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APPROXIMATION BY POISSON LAW

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Abstract

We present here the results of the investigation on approximation by the Poisson law of distributions of sums of random variables in the scheme of series. We give the results pertaining to the behaviour of large deviation probabilities and asymptotic expansions, to the method of cumulants, with the aid of which our results have been obtained.

Keywords: Poisson distribution, compound Poisson distribution, asymptotic expansions, large deviations, cumulants.

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Let us have a scheme of series of random variables (r.v.) $X_{ni},\ n \to \infty$, $i = \overline{1, k_n}$ independent in each row (series).

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Denote

$$S_n = X_{n_1} + \ldots + X_{nk_n}, \quad F_n(x) = \mathbf{P}\{S_n < x\},$$

$$\Pi(x;\lambda) = \sum_{0\le k\le x} \frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}.$$

Let us recall the necessary and satisfactory conditions for the convergence of distributions $\mathbf{P}\{S_n < x\}$ to the limit Poisson law $\Pi(x,\lambda)$.

Theorem 1 ([12]). *For the distribution functions (d.f's) $\bar{F}_n(x) = \mathbf{P}\{S_n - A_n < x\}$ of the center sums consisting of infinitesimal (or contstant in the limit) independent sumands X_{ni} to converge strongly to the Poisson d.f. $\Pi(x;\lambda)$ it is necessary and sufficient that there exist constants a_{nk}, $\sum_k a_{nk} = A_n$, such that the d.f.'s $F_{nk}(x) = \mathbf{P}\{X_{nk} - a_{nk} < x\}$ satisfy the conditions*

$$1) \qquad \sum_{k=1}^{k_n}[1 - P_{nk}(0) - P_{nk}(1)] \to 0,$$

$$2) \qquad \sum_{k=1}^{k_n} P_{nk}(1) \to \lambda,$$

where $P_{nk}(0)$ and $P_{nk}(1)$ are the jumps of F_{nk} at the points 0 and 1, respectively.

Usually, $a_{nk} = 0$ and then $A_n = 0$.

Here the strong convergence of the d.f.'s F_n to the d.f. F means that

$$F_n(x) \to F(x), \quad F_n(x+0) \to F(x+0)$$

for each point.

The necessary and sufficient conditions for the weak convergence $\bar{F}_n(x) \to \Pi(x;\lambda)$ are described in [16].

Let us recall now several results about the rate of convergence. When talking about the problems of the convergence rate, the quantities

$$
\begin{aligned}
d(X,Y) &= \sup_A |\mathbf{P}\{X \in A\} - \mathbf{P}\{Y \in A\}| \\
&= \frac{1}{2}\sum_{k=0}^{\infty} |\mathbf{P}\{X = k\} - \mathbf{P}\{Y = k\}|, \\
d_0(X,Y) &= \sup_{k\ge 0} |\mathbf{P}\{X \le k\} - \mathbf{P}\{Y \le k\}|,
\end{aligned}
$$

were usually taken as a measure of difference between the distributions of two nonnegative integer random variables X and Y. Obviously, $d_0(X,Y) \le d(X,Y)$.

Theorem 2 ([14])**.** *Suppose $X, X_1, X_2, \dots, X_n$ are independent Bernoulli r.v.'s with success probabilities $p_1, p_2, \dots, p_n$, respectively. Let Y be a Poisson r.v. with mean $EY = \sum_{i=1}^n p_i$. Then*

$$d(X,Y) \le \sum_{i=1}^n p_i^2.$$

Theorem 3 ([13])**.** *Let $X_1, X_2, \dots, X_n$ be independent nonnegative integer r.v.'s and Y be a Poisson r.v. with mean $EY = \sum_{i=1}^n EX_i$. Then*

$$d_0\left(\sum_{i=1}^n X_i, Y\right) \le \frac{2}{\pi}\sum_{i=1}^n [E^2 X_i + EX_i(X_i - 1)].$$

