac{(n - r) e^{-\lambda(Q_1)}}{[e^{-\lambda(Q_1)} - 1](n - s + 1) \bar{F}_d(x)} \\
 & \left[M_{X_{s:n-1}^a | X_{r:n-1}}(t | x) - M_{X_{s-1:n-1}^a | X_{r:n-1}}(t | x) \right],
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & E \left[X_{s:n}^a \mid X_{r:n} = x \right] - E \left[X_{s-1:n}^a \mid X_{r:n} = x \right] = \frac{a}{n - s + 1} \\
 (2.16) \quad & E \left[\frac{X_{s:n}^{a-1}}{\lambda'(X_{s:n})} \mid X_{r:n} = x \right] + \frac{(n - r) e^{-\lambda(Q_1)}}{[e^{-\lambda(Q_1)} - 1](n - s + 1) \bar{F}_d(x)} \\
 & \left\{ E \left[X_{s:n-1}^a \mid X_{r:n-1} = x \right] - E \left[X_{s-1:n-1}^a \mid X_{r:n-1} = x \right] \right\}.
 \end{aligned}$$

Remarks.

(1) In the non-truncated case \mathfrak{S} , the characterization condition is the same as (2.13).

(2) A referee has pointed out that relation (2.3) can also be shown to be a consequence of Eq. (2.9) obtained by Ahmad and Fawzy (2003).

2.2 Examples**(1) Doubly truncated Weibull distribution:**

$$\lambda(x) = \beta x^\gamma \text{ and } \lambda'(x) = \beta \gamma x^{\gamma-1}.$$

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{s:n}^a | X_{r:n}}(t | x) - M_{X_{s-1:n}^a | X_{r:n}}(t | x) \\ &= \frac{a t}{\beta \gamma (n - s + 1)} E \left[X_{s:n}^{a-\gamma} e^{t X_{s:n}^a} | X_{r:n} = x \right] \\ &+ \frac{(n - r) Q_2}{(n - s + 1) \bar{F}_d(x)} \left[M_{X_{s:n-1}^a | X_{r:n-1}}(t | x) - M_{X_{s-1:n-1}^a | X_{r:n-1}}(t | x) \right] \end{aligned}$$

and

$$\begin{aligned} & E \left[X_{s:n}^a | X_{r:n} = x \right] - E \left[X_{s-1:n}^a | X_{r:n} = x \right] \\ &= \frac{a}{\beta \gamma (n - s + 1)} E \left[X_{s:n}^{a-\gamma} | X_{r:n} = x \right] \\ &+ \frac{(n - r) Q_2}{(n - s + 1) \bar{F}_d(x)} \left\{ E \left[X_{s:n-1}^a | X_{r:n-1} = x \right] - E \left[X_{s-1:n-1}^a | X_{r:n-1} = x \right] \right\}, \end{aligned}$$

where $\bar{F}_d(x) = \{ \exp[-(x^\gamma - Q_1^\gamma)] - 1 \} / \{ \exp[-\beta(P_1^\gamma - Q_1^\gamma)] - 1 \}$ and $a > \gamma$.

[Recurrence relations for product moments of order statistics under the doubly truncated exponential and doubly truncated Rayleigh distributions can be obtained from the Weibull distribution by setting $\gamma = 1$ and 2, respectively].

**(2) Doubly truncated compound Weibull distribution
(three - parameter Burr type XII distribution):**

$$\lambda(x) = \gamma \ln(1 + x^\theta/\beta) \quad \text{and} \quad \lambda'(x) = \gamma \theta x^{\theta-1}/(\beta + x^\theta).$$

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{s:n}^a | X_{r:n}}(t | x) - M_{X_{s-1:n}^a | X_{r:n}}(t | x) \\ &= \frac{a t}{\gamma \theta (n-s+1)} \left\{ \beta E \left[X_{s:n}^{a-\theta} e^{tX_{s:n}^a} \mid X_{r:n} = x \right] + E \left[X_{s:n}^a \exp^{tX_{s:n}^a} \mid X_{r:n} = x \right] \right\} \\ &+ \frac{(n-r) Q_2}{(n-s+1) \bar{F}_d(x)} \left[M_{X_{s:n-1}^a | X_{r:n-1}}(t | x) - M_{X_{s-1:n-1}^a | X_{r:n-1}}(t | x) \right] \end{aligned}$$

