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ON $S\alpha S$ DENSITY FUNCTION

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Abstract

In this paper, we study some analytical properties of the symmetric α-stable density function.

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1. Introduction

Throughout the paper by $\mathcal{L}(X)$ we denote the distribution of the random variable X. If random variables X and Y have the same distribution we will write $X \stackrel{d}{=} Y$.

We say that a random variable X is strictly stable if for every positive numbers a, b there exists a positive number $c(a,b)$ such that

$$aX + bX' \stackrel{d}{=} c(a,b)X, \tag{1}$$

where X' independent copy of X.

It is known ([**?**, **?**]) that for every strictly stable random variable X, there exists a unique number $\alpha \in (0,2]$, referred to as the index of stability of X, such that

$$c(a,b) = \sqrt[\alpha]{a^\alpha + b^\alpha}.$$

If X is a strictly stable random variable with $\alpha \in (0,1)$ and a characteristic function $\varphi(t) = \exp\{-\sigma^\alpha |t|^\alpha (1 - i\mathrm{sign}(t)\tan(\frac{\pi\alpha}{2}))\}$, $\sigma \geq 0$, then its density function is concentrated on $(0,+\infty)$ and the Laplace transform of X has the form $\mathbf{E}e^{-sX} = e^{-a^\alpha s^\alpha}$ for $s \geq 0$ and $a^\alpha = \frac{\sigma^\alpha}{\cos(\frac{\pi\alpha}{2})}$.

A strictly stable random variable X is called symmetric stable if its distribution is symmetric, that is, if $X \overset{d}{=} -X$. From now on we will use the notation $S\alpha S$ for a symmetric stable distribution with the corresponding index of stability α. We will also say that the random variable X has a symmetric α-stable density function. If a random variable X is $S\alpha S$, then its characteristic function is of the form $\varphi(t) = e^{-C|t|^\alpha}$ for some $C \geq 0$. It is well known ([**?**, **?**]) that no random variable has a characteristic function given by the formula $\varphi(t) = e^{-C|t|^\alpha}$ for $\alpha > 2, C > 0$.

For $\alpha \in (0,2]$, let $f_{\alpha,C}$ denote the density function of the $S\alpha S$ distribution with the characteristic function $\varphi(t) = e^{-C|t|^\alpha}$ for some $C \geq 0$. By the Fourier Inversion Formula

$$f_{\alpha,C}(x) = \frac{1}{\pi}\int_0^\infty \cos(tx) e^{-C|t|^\alpha} dt.$$

Since $f_{\alpha,C}(x) = C^{-\frac{1}{\alpha}} f_{\alpha,1}\left(\frac{x}{C^{\frac{1}{\alpha}}}\right)$ for every $C > 0$, we can restrict our considerations to the function $f_\alpha \overset{def}{=} f_{\alpha,1}$.

2. Properties of SαS density function

In this section, we show some properties of the $S\alpha S$ density function. The explicit formula for this density is known for all $\alpha \in (0,2]$, namely

- for $\alpha = 2$ the distribution $S2S$ is a symmetric Gaussian distribution and

$$f_2(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}},$$

- for $\alpha = 1$ the distribution $S1S$ is the Cauchy distribution:

$$f_1(x) = \frac{1}{\pi(1+x^2)},$$

- for $\alpha \in (0,1)$ we have

$$f_\alpha(x) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma(\alpha k+1)\sin\left(\frac{\alpha k\pi}{2}\right)|x|^{-\alpha k-1},$$

- for $\alpha \in (1,2)$ we have

$$f_\alpha(x) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k!}\Gamma\left(\frac{k}{\alpha}+1\right)|x|^{k-1}.$$

It is easily seen that for $\alpha \in \{1,2\}$ the function f_α has a nice analytical form. However for $\alpha \in (0,1)\cup(1,2)$ the formula of a symmetric α-stable density function is rather complicated and hence it is difficult to find analytical properties of f_α. In [2] we can find asymptotic expansions and integral representations of stable densities and in [4] there are plots of them. The very important property states that for every $\alpha \in (0,2]$ the density function of an $S\alpha S$ random variable is infinitely differentiable (see [4]). The following theorem shows the properties of $S\alpha S$ that are intuitive and some others.