Theorem 4 ([15])**.** *Let $X_1, X_2, \dots, X_n$ be nonnegative (can be dependent as well) integer r.v.'s and let $p_1 = \mathbf{P}\{X_1 = 1\}$ and $p_i = \mathbf{P}\{X_i = 1 | \mathcal{F}_{i-1}\}$, $2 \le i \le n$, where $\mathcal{F}_i$ denotes σ-algebra generated by r.v.'s $X_1, \dots, X_i$. Let Y be a Poisson r.v. with mean $EY = \sum_{i=1}^n Ep_i$. Then*

$$d\left(\sum_{i=1}^{n} X_i, Y\right) \le \sum_{i=1}^{n} E^2(p_i) + \sum_{i=1}^{n} |p_i - Ep_i| + \sum_{i=1}^{n} \mathbf{P}\{X_i \ge 2\}$$

and

$$d_0\left(\sum_{i=1}^{n} X_i, Y\right) \le \frac{2}{\pi}\sum_{i=1}^{n} E^2(p_i) + \sum_{i=1}^{n} E|p_i - Ep_i| + \sum_{i=1}^{n} \mathbf{P}\{X_i \ge 2\}$$

If r.v.'s X_i satisfy additional conditions it is possible to get more precise results, namely, asymptotic expansions and theorems of large deviations for the distributions $F_n(x) = \mathbf{P}\{S_{nk_n} < x\}$, $S_{nk_n} = \sum_{i=1}^{k_n} X_{ni}$, converging to the Poisson law $\Pi(x;\lambda)$, $\lambda = ES_{nk_n} = \sum_{j=1}^{k_n} \lambda_{nj}$, $\lambda_{nj} = EX_{nj}$.

Before stating our results about assymptotic expansions and probabilities of large deviations, we recall the definition of factorial cumulants and their properties.

Let an r.v. X assume nonnegative integer values. If $EX^k < \infty$, then factorial moments and cumulants of the k-th order of the r.v. X are defined as follows:

$$EX_{(k)} = EX(X-1)\dots(X-k+1),$$

$$\Gamma_k(X) = \sum_{\nu=1}^{k} \frac{(-1)^{\nu-1}}{\nu} \sum_{k_1+\dots+k_\nu=k} EX_{(k_1)}\dots EX_{(k_\nu)}.$$

In a special case, when η is a Poisson r.v. with the parameter λ,

$$E\eta_{(k)} = E\eta(\eta-1)\dots(\eta-k+1) = \lambda^k, \quad k = 1, 2, \dots$$

and

$$\Gamma_k(\eta) = \begin{cases} \lambda, & k = 1, \\ 0, & k > 0. \end{cases}$$

The reason why we have used factorial moments and cumulants is explained in the following way.

Denote $z_1 = z_1(it) = \mathrm{e}^{it} - 1$. If, for some integer $s > 0$, the factorial moment $EX_{(s)}$ exists (i.e., $EX_{(s)} < \infty$), then

$$E\,\mathrm{e}^{itX} = E(1 + z_1(it))^X = \sum_{k=0}^{s} \frac{EX_{(k)}}{k!} z_1^k(it) + \mathrm{o}(|t|^s)$$

and

$$\log E\,\mathrm{e}^{itX} = \sum_{k=1}^{s} \frac{\Gamma_k(X)}{k!} z_1^k(it) + \mathrm{o}(|t|^s).$$

Here the coefficients at $z_1^k(it)$ are factorial moments and factorial cumulants.

Also note that for a Poisson r.v. η with the parameter λ

$$\mathrm{e}^{it} - 1 = z_1(it) = \frac{1}{\lambda} \log E\,\mathrm{e}^{it\eta},$$

because $E\,\mathrm{e}^{it\eta} = \mathrm{e}^{\lambda(1 - \mathrm{e}^{it})}$ and $\log E\,\mathrm{e}^{it\eta} = \lambda(\mathrm{e}^{it} - 1)$.

Now consider the asymptotic expansions in the approximation by the Poisson law.