and

$$\begin{aligned} & E \left[X_{s:n}^a \mid X_{r:n} = x \right] - E \left[X_{s-1:n}^a \mid X_{r:n} = x \right] \\ &= \frac{a}{\gamma \theta (n-s+1)} \left\{ \beta E \left[X_{s:n}^{a-\theta} \mid X_{r:n} = x \right] + E \left[X_{s:n}^a \mid X_{r:n} = x \right] \right\} \\ &+ \frac{(n-r) Q_2}{(n-s+1) \bar{F}_d(x)} \left\{ E \left[X_{s:n-1}^a \mid X_{r:n-1} = x \right] - E \left[X_{s-1:n-1}^a \mid X_{r:n-1} = x \right] \right\}, \end{aligned}$$

where $\bar{F}_d(x) = \{[(\beta + Q_1^\theta)/(\beta + x^\theta)]^\alpha - 1\} / \{[(\beta + Q_1^\theta)/(\beta + P_1^\theta)]^\alpha - 1\}$.

[Recurrence relations for product moments of order statistics under the doubly truncated compound exponential, doubly truncated compound Rayleigh and doubly truncated two-parameter Burr type XII distributions can be obtained from the compound Weibull distribution by setting $\alpha = 1$, $\alpha = 2$ and $\beta = 1$, respectively].

(3) Doubly truncated Pareto I distribution:

$$\lambda(x) = -\gamma \ln(\alpha/x) \text{ and } \lambda'(x) = \gamma/x.$$

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{s:n}^a | X_{r:n}}(t | x) - M_{X_{s-1:n}^a | X_{r:n}}(t | x) \\ &= \frac{a t \beta}{\gamma(n-s+1)} E \left[X_{s:n}^a e^{t X_{s:n}^a} | X_{r:n} = x \right] \\ &+ \frac{(n-r) Q_2}{(n-s+1) \bar{F}_d(x)} \left[M_{X_{s:n-1}^a | X_{r:n-1}}(t | x) - M_{X_{s-1:n-1}^a | X_{r:n-1}}(t | x) \right] \end{aligned}$$

and

$$\begin{aligned} & E \left[X_{s:n}^a | X_{r:n} = x \right] - E \left[X_{s-1:n}^a | X_{r:n} = x \right] \\ &= \frac{a}{\gamma(n-s+1)} E \left[X_{s:n}^a | X_{r:n} = x \right] + \frac{(n-r) Q_2}{(n-s+1) \bar{F}_d(x)} \\ & \left\{ E \left[X_{s:n-1}^a | X_{r:n-1} = x \right] - E \left[X_{s-1:n-1}^a | X_{r:n-1} = x \right] \right\}, \end{aligned}$$

where $\bar{F}_d(x) = [(Q_1/x)^\gamma - 1]/[(Q_1/P_1)^\gamma - 1]$.

(4) Doubly truncated beta distribution:

$$\lambda(x) = \beta \ln[1/(1-x)] \text{ and } \lambda'(x) = \beta/(1-x).$$

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{s:n}^a | X_{r:n}}(t | x) - M_{X_{s-1:n}^a | X_{r:n}}(t | x) \\ &= \frac{a t}{\beta(n-s+1)} \left\{ E \left[X_{s:n}^{a-1} e^{tX_{s:n}^a} \mid X_{r:n} = x \right] - E \left[X_{s:n}^a e^{tX_{s:n}^a} \mid X_{r:n} = x \right] \right\} \\ &+ \frac{(n-r)Q_2}{(n-s+1)\bar{F}_d(x)} \left[M_{X_{s:n-1}^a | X_{r:n-1}}(t | x) - M_{X_{s-1:n-1}^a | X_{r:n-1}}(t | x) \right] \end{aligned}$$

and

$$\begin{aligned} & E \left[X_{s:n}^a \mid X_{r:n} = x \right] - E \left[X_{s-1:n}^a \mid X_{r:n} = x \right] \\ &= \frac{a}{\beta(n-s+1)} \left\{ E \left[X_{s:n}^{a-1} \mid X_{r:n} = x \right] - E \left[X_{s:n}^a \mid X_{r:n} = x \right] \right\} \\ &+ \frac{(n-r)Q_2}{(n-s+1)\bar{F}_d(x)} \left\{ E \left[X_{s:n-1}^a \mid X_{r:n-1} = x \right] - E \left[X_{s-1:n-1}^a \mid X_{r:n-1} = x \right] \right\}, \end{aligned}$$

where $\bar{F}_d(x) = \{[(1-x)/(1-Q_1)]^\beta - 1\} / \{[(1-P_1)/(1-Q_1)]^\beta - 1\}$ and $a > 1$.

