OPTIMAL TREND ESTIMATION
IN GEOMETRIC ASSET PRICE MODELS

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Abstract

In the general geometric asset price model, the asset price $P(t)$ at time t satisfies the relation

$$P(t) = P_0 \cdot e^{\alpha \cdot f(t) + \sigma \cdot F(t)}, \quad t \in [0, T],$$

where f is a deterministic trend function, the stochastic process F describes the random fluctuations of the market, α is the trend coefficient, and σ denotes the volatility.

The paper examines the problem of optimal trend estimation by utilizing the concept of kernel reproducing Hilbert spaces. It characterizes the class of trend functions with the property that the trend coefficient can be estimated consistently. Furthermore, explicit formulae for the best linear unbiased estimator $\hat{\alpha}$ of α and representations for the variance of $\hat{\alpha}$ are derived.

The results do not require assumptions on finite-dimensional distributions and allow of jump processes as well as exogeneous shocks.

Keywords: geometric asset price model, trend estimation, Wiener process, Ornstein-Uhlenbeck process, kernel reproducing Hilbert space, exogeneous shocks, compound Poisson process.

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1.   Introduction

In the Black-Scholes model, the asset price dynamics are given by a geometric Brownian motion: the current asset price $P(t)$ at time $t \in [0, T]$ is assumed to satisfy the relation

$$(1) \qquad P(t) = P_0 \, e^{\alpha \, t + \sigma \, W(t)}, \quad t \in [0, T],$$

where α stands for the trend coefficient, $\sigma > 0$ denotes the volatility, P_0 is a positive random variable, and the process W is a standard Wiener process with index set $[0, T]$.

Consider the problem of estimating trend and volatility from historical data; for simplicity, suppose that prices are sampled at n equally spaced intervals of length $h = T/n$, i.e., one observes the quantities $P_k = P(k \, h), 0 \leq k \leq n$. A commonly used estimator is based upon the continuously compounded returns $R_k = \log(P_k/P_{k-1}), 1 \leq k \leq n$, which are stochastically independent and normally distributed random variables with mean $\alpha \, h$ and variance $\sigma^2 \, h$. The maximum likelihood estimator for the parameter (α, σ^2) is readily verified to be

$$(2) \qquad \hat{\alpha} \;=\; \frac{1}{T} \sum_{k=1}^{n} R_k = \frac{1}{T} \, \log\left(\frac{P_n}{P_0}\right),$$

$$(3) \qquad \hat{\sigma}^2 \;=\; \frac{1}{T} \sum_{k=1}^{n} (R_k - \hat{\alpha} \, h)^2,$$

and $\hat{\alpha}$ and $\hat{\sigma}^2$ are asympotically efficient in the class of all consistent and uniformly asymptotically normal estimators, cf. pp. 356–363 in Campbell, Lo and MacKinlay (1997).

Merton (1980) has pointed out that there is an essential difference between $\hat{\alpha}$ and $\hat{\sigma}^2$: under continuous-record asymptotics, $\hat{\sigma}^2$ is consistent while $\hat{\alpha}$ is inconsistent. More precisely, expectation and variance of these estimators are found to be

$$(4) \qquad \mathsf{E}(\hat{\alpha}) = \alpha, \quad \mathsf{E}(\hat{\sigma}^2) = \sigma^2 \left(1 - \frac{h}{T}\right)$$

and

$$(5) \qquad \mathsf{Var}(\hat{\alpha}) = \frac{\sigma^2}{T}, \quad \mathsf{Var}(\hat{\sigma}^2) = \frac{2\sigma^4}{T}\, h\left(1 - \frac{h}{T}\right),$$

in other words, more frequent sampling within the fixed time interval $[0, T]$ yields $\lim_{h \to 0} \mathsf{E}(\,(\hat{\sigma}^2 - \sigma^2)^2\,) = 0$ while $\mathsf{E}(\,(\hat{\alpha} - \alpha)^2\,) = \sigma^2/T$ is independent of h. For further discussions of this topic, see again Campbell, Lo and MacKinlay (1997) and the references cited therein.