Several studies have been devoted to the construction of such expansions. We can mention the papers of P. Franken [11], S. Shorgin [13] and A. Barbour [8]. Here two types of expansions of the distribution F_n are possible. The first type is when the function $F_n(x)$ is expanded in Charlier polynomials (i.e., in the functions $\pi_r(m;\lambda)$). Recall that Charlier polynomials are defined in the following way:

$$\pi(m;\lambda) = \frac{\lambda^m}{m!}\,\mathrm{e}^{-\lambda},\ m = 0,1,2,\ldots, \pi_m(m;\lambda) = 0,\ m = -1,-2,\ldots,$$

$$\pi_1(m;\lambda) = \pi(m;\lambda) - \pi(m-1;\lambda),$$

$$\pi_{k+1}(m;\lambda) = \pi_k(m;\lambda) - \pi_k(m-1;\lambda)\,.$$

Then

$$\sum_{l=0}^{m} \pi_{k+1}(l;\lambda) = \pi_k(m;\lambda),$$

$$\Pi(x;\lambda) = \sum_{l=0}^{[x]} \pi(l;\lambda) = \sum_{l=0}^{[x]} \frac{\lambda^l}{l!}\,\mathrm{e}^{-\lambda},$$

and then the expansion in Charlier polynomials has the form

$$F_n(x) = \Pi([x];\lambda) + \sum_{r=2}^{s} (-1)^r \frac{c_r}{r!} \pi_{r-1}([x];\lambda) + R_s(x), \tag{1}$$

where

$$c_r = \sum_{v=1}^{[r/2]} c_r^{(v)}, \quad c_r^{(v)} = \frac{r!}{v!} \sum_{r_1+\ldots+r_v=r} \frac{\Gamma_{r_1}\ldots\Gamma_{r_v}}{r_1!\ldots t_v!}, \ r_i \geq 2;$$

the quantities c_r are usually called Charlier coefficients, and $\Gamma_l = \Gamma_{nl} = \sum_{j=1}^{k_n} \gamma_{jl};\ \gamma_{jl} = \gamma_{jl}^{(n)} = \frac{d^l}{dz^l} \log(Ez^{X_{nj}})$ are factorial cumulants of X_{nj}. It is known that

$$|\pi_r(m;\lambda)| \leq c \left(\frac{r}{\lambda e}\right)^{(r+1)/2}, \quad r = 1, 2, \ldots, s,$$

$$c = \sqrt{e}\left(1 + \frac{\sqrt{\pi}}{2}\right)/2.$$

Consequently, the order of the "smallness" of the r-th summand of the sum on the right-hand side of relation (1) must be determined by the coefficient c_r. However, c_r is expressed by factorial cumulants $\Gamma_l,\ 2 \leq l \leq r$. It means that the order of "smallness" of the coefficients c_r must be determined by the cumulants $\Gamma_l,\ l = \overline{2, r}$, taking part in the expression of c_r. If we assume that

$$\Gamma_l = \Gamma_{nl} = \mathrm{O}(1/n^{l-1})$$

(in [11] this case is called the normed one), then

$$c_r^{(\nu)} = \mathrm{O}(1/n^{r-\nu}) \quad \text{and } c_r = \mathrm{O}(1/n^{r-[r/2]}).$$

The second type of asympthotic expansions is when the summands on the right-hand side of relation (1) are regrouped in such a way that the entire expansion is written as follows:

$$F_{nk_n}(x) = \Pi(x;\lambda) + \sum_{l=1}^{s} B_l([x];\lambda) + R_s(x),$$

where

$$B_l([x];\lambda) = \sum \frac{c_r^{(\nu)}}{r!}(-1)^l \pi_{r-1}([x];\lambda),$$

and summation is taken over all r and v, for which $r - v = l,\ 1 \le v \le [r/2]$. If condition (2) is fulfilled, then

$$B_l = \mathrm{O}(1/n^l).$$

We have mentioned the papers of P. Franken [11], S. Shorgin [13] and A. Barbour [8]. In the paper of Franken, the general case was investigated by the method of characteristic functions. However, the remaining terms $R_s(x)$ in this work have a too complicated structure. By the same method, in the paper of Shorgin, the final results are obtained in the case, where the r.v.'s X_{nj} assume only two values 0 and 1. Barbour, adapting the Stein-Chen method, has obtained asymptotic expansions for sums of independent non-negative integer r.v.'s.

Now we state our results. At first we will take $s = 3$. This means that only the third moment is finite.