(5) Doubly truncated Gompertz distribution:

$$\lambda(x) = (1/\sigma\gamma)[e^{\gamma x} - 1] \text{ and } \lambda'(x) = (1/\sigma)e^{\gamma x}.$$

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{s:n}^a | X_{r:n}}(t | x) - M_{X_{s-1:n}^a | X_{r:n}}(t | x) \\ &= \left(\frac{a t \sigma}{n - s + 1} \right) E \left[X_{s:n}^{a-1} e^{tX_{s:n}^a - \gamma X_{s:n}} | X_{r:n} = x \right] \\ &+ \frac{(n-r)Q_2}{(n-s+1)\bar{F}_d(x)} \left[M_{X_{s:n-1}^a | X_{r:n-1}}(t | x) - M_{X_{s-1:n-1}^a | X_{r:n-1}}(t | x) \right] \end{aligned}$$

and

$$\begin{aligned} & E \left[X_{s:n}^a | X_{r:n} = x \right] - E \left[X_{s-1:n}^a | X_{r:n} = x \right] \\ &= \frac{a \sigma}{n - s + 1} \left\{ E \left[X_{s:n}^{a-1} e^{-\gamma X_{s:n}} | X_{r:n} = x \right] + E \left[X_{s:n}^a | X_{r:n} = x \right] \right\} \\ &+ \frac{(n-r) Q_2}{(n-s+1) \bar{F}_d(x)} \left\{ E \left[X_{s:n-1}^a | X_{r:n-1} = x \right] - E \left[X_{s-1:n-1}^a | X_{r:n-1} = x \right] \right\}, \end{aligned}$$

where $\bar{F}_d(x) = \{ \exp[-\frac{1}{\sigma\gamma}(e^{\gamma x} - e^{\gamma Q_1})] - 1 \} / \{ \exp[-\frac{1}{\sigma\gamma}(e^{\gamma P_1} - e^{\gamma Q_1})] - 1 \}$ and $a > 1$.

(6) Doubly truncated compound Gompertz distribution:

$$\lambda(x) = \delta \ln[1 + (e^{\gamma x} - 1)/\beta\gamma] \text{ and } \lambda'(x) = \delta\gamma/[1 + (\beta\gamma - 1)e^{-\gamma x}].$$

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{s:n}^a | X_{r:n}}(t | x) - M_{X_{s-1:n}^a | X_{r:n}}(t | x) \\ &= \frac{at}{\gamma\delta(n-s+1)} \left\{ E \left[X_{s:n}^{a-1} e^{tX_{s:n}^a} \mid X_{r:n} = x \right] \right. \\ & \quad \left. + (\beta\gamma - 1) E \left[X_{s:n}^{a-1} e^{tX_{s:n}^a - \gamma X_{s:n}} \mid X_{r:n} = x \right] \right\} \\ & \quad + \frac{(n-r) Q_2}{(n-s+1) \bar{F}_d(x)} \left[M_{X_{s:n-1}^a | X_{r:n-1}}(t | x) - M_{X_{s-1:n-1}^a | X_{r:n-1}}(t | x) \right] \end{aligned}$$

and

$$\begin{aligned} & E \left[X_{s:n}^a \mid X_{r:n} = x \right] - E \left[X_{s-1:n}^a \mid X_{r:n} = x \right] \\ &= \frac{a}{\gamma\delta(n-s+1)} \left\{ E \left[X_{s:n}^{a-1} \mid X_{r:n} = x \right] + (\beta\gamma - 1) E \left[X_{s:n}^{a-1} e^{-\gamma X_{s:n}} \mid X_{r:n} = x \right] \right\} \\ & \quad + \frac{(n-r) Q_2}{(n-s+1) \bar{F}_d(x)} \left\{ E \left[X_{s:n-1}^a \mid X_{r:n-1} = x \right] - E \left[X_{s-1:n-1}^a \mid X_{r:n-1} = x \right] \right\}, \end{aligned}$$

where $\bar{F}_d(x) = \{[(\beta\gamma - 1 + e^{\gamma x})/(\beta\gamma - 1 + e^{\gamma Q_1})]^{-\delta} - 1\} / \{[(\beta\gamma - 1 + e^{\gamma P_1})/(\beta\gamma - 1 + e^{\gamma Q_1})]^{-\delta} - 1\}$ and $a > 1$.

