

BAYESIAN ESTIMATION OF AR(1) MODELS WITH UNIFORM INNOVATIONS

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Abstract

The first-order autoregressive model with uniform innovations is considered. In this paper, we propose a family of BAYES estimators based on a class of prior distributions. We obtain estimators of the parameter which perform better than the maximum likelihood estimator.

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1. INTRODUCTION

Let us consider the following autoregressive model

$$(1.1) \quad X_t = \rho X_{t-1} + \varepsilon_t, \quad t = \dots, -1, 0, 1, \dots,$$

where $0 < \rho < 1$ and the ε_t 's are *i.i.d* and distributed according to uniform distribution $U(0, 1)$. X_1 is assumed to be distributed according to $U(0, 1/(1 - \rho))$ such that the process is mean stationary. The likelihood

function based on the observations $x = (x_1, x_2, \dots, x_n)$ is then

$$p(x|\rho) = (1 - \rho) I_A(x),$$

where $A = \{x : 0 \leq x_1 \leq 1/(1 - \rho), 0 \leq x_t - \rho x_{t-1} \leq 1, t = 2, \dots, n\}$.

Let ρ_1 the maximum likelihood estimator of ρ introduced by Bell and Smith (1986):

$$\rho_1 = \min(1, x_2/x_1, \dots, x_n/x_{n-1}).$$

The problem of bayesian analysis of AR(1) models is studied by Turkmann (1990) and Ibazizen and Fellag (2003) when the errors are exponential.

In this paper, we propose a Bayesian estimator of the parameter of the model (1.1) using a family of prior distributions for the parameter ρ proposed by Ibazizen and Fellag (2003). The estimators obtained with this method under quadratic loss structure appear to be closer to the parameter than the usual maximum likelihood estimator ρ_1 .

2. BAYESIAN ESTIMATION OF THE AR(1) PARAMETER

Consider the following family of prior distributions for the parameter ρ

$$(2.1) \quad p(\rho; \beta) \propto \frac{\rho^{\beta-1}}{1 - \rho} I_{(0,1)}(\rho), \quad \beta > 0.$$

Suppose that our data consists of the segment of the observations $x = (x_1, x_2, \dots, x_n)$. Then, the prior assessment on ρ is transformed via BAYES theorem into

$$p(\rho|x) \propto p(x|\rho) \cdot p(\rho; \beta)$$

and then,

$$p(\rho|x) = C \cdot \rho^{\beta-1} I_{(\rho_0, \rho_1)}(\rho)$$

with ρ_1 given above,

$$\rho_0 = \max\left(0, \frac{x_1 - 1}{x_1}, \frac{x_2 - 1}{x_1}, \dots, \frac{x_n - 1}{x_{n-1}}\right) \quad \text{and} \quad C = \frac{\beta}{\rho_1^\beta - \rho_0^\beta}.$$

Under quadratic loss structure, the BAYES estimator of ρ is the posterior mean and is given by the formula

$$\hat{\rho}_B(\beta) = \int_{\rho_0}^{\rho_1} \rho p(\rho|x) d\rho .$$

This leads to

$$(2.2) \quad \hat{\rho}_B(\beta) = \frac{\beta}{\beta+1} \frac{\rho_1^{\beta+1} - \rho_0^{\beta+1}}{\rho_1^\beta - \rho_0^\beta} \quad , \quad \beta > 0 .$$

The posterior variance is

$$\sigma_B^2(\beta) = E [(\rho - \hat{\rho}_B(\beta))^2 | x] = \frac{\beta}{\beta+2} \frac{\rho_1^{\beta+2} - \rho_0^{\beta+2}}{\rho_1^\beta - \rho_0^\beta} - \hat{\rho}_B(\beta)^2 .$$

In order to illustrate our formulas, consider one segment of 20 observations from the model (1.1) simulated with the true value $\rho = 0.4$. We found

$$\rho_0 = 0.3685 \quad \text{and} \quad \rho_1 = 0.4193$$

Simulated values of the Bayesian estimator and the posterior variance (given in parentheses) of ρ for different values of β are given in the following table