Theorem 5 ([2])**.** *Suppose that independent in each row r.v.'s* $X_{nj},\ j = 1, \ldots, k_n, n = 1, 2, \ldots$, *have three finite moments and*

$$\lambda_j = \lambda_j^{(n)} = EX_{nj} > 0, \quad j = 1, \ldots, k_n,\ n = 1, 2, \ldots.$$

Assume that there exists a constant $\Delta_n > 1$, *satisfying the inequalities*

$$E|X_{(2)}| \le \frac{2\lambda_j}{\Delta_n} \quad \text{and } E|X_{(3)}| \le \frac{3!\lambda_j}{\Delta_n^2}$$

$$j = 1, \ldots, k_n,\ n = 1, 2, \ldots$$

and

$$1 < \Delta_n \le 1/\max_{1 \le j \le k_n} \lambda_j.$$

Then

$$\mathbf{P}\{S_{nk_n} \le x\} = \Pi(x;\lambda) + \frac{1}{2}\Gamma_2\pi_1([x];\lambda) + R_3(x) + R,$$

where $\lambda = \sum_{j=1}^{k_n} \lambda_j$,

$$\sup_x |R_3(x)| < \begin{cases} c_1 \lambda \dfrac{\log \Delta_n}{\Delta_n^2} & \text{for any } \lambda > 0, \\[2ex] \dfrac{1}{\Delta_n^2}\Big(c_2 \log \Delta_n + c_3 \log \lambda\Big) & \text{for } \lambda \geq 1, \end{cases}$$

and

$$|R| \leq 2 \sum_{j=1}^{k_n} \sup_x |\tilde{F}_{nj}(x + \varepsilon_3) - \tilde{F}_{nj}^{(n)}(x)|, \quad \varepsilon_3 = \frac{\log \Delta_n^2}{\Delta_n^2}.$$

Here $\tilde{F}_{nj}(x)$ *is a part of the distribution function* $F_{nj}(x)$ *after rejecting jumps at the points* $0, 1, 2, \ldots$, *i.e.*,

$$\tilde{F}_{nj}(x) = F_{nj}(x) - \sum_{m \leq x} p_{nj}(m),$$

$$p_{nj}(m) = F_{nj}(m+0) - F_{nj}(m-0).$$

Moreover, if r.v.'s X_{nj} *are integer and non-negative, then* $R = 0$. *If* $\Delta_n \geq 10$ *(usually* $\Delta_n = n$*) and r.v.'s* X_{nj} *are integer, then*

$$\sup_x |R_3(x)| < \begin{cases} \dfrac{8\lambda}{\Delta_n^2}, & \lambda > 0, \\[2ex] \dfrac{1}{\Delta_n^2}\left(6.24 + \frac{1}{2}\log \lambda\right), & \lambda \geq 1. \end{cases}$$

Obviously, Theorem 5 is not trivial only if $|R| \to 0$ as $\Delta_n \to \infty$.
In the general case we have

Theorem 6 ([2]). *Suppose that independent r.v.'s* X_{nj}, $j = \overline{1, k_n}$, $n = 1, 2, \ldots$ *have* $s+1$ *finite moments, where the* $s \geq 3$, *condition*

$$\lambda_j = \lambda_j^{(n)} = EX_{nj} > 0, \quad j = \overline{1, k_n}, \ n = 1, 2, \ldots$$

is fulfilled, and there exists a constant $\Delta_n > 1$, satisfying inequalities

$$1 < \Delta_n \le 1/\max_{1\le j\le k_n} \lambda_j$$

and

$$E|X_{nj}(X_{nj}-1)\dots(X_{nj}-l+1)| \le \lambda_j l!/\Delta_n^{l-1},$$
$$l = \overline{2, s+1},\ \ j = \overline{1, k_n}.$$

Then there exist constants c_{1s}, c_{2s}, and c_{3s} such that

$$\mathbf{P}\{S_{nk_n} \le x\} = \Pi(x;\lambda) + \sum_{\nu=1}^{s-1} B_\nu([x]) + R_s(x) + |\bar{R}|,$$

where $\lambda = \sum\limits_{j=1}^{k_n} \lambda_j$,

$$|\bar{R}| \le 2\sum_{j=1}^{k_n} \sup_x \left(\bar{F}_{nj}(x+\varepsilon_s) - \bar{F}_{nj}(x)\right), \quad \varepsilon_s = \log\Delta_n^s/\Delta_n^s,$$

$$\sup_x |R_s(x)| \le \begin{cases} \lambda \dfrac{c_{1s}\log\Delta_n^s}{\Delta_n^s}, & \lambda > 0, \\[2ex] \dfrac{c_{2s}\log\Delta_n^s + c_{3s}\log\lambda}{\Delta_n^s}, & \lambda > 1. \end{cases}$$