3. RECURRENCE RELATION FOR CONDITIONAL MOMENT GENERATING FUNCTION OF RECORD VALUES

A different type of ordering is that of records. Suppose that X_1, X_2, \dots is a sequence of i.i.d. random variables as a random variable X having a *df* $F_d(x)$. Let, for $n \geq 1$, $X_{U(n)} = \max\{X_1, \dots, X_n\}$.

We say that $X_{U(n)}$ is an upper record value of $\{X_n, n \geq 1\}$, if $X_{U(j)} > X_{U(j-1)}$, for $j > 1$. The sequence $\{U(n), n \geq 1\}$ is called upper record times, where $U(1) = 1$ and $U(n) = \min\{j : j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$. Lower record times and values are similarly defined. For details, see Arnold, Balakrishnan and Nagaraja (1998). In this book, it was shown that the conditional density function $f_{U(n)|U(m)}(y | x)$ is given by

$$f_{U(n)|U(m)}(y | x) = \frac{[R(y) - R(x)]^{n-m-1} f(y)}{(n-m-1)! \bar{F}(x)}, \quad y > x.$$

The conditional density function based on the doubly truncated distribution $\bar{F}_d(\cdot)$ (and density $f_d(\cdot)$) is then given by

$$(3.1) \quad f_{U(n)|U(m)}(y | x) = \frac{[R_d(y) - R_d(x)]^{n-m-1} f_d(y)}{(n-m-1)! \bar{F}_d(x)}, \quad P_1 \leq x < y \leq Q_1,$$

where

$$(3.2) \quad R_d(\cdot) = -\ln[\bar{F}_d(\cdot)].$$

For a given record value, we may be interested in knowing what is expected in the next record. The following theorem gives recurrence relations for the conditional moment generating function or conditional moments of record values.

Theorem 2. *The necessary and sufficient condition for a random variable X to be distributed as (1.4), is that, for integers $1 \leq m < n$ and $a \geq 1$,*

$$\begin{aligned}
 & M_{X_{U(n+1)}^a | X_{U(m)}}(t | x) - M_{X_{U(n)}^a | X_{U(m)}}(t | x) \\
 (3.3) \quad & = atE \left[\frac{X_{U(n+1)}^{a-1} e^{tX_{U(n+1)}^a}}{\lambda'(X_{U(n+1)})} \left\{ 1 - e^{[\lambda(X_{U(n+1)}) - \lambda(Q_1)]} \right\} \mid X_{U(m)} = x \right],
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & E \left[X_{U(n+1)}^a \mid X_{U(m)} = x \right] - E \left[X_{U(n)}^a \mid X_{U(m)} = x \right] \\
 (3.4) \quad & = aE \left[\frac{X_{U(n+1)}^{2a-1}}{\lambda'(X_{U(n+1)})} \left\{ 1 - e^{[\lambda(X_{U(n+1)}) - \lambda(Q_1)]} \right\} \mid X_{U(m)} = x \right].
 \end{aligned}$$

It is assumed that all of the conditional moment generating functions and conditional moments involved exist.

Proof. By definition,

$$\begin{aligned}
 & M_{X_{U(n)}^a | X_{U(m)}}(t | x) = E \left[e^{tX_{U(n)}^a} \mid X_{U(m)} = x \right] \\
 (3.5) \quad & = \int_x^{Q_1} e^{ty^a} f_{X_{U(n)} | X_{U(m)}}(y | x) dy \\
 & = B \int_x^{Q_1} e^{ty^a} \left[R_d(y) - R_d(x) \right]^{n-m-1} f_d(y) dy,
 \end{aligned}$$

where

$$(3.6) \quad B = 1 / \left[(n - m - 1)! \bar{F}_d(x) \right], \quad R_d(\cdot) = -\ln \left[\bar{F}_d(\cdot) \right],$$