It follows from (1) that the log-price process $p(t) = \log(P(t)/P_0)$ has a linear trend, and the random fluctuations of the market are described by a Wiener process. Though (1) is regarded as a satisfactory approximation of real asset prices, non-linear trends and other classes of processes are more efficient for certain assets. In the literature, numerous extensions and alternatives have been proposed, cf. the discussion and the references given on pp. 32–33 of Karatzas and Shreve (1998). See also Eberlein and Keller (1995).

Consider a generalized geometric asset price process by assuming that the expression $\alpha\,t + \sigma\,W(t)$ in (1) is replaced by $\alpha\,f(t) + \sigma\,F(t)$ where f is a possibly non-linear trend function and F stands for a prescribed stochastic process which is not necessarily a Wiener process. It is the purpose of this paper to examine the problem of optimal estimation of the trend coefficient α by reasonable estimators where 'reasonable' means 'linear in the observations and unbiased'. In particular, the following questions are to be answered:

- Can one characterize those trend functions permitting of a consistent estimation of α?

- Let f be a trend function with the property that a consistent sequence of linear unbiased estimators does not exist. Is it possible to derive an explicit formula for the best estimator having minimum variance among all linear unbiased estimators?

- If the best estimator is based on a whole trajectory how can one find a satisfactory discretization?

2. Basic assumptions

Suppose the asset price $P(t)$ at time $t \in [0, T]$ can be represented as a generalized geometric process according to

(6) $$P(t) = P_0 \, e^{\alpha \, f(t) + \sigma \, F(t)}, \quad t \in [0, T],$$

where P_0 is a positive random variable, f denotes a (deterministic) trend function, and the random fluctuations of the market are described by the stochastic process F having vanishing mean values $\mathsf{E}(\, F(t)\,) = 0$ and finite covariance kernel $\gamma(s, t) = \mathsf{E}(\, F(s) \cdot F(t)\,)$. Trend function f and covariance kernel γ are assumed to be known while the volatility parameter $\sigma > 0$ as well as the real trend coefficient α are unknown. Finally, suppose that f is a continuous function and F is a mean square continuous process. Note that numerous random functions such as Poisson-type processes meet the requirement of mean square continuity though their sample paths have jump discontinuities.

The classical model (1) and related price processes discussed in the literature such as the geometric Ornstein-Uhlenbeck process are special cases of (6). For suitable processes, (6) can of course be written in differential form; e.g., if f, g are continuously differentiable functions, P_0 is a constant, and F stems from the standard Wiener process according to $F(t) = g(t)\, W(t)$, then the price dynamics are expressible in the differential form

$$\frac{dP(t)}{P(t)} = \left(\alpha \, f'(t) + \sigma \, g'(t) \, W(t) + \frac{\sigma^2}{2} \, g^2(t) \right) dt + \sigma \, g(t) \, dW(t).$$

3. Optimal trend estimation: continuous-record asymptotics

The first and the second question formulated above can be answered by considering so-called kernel reproducing Hilbert spaces. This concept is essentially due to Parzen (1961) as well as Sacks and Ylvisaker (1966) and is an important tool in time-continuous regression analysis with dependent observations. See also Su and Cambanis (1993) and the overview by Cambanis (1985).

Let N be an arbitrary mean square continuous stochastic process with index set $[0, T]$, vanishing mean values $\mathsf{E}(N(t)) = 0$ and covariance kernel $R(s, t) = \mathsf{E}(N(s) \cdot N(t))$. Then the kernel reproducing Hilbert space $H(R)$ associated with R is a Hilbert space of real-valued functions h which are defined on the interval $[0, T]$. $H(R)$ is uniquely determined by the following properties: for each fixed $t \in [0, T]$, the function $R(\cdot, t)$ is an element of $H(R)$, and the real scalar product $\langle \cdot, \cdot \rangle$ on $H(R)$ satisfies $\langle h, R(\cdot, t) \rangle = h(t)$ for each $h \in H(R)$ and each $t \in [0, T]$. See Atteia (1992) for a more detailed treatment.