If $\Delta_n \ge 10$, the r.v.'s X_{nj} are integer and non-negative, then $\bar{R} = 0$ and

$$\sup_x |R_s(x)| \le \begin{cases} 2^{3s-2}\lambda \dfrac{2.3\log\Delta_n^s}{\Delta_n^s}, & \lambda > 0, \\[2ex] 2^{3s-2}\dfrac{1.7\log\Delta_n^s + 1.75\log\lambda}{\Delta_n^s}, & \lambda \ge 1. \end{cases}$$

We have defined the polynomials $B_\nu(m)$ earlier:

$$B_\nu(m) = \sum_{r-v=\nu} \frac{c_r^{(v)}}{r!}(-1)^r \pi_{r-1}(m);$$

$$c_r^{(v)} = \frac{r!}{v!} \sum_{\substack{r_1+\ldots+r_v=r \\ r_i \geq 2}} \frac{\Gamma_{r_1} \ldots \Gamma_{r_v}}{r_1! \ldots r_v!}$$

In particular, we have

$$B_1(m) = \frac{\Gamma_2}{2!}\pi_1(m;\lambda),$$

where $\pi_1(m;\lambda) = \pi(m;\lambda) - \pi(m-1;\lambda)$; $\pi(m;\lambda) = \dfrac{\lambda^m}{m!}\,\mathrm{e}^{-\lambda}$,

$$B_2(m) = \frac{\Gamma_3}{3!}\pi_2(m;\lambda) + \frac{1}{2}\left(\frac{\Gamma_2}{2!}\right)^2 \pi_3(m;\lambda),$$

$$B_3(m) = \frac{\Gamma_4}{4!}\pi_3(m;\lambda) + \frac{\Gamma_3}{3!}\frac{\Gamma_2}{2!}\pi_4(m;\lambda) + \frac{1}{3}\left(\frac{\Gamma_2}{2!}\right)^3 \pi_5(m;\lambda), \ldots$$

To prove our theorems, we used the theorem obtained by the authors [7].

Theorem 7 ([7]). *Suppose F is a distribution function, defined on R, the set of jump points of which is A_F. Let G be a discrete (jumps) function of bounded variance, defined on R, too. Let*

$$A_F \supseteq A_G = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$$

and $G(-\infty) = F(-\infty) = 0$. Then

$$\sup_x |F(x) - G(x)| \leq \frac{I_T + \delta_F U(x)}{2U(x) - 1},$$

where

$$I_T = \frac{1}{2\pi} \int_{-T}^{T} \frac{|f(t) - g(t)|}{|t|} \, \mathrm{d}t,$$

$$f(t) = \int_{-\infty}^{\infty} \mathrm{e}^{itx} \, \mathrm{d}F(x), \quad g(t) = \sum_{x_j \in A_G} \mathrm{e}^{itx_j} (G(x_j + 0) - G(x_j - 0)),$$

$$\delta_F = \max_j \left(F(x_{j+1} - 0) - F(x_j + 0) \right),$$

$$U(x) = \int_{|u|<x} p(u) \, \mathrm{d}u, \quad p(u) = \frac{1}{2\pi} \Big(\frac{\sin u/2}{u/2} \Big)^2.$$

In our case, the Poisson distribution $\Pi(x; \lambda)$ was taken instead of G.

Further we investigated probabilities of large deviations.

Recall that the studies of large deviation probabilities follow two main trends. The works following the first trend are associated with the large deviation principle. In this case one considers the behaviour of large deviation probabilities within an accuracy of logarithmic equivalence, mainly in functional limit theorems and empiric processes.

The other trend in which we worked is simply the asymptotic analysis of distribution tails in various integral limit theorems, when the approximating distributions are mainly Gaussian ones. Later on Poisson approximations of large deviations started to be investigated (also apprpximations by χ^2 as well as infinitely divisible distributions).