and $\bar{F}_d(\cdot)$, $f_d(\cdot)$ are given by (1.4) and (1.1). Therefore

$$\begin{aligned}
M_{X_{U(n)}^a | X_{U(m)}}(t | x) &= B^* \int_x^{Q_1} e^{ty^a} \bar{F}_d(y) d[R_d(y) - R_d(x)]^{n-m} \\
&= -B^* \int_x^{Q_1} [R_d(y) - R_d(x)]^{n-m} \left\{ e^{ty^a} [-f_d(y)] + a t y^{a-1} e^{ty^a} \bar{F}_d(y) \right\} dy \\
(3.7) \quad &= B^* \int_x^{Q_1} e^{ty^a} [R_d(y) - R_d(x)]^{n-m} f_d(y) dy \\
&\quad - a t B^* \int_x^{Q_1} y^{a-1} e^{ty^a} \left[\frac{\bar{F}_d(y)}{f_d(y)} \right] [R_d(y) - R_d(x)]^{n-m} f_d(y) dy,
\end{aligned}$$

where $B^* = B / (n - m)! = 1 / [(n - m)! \bar{F}_d(x)]$.

It may be observed that the first term in (3.7) is the same as (3.5) if n is replaced by $n - 1$. In the second term of (3.7),

$$\frac{\bar{F}_d(y)}{f_d(y)} = \frac{1}{\lambda'(y)} \left[1 - e^{[-\lambda(Q_1) - \lambda(y)]} \right].$$

Therefore, (3.7) can be rewritten in the form

$$\begin{aligned}
&M_{X_{U(n+1)}^a | X_{U(m)}}(t | x) - M_{X_{U(n)}^a | X_{U(m)}}(t | x) \\
&= a t E \left[\frac{X_{U(n+1)}^{a-1} e^{tX_{U(n+1)}^a}}{\lambda'(X_{U(n+1)})} \left\{ 1 - e^{[\lambda(X_{U(n+1)}) - \lambda(Q_1)]} \right\} \mid X_{U(m)} \right].
\end{aligned}$$

On the other hand, if condition (3.3) is satisfied, then its left hand side is given from (3.7) by

$$(3.8) \quad a t B^* \int_x^{Q_1} y^{a-1} e^{ty^a} \left[\frac{\bar{F}_d(y)}{f_d(y)} \right] [R_d(y) - R_d(x)]^{n-m} f_d(y) dy.$$

The right hand side of condition (3.3) is given by

$$(3.9) \quad {}_a t B^* \int_x^{Q_1} y^{a-1} e^{ty^a} \left[\frac{1 - e^{-[\lambda(Q_1) - \lambda(y)]}}{\lambda'(y)} \right] [R_d(y) - R_d(x)]^{n-m} f_d(y) dy.$$

Equating (3.8) and (3.9) we then have

$$0 = \int_x^{Q_1} y^{a-1} e^{ty^a} \left[\frac{\bar{F}_d(y)}{f_d(y)} - \frac{1 - e^{-[\lambda(Q_1) - \lambda(y)]}}{\lambda'(y)} \right] [R_d(y) - R_d(x)]^{n-m} f_d(y) dy.$$

It then follows from the extension of Müntz-Szás theorem [see, Hwang and Lin (1984)] that

$$\frac{\bar{F}_d(y)}{f_d(y)} = \frac{1 - e^{-[\lambda(Q_1) - \lambda(y)]}}{\lambda'(y)} = \frac{e^{-\lambda(y)} - e^{-\lambda(Q_1)}}{\lambda'(y) e^{-\lambda(y)}},$$

which has a solution given by

$$\bar{F}_d(y) = A[e^{-\lambda(Q_1)} - e^{-\lambda(y)}],$$

so that

$$f_d(y) = A [\lambda'(y) e^{-\lambda(y)}], \quad P_1 \leq y \leq Q_1,$$

where A is such that $\bar{F}_d(y)$ is a survival function, or $f_d(y)$ is a pdf.

Differentiating both sides of (3.3) with respect to t and then setting $t = 0$, recurrence relation (3.4), for conditional moments, is obtained.

Remark. A referee has pointed out that relation (3.3) can also be shown to be a consequence of Eq. (2.7) obtained by Ahmad and Fawzy (2003).