Theorem 1. *Let $\alpha \in (0,2]$. The density function $f_\alpha : \mathbb{R} \to [0,+\infty)$ of a symmetric α -stable distribution with a corresponding characteristic function $\varphi(t) = e^{-|t|^\alpha}$ has the following properties:*

(i) $f_\alpha^{(2n)}(x) = f_\alpha^{(2n)}(-x)$, $f_\alpha^{(2n+1)}(x) = -f_\alpha^{(2n+1)}(-x)$ *for every* $x \in \mathbb{R}$, $n \in \mathbb{N}$ *(we assume* $f_\alpha^{(0)} = f_\alpha$*);*

(ii) $f_\alpha^{(2n)}(0) = \frac{(-1)^n}{\pi\alpha}\Gamma(\frac{2n+1}{\alpha})$, $f_\alpha^{(2n+1)}(0) = 0$ *for every* $n \in \mathbb{N}$*;*

(iii) *if* $g(x) \overset{def}{=} f_\alpha(\sqrt{x})$ *for* $x \geq 0$*, then* g *is completely monotonic;*

(iv) $f_\alpha(x) > 0$ *for every* $x \in \mathbb{R}$*;*

(v) f_α *is strictly decreasing on* $(0, \infty)$*;*

(vi) *there exists only one number* $y > 0$ *such that* f_α *is concave on the interval* $(0, y)$ *and convex on* (y, ∞)*;*

(vii) $\lim_{x\to\infty} f_\alpha^{(n)}(x) = 0$ *for every* $n \in \mathbb{N}$.

(viii) *Let* $n, k \in \mathbb{N}$, $n > k$.

- *If* $\alpha \neq 2$ *and* $n + k = 2l + 1$ $(n + k = 2l)$ *for some* $l \in \mathbb{N}$*, then there exists a number* $z \geq 0$ *such that the function* $\frac{f_\alpha^{(n)}}{f_\alpha^{(k)}}$ *is strictly increasing (decreasing) on* (z, ∞).

- *If* $\alpha = 2$ *then*

$$\frac{f_2^{(n)}(x)}{f_2^{(k)}(x)} = \left(-\frac{x}{2}\right)^{n-k} + Q_{n-k}(x),$$

where Q_{n-k} *is an algebraic expression of power fewer than* $n - k$.

Proof. The proof of properties (i)-(vii) for the cases $\alpha \in \{1, 2\}$ is a matter of simple calculations. In (vi) we obtain $y = \sqrt{2}$ for $\alpha = 2$ and $y = \frac{\sqrt{3}}{3}$ for $\alpha = 1$.

Since the proof of these properties can be the same for all $\alpha \in (0, 2]$, on the other hand there is no need to separate these cases. Let $\alpha \in (0, 2]$.

The first property is obvious in view of the previous remark.

The second one is obtained out of the property (i), the Fourier Inversion Theorem and the properties of Gamma distribution, namely

$$f_\alpha^{(2n)}(0) = \frac{\partial^{2n}}{\partial x^{2n}} \frac{1}{\pi} \int_0^\infty \cos(tx) e^{-|t|^\alpha} dt|_{x=0}$$

$$= \frac{(-1)^n}{\pi} \int_0^\infty t^{2n} e^{-t^\alpha} dt$$

$$= \frac{(-1)^n}{\pi\alpha} \int_0^\infty t^{\frac{2n+1}{\alpha}-1} e^{-t} dt = \frac{(-1)^n}{\pi\alpha} \Gamma\left(\frac{2n+1}{\alpha}\right).$$