We mention several papers ([1], [10], [9]), in which large deviation theorems for sums $S_n = X_{n1} + \ldots + X_{nk_n}$ of independent in each row (series) random variables $X_i^{(n)}$, $i = 1, 2, \ldots, k_n$ are studied, and in which the usual normal approximation for the sum S_n is replaced by a Poisson approximation. As mentioned above we have also investigated the probabilities of large deviations in the approximation by the Poisson law. But instead of the sum S_n, we have studied an r.v. X, the factorial cumulants of which satisfy some growth conditions. Of course, instead of X we can take a sum S_n of a row (series) of independent r.v.'s or some statistics, or linear forms of such r.variables.

Lemma 8 ([3]). *If the random variable X takes non-negative integer values, $EX = \lambda > 0$, and*

$$\text{(S)} \qquad |\Gamma_k(X)| \leq \frac{k!\lambda}{\Delta^{k-1}}$$

for all $k \geq 2$ and some $\Delta > 1$, then in the interval

$$\lambda \leq x < \frac{1}{6e}\lambda\Delta$$

the relation of large deviations

$$\frac{\mathbf{P}\{X \geq x\}}{1 - \Pi(x;\lambda)} = \mathrm{e}^{L(x)}\left(1 + \theta_1 \frac{x}{\Delta}\right)$$

holds. Here

$$\theta_1 = \theta\left(22 + \max\left(\frac{20}{\lambda}, \frac{121}{\sqrt{x}}\right)\right), \quad |\theta| \leq 1,$$

(here θ_1 is calculated for $\Delta > 5\max(1, 1/\lambda)$).

$$L(x) = -\frac{(x-\lambda)^2}{\lambda^*\Delta}\left\{\frac{\Delta\Gamma_2}{2\lambda^*} + \sum_{k=1}^{\infty} b_k \left(\frac{x-\lambda}{\lambda^*\Delta}\right)^k\right\}$$

$$-x\log\left\{1 + \sum_{k=1}^{\infty} a_k \left(\frac{x-\lambda}{\lambda^*\Delta}\right)\right\},$$

$$\lambda^* = \lambda + \Gamma_2 = \lambda\left(1 + \theta\frac{2}{\Delta}\right), \quad |\theta| \leq 1,$$

both series on the right-hand side of the latter equality converge as

$$\frac{x-\lambda}{\lambda^*} < \frac{1}{6e}\Delta,$$

and the coefficients a_k and b_k are expressible in terms of the first $k+2$ and $k+1$ factorial cumulants, respectively.

Remark. It could seem that condition (S) is hardly verifiable. But here the next auxiliary lemma, proved by the authors, can be used.

Lemma 9 ([4]). *Suppose that for r.v.'s X_{nj} from the sequence of series with means $EX_{nj} = \lambda_j^{(n)} > 0$, $j = \overline{1, k_n}$, there exists a constant $\Delta_n > 1$ such that*

$$E|X_{(l)}| \le \lambda_j^{(n)} l!/\Delta_n^{l-1}, \quad l = \overline{2, s},\ j = \overline{1, k_n}.$$

Then for factorial cumulants $\Gamma_{jl} = \Gamma_l(X_{nj})$ of the r.v. X_{nj} the estimates

$$|\Gamma_l(X_{nj})| \le 2\lambda_j^{(n)} l!/\left(\frac{\Delta_n}{2}\right)^{l-1}, \quad l = \overline{2, s},\ j = \overline{1, k_n}$$

hold.

The conclusion of this lemma is as follows. To know the growth rate of factorial cumulants it suffices to know the upper estimates of factorial moments. But we know that it is not very difficult to estimate the factorial moments, as well as simple moments from above. Therefore, if we can estimate the factorial moments from above of one or another quantity which stands in place of X, then we can estimate probabilities of large deviations of this quantity at once.

So we can also rewrite Lemma 8 for that quantity at once.