3.1. Left, right and nontruncated cases

In the left truncated or nontruncated cases, conditions (3.3) and (3.4) become, for integers $n < m$ and $a \geq 1$,

$$\begin{aligned}
& M_{X_{U(n+1)}^a | X_{U(m)}}(t | x) - M_{X_{U(n)}^a | X_{U(m)}}(t | x) \\
&= a t E \left[\frac{X_{U(n+1)}^{a-1} e^{tX_{U(n+1)}^a}}{\lambda'(X_{U(n+1)})} \mid X_{U(m)} \right],
\end{aligned}$$

and

$$\begin{aligned}
& E \left[X_{U(n+1)}^a \mid X_{U(m)} = x \right] - E \left[X_{U(n)}^a \mid X_{U(m)} = x \right] \\
&= a E \left[\frac{X_{U(n+1)}^{2a-1}}{\lambda'(X_{U(n+1)})} \mid X_{U(m)} \right],
\end{aligned}$$

In the left truncated case, $x \geq P_1$, ($P_1 > 0$) and $\lambda(x) \rightarrow \infty$ as $x \rightarrow \infty$.

In the non-truncated case, $x > 0$, ($P_1 > 0$) and $\lambda(x) \rightarrow 0$ as $x \rightarrow 0^+$ and $\lambda(x) \rightarrow \infty$ as $x \rightarrow \infty$.

In the right truncated case. conditions (3.3) and (3.4) remain the same, provided that $0 \leq x \leq Q_1$, $Q_1 < \infty$ and $\lambda(x) \rightarrow 0$ as $x \rightarrow 0^+$.

3.2 Examples

(1) Doubly truncated Weibull distribution:

$$\lambda(x) = \beta x^\gamma \text{ and } \lambda'(x) = \beta \gamma x^{\gamma-1}.$$

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$\begin{aligned}
& M_{X_{U(n+1)}^a | X_{U(m)}}(t | x) - M_{X_{U(n)}^a | X_{r:n}}(t | x) \\
&= \frac{a t}{\beta \gamma} E \left[X_{U(n+1)}^{a-\gamma} \exp^{tX_{U(n+1)}^a} \left(1 - e^{\beta(x^\gamma - Q_1^\gamma)} \right) \mid X_{U(m)} = x \right],
\end{aligned}$$

where $a > \gamma$ and

$$\begin{aligned} & E\left[X_{U(n+1)}^a \mid X_{U(m)} = x\right] - E\left[X_{U(n)}^a \mid X_{U(m)} = x\right] \\ &= \frac{a}{\beta\gamma} E\left[X_{U(n+1)}^{2a-\gamma} \left(1 - e^{\beta(x^\gamma - Q_1^\gamma)}\right) \mid X_{U(m)} = x\right], \end{aligned}$$

where $2a > \gamma$.

[Recurrence relations for product moments of record values under the doubly truncated exponential and doubly truncated Rayleigh distributions can be obtained from the Weibull distribution by setting $\alpha = 1$ and 2 , respectively].

**(2) Doubly truncated compound Weibull distribution
(three - parameter Burr type XII distribution):**

$$\lambda(x) = \gamma \ln(1 + x^\theta/\beta) \quad \text{and} \quad \lambda'(x) = \gamma \theta x^{\theta-1}/(\beta + x^\theta).$$

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{U(n+1)}^a \mid X_{U(m)}}(t \mid x) - M_{X_{U(n)}^a \mid X_{U(m)}}(t \mid x) \\ &= \frac{a t}{\gamma\theta} E\left[\left(\beta X_{U(n+1)}^{a-\theta} + X_{U(n+1)}^a\right) e^{tX_{U(n+1)}^a} \left[1 - \left(\frac{\beta + X_{U(n+1)}^\theta}{\beta + Q_1^\theta}\right)^\gamma\right] \mid X_{r:n} = x\right], \end{aligned}$$

where $a \geq \theta$ and

$$\begin{aligned} & E\left[X_{U(n+1)}^a \mid X_{U(m)} = x\right] - E\left[X_{U(n)}^a \mid X_{U(m)} = x\right] \\ &= \frac{a}{\gamma\theta} E\left[\left(\beta X_{U(n+1)}^{2a-\theta} + X_{U(n+1)}^{2a}\right) \left[1 - \left(\frac{\beta + X_{U(n+1)}^\theta}{\beta + Q_1^\theta}\right)^\gamma\right] \mid X_{r:n} = x\right], \end{aligned}$$

where $2a \geq \theta$.