In order to see (iii) let us notice that if X is a random variable with the density function f_α, then $X \stackrel{d}{=} Y\sqrt{\Theta}$, where Y is a standard Gaussian random variable, for $\alpha \in (0,2)$ a random variable Θ is positive strictly stable independent of Y with the Laplace transform $\mathbf{E}e^{-s\Theta} = e^{-(2s)^{\frac{\alpha}{2}}}$ for $s \geq 0$ and for $\alpha = 2$ we have $\mathbf{P}\{\Theta = 2\} = 1$. If we denote $\mathcal{L}(\Theta) = \lambda$ then we obtain

$$f_\alpha(x) = \int_0^\infty \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} \lambda(ds) \tag{2}$$

and hence

$$g(x) = \int_0^\infty \frac{1}{\sqrt{2\pi s}} e^{-\frac{x}{2s}} \lambda(ds) \text{ for } x \geq 0.$$

Now it is easily seen that $g^{(2n)}(x) > 0$ and $g^{(2n+1)}(x) < 0$ for every $x > 0$ and $n \in I\!N$, which implies complete monotonicity of the function g.

From the property (iii) we obtain $f_\alpha(x) = g(x^2) > 0$ for every $x > 0$ and from the property (i) we get (iv).

From (iii) we obtain also that $f'_\alpha(x) = 2xg'(x^2) < 0$ for every $x > 0$, which implies (v).

We now show the property (vi). Because of the symmetry of the function $f_\alpha(x)$ we consider only $x \geq 0$. From the equation (**??**) we obtain

$$\sqrt{2\pi} f''_\alpha(x) = \int_0^\infty \left(\frac{x^2}{s} - 1\right) s^{-\frac{3}{2}} e^{-\frac{x^2}{2s}} \lambda(ds)$$

$$= \int_0^{x^2} \left(\frac{x^2}{s} - 1\right) s^{-\frac{3}{2}} e^{-\frac{x^2}{2s}} \lambda(ds) - \int_{x^2}^\infty \left(1 - \frac{x^2}{s}\right) s^{-\frac{3}{2}} e^{-\frac{x^2}{2s}} \lambda(ds).$$

Since $f''_\alpha(x)$ exists for every $x \geq 0$, then in the second expression each of the integrals is finite and takes a nonnegative value for every $x \geq 0$. Moreover, these integrals are monotonic functions of the variable x - the first is increasing and the other is decreasing. This implies that we can find only one number $y \geq 0$ such that $\sqrt{2\pi} f''_\alpha(x) > 0$ for every $x > y$.

Since the α-stable density function is infinitely differentiable in $\mathbb{R}$, in view of (v), the point $x = 0$ is a local maximum of the function f_α. This implies that y must be positive and that $\sqrt{2\pi} f''_\alpha(x) < 0$ for every $0 < x < y$, which completes the proof of (vi).

The property (vii) follows from the equality

$$\lim_{x\to\infty} x^\alpha \mathbf{P}\{X > x\} = c_\alpha = \begin{cases} \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{1}{2}\pi\alpha\right)} & \text{for} \quad \alpha \neq 1 \\ \dfrac{1}{\pi} & \text{for} \quad \alpha = 1 \end{cases}.$$

For $\alpha \in (0,2)$ we have $c_\alpha > 0$ and for $\alpha = 2$ there is $c_\alpha = 0$. If $x \to \infty$, then $x^\alpha \to \infty$ and $\mathbf{P}\{X > x\} \to 0$. Using the de l'Hospital theorem we obtain

(3)
$$\begin{aligned} c_\alpha = \lim_{x\to\infty} \frac{\mathbf{P}\{X > x\}}{\frac{1}{x^\alpha}} &\overset{H}{=} \lim_{x\to\infty} \frac{f_\alpha(x)}{\alpha\frac{1}{x^{\alpha+1}}} \\ \overset{H}{=} \lim_{x\to\infty} \frac{f'_\alpha(x)}{-\alpha(\alpha+1)\frac{1}{x^{\alpha+2}}} &\overset{H}{=} \lim_{x\to\infty} \frac{f''_\alpha(x)}{\alpha(\alpha+1)(\alpha+2)\frac{1}{x^{\alpha+3}}} \\ \overset{H}{=} \qquad \ldots \qquad &\overset{H}{=} \lim_{x\to\infty} \frac{f^{(n)}_\alpha(x)}{(-1)^n\alpha(\alpha+1)\ldots(\alpha+n)\frac{1}{x^{\alpha+n+1}}}, \end{aligned}$$

which means $\lim_{x\to\infty} f^{(n)}_\alpha(x) = 0$ for every $n \in \mathbb{N}$.