For example, if $\mathbf{P}\{X_{nj} = 1\} = \frac{\lambda}{n}$, $\mathbf{P}\{X_{nj} = 0\} = 1 - \frac{\lambda}{n}$, $j = \overline{1, n}$, then it is easy to check that

$$\Gamma_k(S_n) = \frac{(-1)^{k-1}(k-1)!\lambda^k}{n^{k-1}}, \quad \forall\ k \ge 1.$$

We see that here $\Delta = \Delta_n = \frac{n}{\lambda}$ and, consequently,

$$\frac{\mathbf{P}\{S_n > x\}}{1 - \Pi(x;\lambda)} = \mathrm{e}^{L(x)} \left(1 + \theta_1 \frac{x}{\Delta}\right), \quad \lambda \le x < \frac{1}{6e} n.$$

Next we proved the following inequality of large deviations.

Lemma 10 ([3])**.** *Let the r.v. X take non-negative integer values, $EX = \lambda > 0$, and*

$$|\Gamma_n(X)| \le \frac{(k-1)!\lambda}{\Delta^{k-1}}$$

for all $k \ge 2$ and some $\Delta > 1$. Then

$$\begin{aligned} \mathbf{P}\{X \ge x\} &\le \exp\left\{-x \log \frac{x}{\lambda} + x - \lambda + \frac{x-\lambda}{\lambda\Delta}\right\} \\ &\le \pi(x;\lambda)\, \mathrm{e}^{x(x-\lambda)/\lambda\Delta} \sqrt{2\pi x}\, \mathrm{e}^{1/12x} \end{aligned}$$

for $0 < x - \lambda \le \lambda\Delta$, and

$$\mathbf{P}\{X \ge x\} \le \exp\left\{-x \log \frac{x}{\lambda} + \lambda\Delta \log\left(1 + \frac{x-\lambda}{\lambda\Delta}\right) + x \log\left(1 + \frac{x-\lambda}{\lambda\Delta}\right)\right\}$$

for $x - \lambda \ge \lambda\Delta$. Here $\pi(x;\lambda) = \frac{\mathrm{e}^{-\lambda}\lambda^x}{x!}$, $x > 0$.

Our next result belongs to the case, where instead of condition (S), a weaker condition is satisfied.

Theorem 11 ([4], [5])**.** *Let X be a nonnegative r.v. with $\mathbf{E}X = \lambda > 0$ and let*

$$(\mathrm{S}_\gamma) \qquad |\Gamma_k(X)| \le \frac{\lambda(k!)^{1+\gamma}}{\Delta^{k-1}}, \quad \gamma > 0,$$

for all $k \ge 2$ *and some* Δ*. Then, in the interval* $1 < x < \lambda\Delta_\gamma$*, where*

$$\Delta_\gamma = \frac{0,3}{2\,\mathrm{e}}\Big(\frac{\Delta}{3}\Big)^{1/(1+2\gamma)}$$

the relation of large deviations

$$\frac{\mathbf{P}\{X > x\}}{1 - \Pi(x;\lambda)} = \mathrm{e}^{L_\gamma(x)}\Big(1 + \theta_1\frac{x}{\lambda} + \theta_2\sqrt{x}\max_{k\ge 0}\mathbf{P}\{k < X < k+1\}\Big)$$

holds.

Moreover, if the r.v. X *is nonnegative and integer, then*

$$\frac{\mathbf{P}\{X > x\}}{1 - \Pi(x;\lambda)} = \mathrm{e}^{L_\gamma(x)}\Big(1 + \theta_1\frac{x}{\lambda}\Big).$$

Here the power series

$$L_\gamma(x) \;=\; -\frac{(x-\lambda)^2}{\lambda^*\Delta}\left\{\frac{\delta\Gamma_2}{2\lambda^*} + \sum_{k=1}^{p} b_k\Big(\frac{x-\lambda}{\lambda^*\Delta}\Big)^k\right\}$$

$$-x\log\left\{1 + \sum_{k=1}^{p} a_k\Big(\frac{x-\lambda}{\lambda^*\Delta}\Big)^k\right\}, \quad p = 2 + \frac{1}{2\gamma}.$$