[Recurrence relations for product moments of record values under the doubly truncated compound exponential, doubly truncated compound Rayleigh and doubly truncated Burr type XII distributions can be obtained from the compound Weibull distribution by setting $\gamma = 1, \gamma = 2$ and $\beta = 1$, respectively].

(3) Doubly truncated Pareto distribution:

$$\lambda(x) = -\gamma \ln(\alpha/x) \text{ and } \lambda'(x) = \gamma/x.$$

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{U(n+1)}^a | X_{U(m)}}(t | x) - M_{X_{U(n)}^a | X_{U(m)}}(t | x) \\ &= \frac{a t}{\gamma} \left\{ E \left[X_{U(n+1)}^a e^{t X_{U(n+1)}^a} \left(1 - \left(\frac{X_{U(n+1)}}{Q_1} \right)^\gamma \right) \mid X_{U(m)} = x \right] \right\} \end{aligned}$$

and

$$\begin{aligned} & E \left[X_{U(n+1)}^a \mid X_{U(m)} = x \right] - E \left[X_{U(n)}^a \mid X_{U(m)} = x \right] \\ &= \frac{a}{\gamma} \left\{ E \left[X_{U(n+1)}^a \left(1 - \left(\frac{X_{U(n+1)}}{Q_1} \right)^\gamma \right) \mid X_{U(m)} = x \right] \right\}. \end{aligned}$$

(4) Doubly truncated beta distribution:

$$\lambda(x) = \beta \ln[1/(1-x)] \text{ and } \lambda'(x) = \beta/(1-x).$$

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{U(n+1)}^a | X_{U(m)}}(t | x) - M_{X_{U(n)}^a | X_{U(m)}}(t | x) \\ &= \frac{at}{\beta} E \left[\left(X_{U(n+1)}^{a-1} + X_{U(n+1)}^a \right) e^{t X_{U(n+1)}^a} \left(1 - \left(\frac{1 - Q_1}{1 - X_{U(n+1)}} \right)^\beta \right) \mid X_{U(m)} = x \right] \end{aligned}$$

and

$$\begin{aligned}
 & E\left[X_{U(n+1)}^a \mid X_{U(m)} = x\right] - E\left[X_{U(n)}^a \mid X_{U(m)} = x\right] \\
 &= \frac{a}{\beta} E\left[\left(X_{U(n+1)}^{a-1} + X_{U(n+1)}^a\right) \left(1 - \left(\frac{1 - Q_1}{1 - X_{U(n+1)}}\right)^\beta\right) \mid X_{U(m)} = x\right],
 \end{aligned}$$

where $a > 1$.

(5) Doubly truncated Gompertz distribution:

$$\lambda(x) = (1/\sigma\gamma)[e^{\gamma x} - 1] \text{ and } \lambda'(x) = (1/\sigma)e^{\gamma x}.$$

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$\begin{aligned}
 & M_{X_{U(n+1)}^a \mid X_{U(m)}}(t \mid x) - M_{X_{U(n)}^a \mid X_{U(m)}}(t \mid x) = a t \sigma \\
 & \times E\left[X_{U(n+1)}^{a-1} e^{tX_{U(n+1)}^a - \gamma X_{U(n+1)}}\right. \\
 & \left. \times \left\{1 - \exp\left(\frac{1}{\sigma \gamma} (e^{\gamma X_{U(n+1)}} - e^{\gamma Q_1})\right)\right\} \mid X_{U(m)} = x\right]
 \end{aligned}$$

and

$$\begin{aligned}
 & E\left[X_{U(n+1)}^a \mid X_{U(m)} = x\right] - E\left[X_{U(n)}^a \mid X_{U(m)} = x\right] \\
 &= a \sigma E\left[X_{U(n+1)}^{a-1} e^{-\gamma X_{U(n+1)}}\right. \\
 & \left. \times \left\{1 - \exp\left(\frac{1}{\sigma \gamma} (e^{\gamma X_{U(n+1)}} - e^{\gamma Q_1})\right)\right\} \mid X_{U(m)} = x\right],
 \end{aligned}$$

where $a > 1$.