In order to obtain (viii) for $\alpha \in (0,2)$ we use the equality (**??**). It implies that for $\epsilon = \frac{c_\alpha}{n-k}\left(\sqrt{\alpha+n+1} - \sqrt{\alpha+k+1}\right)^2$, $0 < \epsilon < c_\alpha$, there exists a number $z = z(\epsilon) \geq 0$ such that for every number $x > z$ and for every $n \in \mathbb{N}$ we have

$$\left| \frac{1}{B_n} x^{\alpha+n+1}(-1)^n f^{(n)}_\alpha(x) - c_\alpha \right| < \epsilon,$$

where $B_n = \alpha(\alpha+1)\dots(\alpha+n)$. This means

$$0 < \frac{B_n(c_\alpha - \epsilon)}{x^{\alpha+n+1}} < (-1)^n f_\alpha^{(n)}(x) < \frac{B_n(c_\alpha + \epsilon)}{x^{\alpha+n+1}} \qquad \text{for every } x > z$$

and since $\frac{B_{n+1}}{B_n} = \alpha + n + 1$ then for $k \in I\!N$, $n > k$, we obtain

$$-\frac{f_\alpha^{(k+1)}(x)}{f_\alpha^{(k)}(x)} < \frac{(\alpha+k+1)(c_\alpha+\epsilon)}{(c_\alpha-\epsilon)x} < \frac{(\alpha+n+1)(c_\alpha-\epsilon)}{(c_\alpha+\epsilon)x} < -\frac{f_\alpha^{(n+1)}(x)}{f_\alpha^{(n)}(x)}.$$

Hence for $x > z$ we have

$$-\frac{f_\alpha^{(k+1)}(x)}{f_\alpha^{(k)}(x)} = \frac{(-1)^{(k+1)} f_\alpha^{(k+1)}(x)}{(-1)^k f_\alpha^{(k)}(x)} < \frac{(-1)^{(n+1)} f_\alpha^{(n+1)}(x)}{(-1)^n f_\alpha^{(n)}(x)} = -\frac{f_\alpha^{(n+1)}(x)}{f_\alpha^{(n)}(x)}$$

and since $(-1)^n f_\alpha^{(n)}(x) > 0$ for every $n \in I\!N$, then

$$(-1)^{n+k+1}\left(f_\alpha^{(n+1)}(x) f_\alpha^{(k)}(x) - f_\alpha^{(k+1)}(x) f_\alpha^{(n)}(x)\right) > 0.$$

Equivalently, if $n + k = 2l + 1$ for some $l \in I\!N$, then the function $\frac{f_\alpha^{(n)}}{f_\alpha^{(k)}}$ is strictly increasing on (z, ∞) and if $n + k = 2l$, then that function is strictly decreasing.

To see (viii) for $\alpha = 2$ let us notice that

$$f_2^{(n)}(x) = \frac{(-1)^n}{2^{n+1}\sqrt{\pi}} e^{-\frac{x^2}{4}} H_n\left(\frac{x}{2}\right),$$

where $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$ is the Hermite polynomial. Using the property

$$H_{n+1}\left(\frac{x}{2}\right) = xH_n\left(\frac{x}{2}\right) - H_n'\left(\frac{x}{2}\right)$$

$n-k$ times we obtain that

$$H_n\left(\frac{x}{2}\right) = H_k\left(\frac{x}{2}\right)\left(x^{n-k} + Q_{n-k}(x)\right),$$

which ends the proof. ■

3. Negative moments of stable distribution

In [4] we can find the very well known property of nonnegative moments of an α-stable distribution. It states that for every α-stable density function g_α, $0 < \alpha < 2$, we have

$$\int_{\mathbb{R}} |x|^r g_\alpha(x)dx < +\infty \quad \text{for } 0 < r < \alpha,$$

$$\int_{\mathbb{R}} |x|^r g_\alpha(x)dx = +\infty \quad \text{for } r \geq \alpha.$$

In the following remarks we give some information about negative moments of the strictly positive and symmetric α-stable distribution.