Suppose one wishes to estimate the regression coefficient β in the regression model

$$(7) \qquad X(t) = \beta \cdot h(t) + N(t), \quad t \in [0, T],$$

by means of observations from X at possibly infinitely many time points $s \in [0, T]$ where $h \not\equiv 0$ is a deterministic regression function. The problem whether consistent estimation of β by 'reasonable' estimators is possible or not can be solved as follows: if h is *not* an element of $H(R)$ then there exists a sequence $\hat{\beta}_1, \hat{\beta}_2, \ldots$ of linear unbiased estimators of β with the property $\lim_{n \to \infty} \mathsf{E}\left((\hat{\beta}_n - \beta)^2\right) = 0$; on the other hand, if h lies in $H(R)$ then even the best linear unbiased estimator $\hat{\beta}$ of β has (minimum) variance $\mathsf{Var}(\hat{\beta}) = \langle h, h \rangle^{-1} > 0$, cf. Theorem 2.3 in Sacks and Ylvisaker (1966). Consequently, consistent estimation of β by linear unbiased estimators is impossible for $h \in H(R)$. In the latter case, however, an explicit formula for $\hat{\beta}$ may be established provided h has the representation

$$(8) \qquad h(t) = \int_0^T \phi(s) \cdot R(s, t) \, ds + \sum_{k=1}^K c_k \cdot R(s_k, t)$$

with a continuous function ϕ, real constants $c_1, c_2, \ldots, c_K$ and fixed time points $s_1, s_2, \ldots, s_K \in [0, T]$. For (8) implies that the best linear unbiased estimator $\hat{\beta}$ is given by

$$(9) \qquad \hat{\beta} = \frac{\int_0^T \phi(t) \cdot X(t) \, dt + \sum_{k=1}^K c_k \cdot X(s_k)}{\int_0^T \phi(t) \cdot h(t) \, dt + \sum_{k=1}^K c_k \cdot h(s_k)},$$

and the variance of $\hat{\beta}$ reads

$$(10) \qquad \mathsf{Var}(\hat{\beta}) = \left(\int_0^T \phi(t) \cdot h(t) \, dt + \sum_{k=1}^K c_k \cdot h(s_k) \right)^{-1},$$

see Cambanis (1985). (Integrals of stochastic processes are to be understood in the mean square sense, cf. chapter XI in Loève (1978).)

These facts may immediately be applied on trend estimation in geometric asset price models because (6) is a special case of (7) according to

$$(11) \qquad \beta = \alpha, \; h = f, \; N = \sigma \cdot F, \; R = \sigma^2 \cdot \gamma, \; X = p,$$

where $p(t) = \log(P(t)/P_0)$ again denotes the log-price process.

The kernel reproducing Hilbert space of a prescribed covariance kernel characterizes all functions being comparable to that peculiar kernel. From the viewpoint of financial economics, the above results therefore admit the following interpretation: if the trend structure of the asset price is 'similar' to the structure of the random fluctuations of the market then even the best analysis cannot completely distinguish between trend and fluctuation. On the other hand, if the trend function of the asset price is 'essentially different' from the fluctuations then a sufficiently expensive observation will recover the trend coefficient with arbitrary accuracy. (This interpretation complements a statement given on p. 365 of Campbell, Lo and MacKinlay (1997). Here, the inconsistency of the trend estimator is related to the non-differentiability of diffusion sample paths. In the light of the above considerations, membership or non-membership of the trend function in the kernel reproducing Hilbert space is the reason for inconsistency or consistency, respectively.)