Theorem 12 ([5])**.** *Let* X *be a nonnegative r.v. with* $\mathbf{E}X = \lambda > 0$ *and the condition*

$$|\Gamma_k(X)| \le \frac{\lambda((k-1)!)^{1+\gamma}}{\Delta^{k-1}}$$

be fulfilled for all $k \geq 2$ and some $\Delta > 1$ and $\gamma \geq 0$. Then for all $x > \lambda$,

$$\mathbf{P}\{X \geq x\} \leq \exp\left\{ -x \log \frac{x}{\lambda} + x \log \frac{(1 + \frac{x}{\lambda\Delta})x^{\gamma/(1+\gamma)}}{1 - \frac{\lambda}{x} + (\frac{\lambda}{x} + \frac{1}{\Delta})x^{\gamma/(1+\gamma)}} + \lambda\Delta \log \frac{\lambda + \frac{x}{\Delta}}{\lambda + \frac{\lambda}{\Delta}} \right\}$$

holds.

We have obtained ([6]) similar, only more complicated results for probabilities of large deviations in the approximation by a compound Poisson law, the characteristic function of which is

$$(2) \quad \log f_Y(t) = \lambda \sum_{m=1}^{k} (\mathrm{e}^{itm} - 1)p_m, \quad \lambda > 0, \quad p_m > 0, \quad \sum_{m=1}^{N} p_m = 1.$$

It is possible to express such an r.v. Y with the distribution, whose logarithm of the characteristic function is (2), as the sum

$$Y \stackrel{d}{=} \xi_1 + \cdots + \xi_\eta$$

of a random number of iid r.v.'s $\xi_1, \xi_2, \ldots$, where

$$\mathbf{P}\{\xi_1 = m\} = p_m, \quad m = 1, \ldots, N$$

and η has the Poisson distribution with the parameter $\lambda > 0$:

$$\mathbf{P}\{\eta = k\} = \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 1, 2, \ldots .$$

Evidently, $\mathbf{E}Y = \lambda \mathbf{E}\xi_1 = \lambda\alpha_1$, $\alpha_1 = \mathbf{E}\xi_1$.

Let X be an r.v. taking integer nonnegative values with $\mathbf{E}X = \mathbf{E}Y$ and $\mathbf{E}X^s < \infty$ with $s > 0$. We wish to approximate the distribution of X by the distribution of Y.

How to select cumulants $\widetilde{\Gamma}_k(X)$, $k = 1, 2, \ldots$ in the approximation by the compound Poisson distribution?

Let us consider the following example.

Let $X = X_{n1} + \cdots + X_{nn}$, where X_{ni} are iid r.v.'s and

$$\mathbf{P}\{X_{n1} = m\} = \frac{\lambda}{n} p_m, \quad m = \overline{1, N},$$

$$\mathbf{P}\{X_{n1} = 0\} = 1 - \frac{\lambda}{n}.$$

Hence

$$\begin{aligned}
\log f_X(t) &= n \log \left(1 - \frac{\lambda}{n} + \sum_{m=1}^{N} \mathrm{e}^{itm} \frac{\lambda p_m}{n} \right) \\
&= n \log \left(1 + \frac{\lambda}{n} \sum_{m=1}^{N} (\mathrm{e}^{itm} - 1) p_m \right) \\
&= n \log \left(1 + \frac{\lambda}{n} z(it) \right), \quad z(it) = \sum_{m=1}^{N} (\mathrm{e}^{itm} - 1) p_m,
\end{aligned}$$

or

$$\log f_X(t) = \sum_{k=1}^{\infty} \frac{\widetilde{\Gamma}_k}{k!} z^k(it),$$

where

$$\widetilde{\Gamma}_k(X) = \frac{(-1)^{k-1} \lambda^k (k-1)!}{n^{k-1}}, \quad k = 1, 2, \ldots$$

This example shows that one ought to take coefficients in the expansion of $\log f_X(t)$ as cumulants $\widetilde{\Gamma}_k$ on the base $z(it)$.

Then, if we want to obtain the theorem of large deviations for X when approximating by Y, the condition

$$(\widetilde{S}) \qquad \left| \widetilde{\Gamma}_k(X) \right| \leq \frac{\lambda k!}{\Delta^{k-1}}, \quad k = 2, 3, \ldots, \quad \widetilde{\Gamma}_1(X) = \lambda$$

must be fulfilled.

In our example we may assume $\Delta = n/\lambda$.

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