Remark 1. If a random variable Θ is positive strictly stable with the Laplace transform $\mathbf{E}e^{-s\Theta} = e^{-(2s)^{\frac{\alpha}{2}}}$ for $s \geq 0$, $\alpha \in (0,2)$, and with the density function $\mathcal{L}(\Theta) = \lambda$, then for every $r < 0$ we have

$$\int_0^\infty x^r \lambda(dx) = \frac{2^{r+1}}{\alpha\Gamma(-r)}\Gamma\left(-\frac{2r}{\alpha}\right).$$

Proof. From the properties of the Gamma distribution for $r < 0$ and $x > 0$ we obtain that $\frac{1}{\Gamma(-r)}\int_0^\infty s^{-r-1}e^{-xs}ds = x^r$ ([**?**]), so we can write

$$\int_0^\infty x^r \lambda(dx) = \frac{1}{\Gamma(-r)} \int_0^\infty s^{-r-1} \int_0^\infty e^{-xs} \lambda(dx)\, ds$$

$$= \frac{1}{\Gamma(-r)} \int_0^\infty s^{-r-1} e^{-(2s)^{\frac{\alpha}{2}}} ds$$

$$= \frac{1}{\Gamma(-r)} \int_0^\infty \frac{2^{r+1}}{\alpha} u^{-\frac{2r}{\alpha}-1} e^{-u} du = \frac{2^{r+1}}{\alpha\Gamma(-r)} \Gamma\left(-\frac{2r}{\alpha}\right).$$

■

Remark 2. If an $S\alpha S$ random variable X has the density function f_α, $\alpha \in (0,2]$, then

$$\int_{\mathbb{R}} |x|^r f_\alpha(x) dx = \frac{1}{\alpha \cos(0.5\pi r)\Gamma(-r)} \Gamma\left(-\frac{r}{\alpha}\right) \quad \text{for } -1 < r < 0,$$

$$\int_{\mathbb{R}} |x|^r f_\alpha(x) dx = +\infty \qquad \text{for } r \leq -1.$$

Proof. The symmetry of the function f_α allows us to consider only the case $x > 0$. Using the same trick as in the proof of the previous remark for $r < 0$ we obtain

$$\int_0^\infty x^r f_\alpha(x) dx = \frac{1}{\Gamma(-r)} \int_0^\infty s^{-r-1} \int_0^\infty e^{-xs} f_\alpha(x) dx\, ds.$$

Consider the random variable ZX, where Z is $S1S$ and X is $S\alpha S$, Z and X are independent, then

$$
\begin{aligned}
\mathbf{E}e^{isZX} &= \int_{\mathbb{R}} e^{-|xs|} f_\alpha(x)dx \\
&= 2\int_0^\infty e^{-xs} f_\alpha(x)dx \\
&= 2\int_0^\infty e^{-(xs)^\alpha} f_1(x)dx.
\end{aligned}
$$

Using this equality for a nonnegative s we obtain

$$
\begin{aligned}
\int_0^\infty x^r f_\alpha(x)dx &= \frac{1}{\Gamma(-r)}\int_0^\infty s^{-r-1}\int_0^\infty e^{-(xs)^\alpha} f_1(x)dx\, ds \\
&= \frac{1}{\Gamma(-r)}\int_0^\infty f_1(x)\int_0^\infty s^{-r-1}e^{-(xs)^\alpha} ds\, dx \\
&= \frac{1}{\alpha\Gamma(-r)}\Gamma\left(-\frac{r}{\alpha}\right)\int_0^\infty x^r f_1(x)dx \\
&= \frac{1}{\pi\alpha\Gamma(-r)}\Gamma\left(-\frac{r}{\alpha}\right)\int_0^\infty \frac{x^r}{1+x^2}dx.
\end{aligned}
$$

The integral $\int_0^\infty \frac{x^r}{1+x^2}dx$ is finite only for $r \in (-1,1)$ and then it is equal $\frac{\pi}{2\cos(0.5\pi r)}$. This ends the proof. ∎

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