In the sequel the correspondence (11) is always assumed. The first and the second question asked in the introduction can now be answered: one has to check the relation $f \in H(R)$. If f is not an element of $H(R)$ then consistent trend estimation is always possible. In the case $f \in H(R)$, however, consistent linear unbiased estimation is impossible but if f is verified to admit representation (8) then the best linear unbiased estimator $\hat{\alpha}$ and the minimum variance $\mathsf{Var}(\alpha)$ are given by (9) and (10), respectively. For several covariance kernels R, the associated kernel reproducing Hilbert spaces $H(R)$

are well-known. It will turn out that many - but not all - commonly used trend functions lie in $H(R)$ and are expressible in the form of (8). Note that the above considerations exclusively depend upon the covariance structure while assumptions on finite-dimensional distributions or analytic properties of sample paths are not required.

3.1. Geometric Wiener processes

Suppose F is a standard Wiener process which implies that $N = \sigma F$ has the covariance kernel $R(s,t) = \sigma^2 \cdot \min(s,t)$. The kernel reproducing Hilbert space $H(R)$ consists of all real-valued functions h on $[0,T]$ having the following properties: h is absolutely continuous, $h(0) = 0$, and the derivative h' is square integrable on $[0,T]$ with respect to the Lebesgue measure. The scalar product $\langle g, h \rangle$ is defined by

$$(12) \qquad \langle g, h \rangle = \frac{1}{\sigma^2} \int_0^T g'(t) \, h'(t) \, dt, \quad g, h \in H(R).$$

It is easily checked that $f(t) = t^m$ lies in $H(R)$ for each integer with $m \geq 1$; trend functions such as $f(t) = \sin \lambda t$ and $f(t) = \log(1 + t)$ are elements of $H(R)$ as well. Therefore, consistent estimation of the trend coefficient is impossible for many 'natural' trend functions. On the other hand, $H(R)$ does not contain $f(t) = \sqrt{t}$ or related mappings.

$\hat{\alpha}$ and $\mathsf{Var}(\hat{\alpha})$ are to be computed for several trend functions. It should be emphasized that $\hat{\alpha}$ itself will not depend upon the unknown volatility parameter σ, as opposed to the distribution and variance of $\hat{\alpha}$.

(i) Linear trend. Consider the linear trend function $f(t) = t$. It is pointed out on pp. 364–365 of Campbell, Lo and MacKinlay (1997) that more frequent sampling within a fixed time span does not increase the precision of the maximum likelihood estimator (2), and Table 9.2a presented on p. 365 of this book is a lucid numerical illustration of this effect. The statement of the authors can slightly be sharpened: f obviously has the representation

$$(13) \qquad f(t) = \frac{1}{\sigma^2} \, R(T, t)$$

whence it follows by (9) and (10) that the best linear unbiased estimator and its variance are given by

$$(14) \qquad \hat{\alpha} = \frac{p(T)}{T}, \quad \mathsf{Var}(\hat{\alpha}) = \frac{\sigma^2}{T}.$$

Thus $\hat{\alpha}$ is identical with the maximum likelihood estimator (2), and one can conclude that even complete knowledge of a whole trajectory does not increase the precision. The representation (13) of f also explains the reason: since (13) depends only upon $R(T,t)$ formula (14) shows that the best linear unbiased estimator requires only a single observation of the log-price process at time T.

(ii) Polynomial trend. Let f be the trend function $f(t) = t^m$ for some prescribed integer $m \geq 2$. One finds

$$(15) \qquad f(t) = \frac{m}{\sigma^2} \left(T^{m-1} R(T,t) - (m-1) \int_0^T s^{m-2} R(s,t)\, ds \right),$$

hence (9), (10) imply

$$(16) \qquad \hat{\alpha} = \frac{2 - \frac{1}{m}}{T^{m-1}} \left(\frac{p(T)}{T} - \frac{m-1}{T^m} \int_0^T t^{m-2} p(t)\, dt \right)$$

and

$$(17) \qquad \mathsf{Var}(\hat{\alpha}) = \frac{(2m-1)\,\sigma^2}{m^2\, T^{2m-1}}.$$

While best estimation in case of a linear trend requires only $p(T)$, the situation is different for polynomial trends because (16) needs a whole trajectory of the log-price process.