(6) Doubly truncated compound Gompertz distribution:

$$\lambda(x) = \delta \ln[1 + (e^{\gamma x} - 1)/\beta\gamma] \text{ and } \lambda'(x) = \delta\gamma/[1 + (\beta\gamma - 1)e^{-\gamma x}].$$

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$\begin{aligned} & M_{X_{U(n+1)}^a | X_{U(m)}}(t | x) - M_{X_{U(n+1)}^a | X_{U(m)}}(t | x) \\ &= \frac{at}{\gamma\delta} E \left[X_{U(n+1)}^{a-1} e^{tX_{U(n+1)}^a} \left(1 + (\beta\gamma - 1)e^{-\gamma X_{U(n+1)}} \right) \right. \\ & \quad \left. \times \left[1 - \left(\frac{\beta\gamma + e^{\gamma X_{U(n+1)}} - 1}{\beta\gamma + e^{\gamma Q_1} - 1} \right)^\delta \right] \mid X_{U(m)} = x \right] \end{aligned}$$

and, for $a > 1$,

$$\begin{aligned} & E \left[X_{U(n+1)}^a \mid X_{U(m)} = x \right] - E \left[X_{U(n)}^a \mid X_{U(m)} = x \right] \\ &= \frac{a}{\gamma\delta} E \left[X_{U(n+1)}^{a-1} \left(1 + (\beta\gamma - 1)e^{-\gamma X_{U(n+1)}} \right) \right. \\ & \quad \left. \times \left[1 - \left(\frac{\beta\gamma + e^{\gamma X_{U(n+1)}} - 1}{\beta\gamma + e^{\gamma Q_1} - 1} \right)^\delta \right] \mid X_{U(m)} = x \right]. \end{aligned}$$

Acknowledgement

The authors appreciate the comments made by a referee.

REFERENCES

- [1] A.A. Ahmad and M.A. Fawzy, *Recurrence relations for single moments of generalized order statistics from doubly truncated distributions*, J. Statist. Plann. Inf. **117** (2003), 241–249.

- [2] M. Ahsanullah, *Record Statistics*, Nova Science Publishers, Inc., Huntington 1995.
- [3] M. Ahsanullah and V. Nevzorov, *Ordered Random Variables*, Nova Science Publishers, Inc., Huntington 2001.
- [4] E.K. AL-Hussaini, *Predicting observables from a general class of distributions*, J. Statist. Plann. Inf. **79** (1999), 79–91.
- [5] E.K. AL-Hussaini and A.A. Ahmad, *On Bayesian interval prediction of future records*, Test **12** (2003a), 79–99.
- [6] E.K. AL-Hussaini and A.A. Ahmad, *On Bayesian predictive distributions of generalized order statistics*, Metrika **57** (2003b), 165–176.
- [7] E.K. AL-Hussaini, A.A. Ahmad and H.H. El-Boghdady, *Recurrence relations for moment generating functions of order statistics*, Metron **LXII** (2004a), 85–99.
- [8] E.K. AL-Hussaini, A.A. Ahmad and H.H. El-Boghdady, *Recurrence relations for moment generating functions of record values*, (2004b) submitted.
- [9] E.K. AL-Hussaini and M.I. Osman, *On the median of finite mixtures*, J. Statist. Comput. Simul. **58** (1997), 121–144.
- [10] B.C Arnold, N. Balakrishnan and H.N. Nagaraja, *A First Course in Order Statistics*, Wiley, New York 1992.
- [11] B.C Arnold, N. Balakrishnan and H.N. Nagaraja, *Records*, Wiley, New York 1998.
- [12] N. Balakrishnan and A.C. Cohen, *Order Statistics Estimation Methods*, Academic Press, Boston 1991.
- [13] David, *Order Statistics*, First Edition, Wiley, New York 1970.
- [14] David, *Order Statistics*, Second Edition, Wiley, New York 1981.
- [15] H.L. Harter, *Order Statistics and Their Use in Testing and Decision*, U.S. Government Printing Office, Washington D.C. (Vol. 1 and 2, 1970).
- [16] J.S. Hwang and G.D. Lin, *Extensions of Müntz Szász theorem and applications*, Analysis **4** (1984), 143–160.
- [17] R.D. Reiss, *Approximate Distributions of Order Statistics with Applications to Nonparametric Statistics*, Springer Verlag, Berlin 1989.
- [18] A.E. Sarhan and B.G. Greenberg, *Contributions to Order Statistics*, (eds) Wiley, New York 1962.

Received 20 April 2004