Table 1. Simulated values of the Bayesian estimator and posterior variance of ρ for n=20 and $\rho = 0.4$

β	0.2	0.6	1.0	2.0	5.0
$\hat{\rho}_B(\beta)$	0.3934	0.3936	0.3939	0.3944	0.3960
$ \hat{\rho}_B - \rho $	0.00653725	0.0063186	0.0061	0.00555404	0.00392878
$\sigma_B^2(\beta)$	0.000215034	0.000215072	0.000215053	0.000214755	0.000211759

Since $|\rho_1 - \rho| = 0.0193$, one can note that, for every β , the estimator $\hat{\rho}_B(\beta)$ is closer than ρ_1 to the parameter ρ . Also, the value of $\hat{\rho}_B$ tends to the true value when β grows . The posterior variance is near zero for every β and changes slightly when β increases.

3. SIMULATION STUDY

Consider the following exhaustive simulation study. We simulate samples from the model (1.1) for $\beta = 0.5, 1.0, 1.5, 2.0$ and for $n = 10, 20$. The value of $\hat{\rho}_B(\beta)$ and its posterior variance are calculated. The computations are based on 100000 replications of the process. The results are given in Table 2.

Table 2. Simulated values $\hat{\rho}_B(\beta)$ and its variance for $\rho=0.1,0.4,0.9$, $n=10,20$ and $\beta=0.5,1.0,1.5,2.0$. Variances are given in parentheses

n	ρ	ρ_1	$\beta = 0.5$	$\beta = 0.8$	$\beta = 1.0$	$\beta = 1.5$	$\beta = 2.0$
10	0.1	0.1052	0.0988	0.1030	0.1134	0.1237	0.1315
		(0.0060)	(0.0018)	(0.0185)	(0.0019)	(0.0022)	(0.0026)
	0.4	0.4211	0.3954	0.3960	0.3978	0.4000	0.4022
		(0.0024)	(0.0017)	(0.0017)	(0.0016)	(0.0014)	(0.0014)
	0.9	0.9101	0.8999	0.8999	0.8999	0.9000	0.9000
		($1.08 \cdot 10^{-4}$)	($0.5525 \cdot 10^{-4}$)	($0.5522 \cdot 10^{-4}$)	($0.5521 \cdot 10^{-4}$)	($0.5517 \cdot 10^{-4}$)	($0.5514 \cdot 10^{-4}$)
20	0.1	0.1833	0.0981	0.1087	0.1127	0.1228	0.1305
		(0.0051)	(0.0017)	(0.0017)	(0.0016)	(0.0019)	(0.0022)
	0.4	0.4540	0.3924	0.3938	0.3949	0.3972	0.3995
		(0.0023)	(0.0018)	(0.0017)	(0.0016)	(0.0015)	(0.0014)
	0.9	0.9100	0.8992	0.8992	0.8992	0.8993	0.8993
		($1.0434 \cdot 10^{-4}$)	($0.7808 \cdot 10^{-4}$)	($0.778 \cdot 10^{-4}$)	($0.7689 \cdot 10^{-4}$)	($0.7574 \cdot 10^{-4}$)	($0.7464 \cdot 10^{-4}$)

We can remark that the Bayesian estimator $\hat{\rho}_B(\beta)$ has smaller standard deviation than the maximum likelihood estimator ρ_1 . In our exhaustive simulation, we remark that, when n is not very small, the Bayesian estimator $\hat{\rho}_B(\beta)$ is better than the maximum likelihood estimator of ρ for every β . Also, this is true if n is too small (e.g; $n = 10$) and ρ not near zero. However, when ρ is near zero, $\hat{\rho}_B(\beta)$ is the best only if $0 < \beta < 1$. So our conclusion can be as follows : if we choose a prior distribution given by the formula (2.1) with $0 < \beta < 1$, then the Bayesian estimator obtained under quadratic loss structure performs better than the maximum likelihood estimator.

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