(iii) Cyclical trend. Suppose $f(t) = \sin \lambda t$ with a known frequency $\lambda \neq 0$. f has the representation

$$(18) \qquad f(t) = \frac{\lambda}{\sigma^2} \left(\cos \lambda T \cdot R(T, t) + \lambda \int_0^T \sin \lambda s \cdot R(s, t) \, ds \right),$$

and (9), (10) guarantee the formulae

$$(19) \qquad \hat{\alpha} = \frac{2}{\lambda T + \sin 2\lambda T} \left(\cos \lambda T \cdot p(T) + \lambda \int_0^T \sin \lambda t \cdot p(t) \, dt \right)$$

as well as

$$(20) \qquad \mathsf{Var}(\hat{\alpha}) = \frac{2 \, \sigma^2}{\lambda^2 T + \frac{\lambda}{2} \sin 2\lambda T}.$$

Similar to polynomial trends a whole trajectory is required for optimal estimation.

3.2. Geometric Ornstein-Uhlenbeck processes

If F stands for an Ornstein-Uhlenbeck process with parameter $\lambda > 0$ the process $N = \sigma F$ has the covariances $R(s, t) = \sigma^2 \exp(-\lambda |s - t|)$. The kernel reproducing Hilbert space $H(R)$ is the collection of all absolutely continuous functions h on $[0, T]$ having a square integrable derivative h' with respect to the Lebesgue measure. (Comparing $H(R)$ with the corresponding space in 3.1 the condition $h(0) = 0$ has been dropped.) $H(R)$ is endowed with the scalar product

$$(21) \quad 2\sigma^2 \cdot \langle g, h \rangle =$$

$$g(0)h(0) + g(T)h(T) + \lambda \int_0^T g(t)h(t) \, dt + \frac{1}{\lambda} \int_0^T g'(t)h'(t) \, dt.$$

Again, 'natural' trend functions are contained in $H(R)$ but at least admit representations in the form of (8); furthermore, $\hat{\alpha}$ will turn out to be independent from σ. For brevity, only $\hat{\alpha}$ and its variance are listed below.

(i) Constant trend. The trend function $f(t) = 1$ lies in $H(R)$, as opposed to the Wiener process. One finds

$$(22) \qquad \hat{\alpha} = \frac{1}{\lambda T + 2} \left(p(0) + p(T) + \lambda \int_0^T p(t)\, dt \right)$$

as well as

$$(23) \qquad \mathsf{Var}(\hat{\alpha}) = \frac{2\sigma^2}{\lambda T + 2}.$$

(ii) Linear trend. Consider $f(t) = t$. In case of the Wiener process optimal estimation is achieved by means of a single observation, cf. (14). The Ornstein-Uhlenbeck process, however, needs a whole trajectory due to

$$(24) \quad \hat{\alpha} = \frac{3}{T(\lambda^2 T^2 + 3\lambda T + 3)} \left(-p(0) + (\lambda T + 1)p(T) + \lambda^2 \int_0^T t\, p(t)\, dt \right).$$

Moreover, the relation

$$(25) \qquad \mathsf{Var}(\hat{\alpha}) = \frac{6\lambda\sigma^2}{T(\lambda^2 T^2 + 3\lambda T + 3)}$$

is valid.

(iii) Polynomial trend. Let f be the trend function $f(t) = t^m$ for some integer $m \geq 2$. The following formulae can eventually be verified:

$$(26) \quad \hat{\alpha} = \frac{4m^2 - 1}{T^{2m-1} \left((2m-1)\lambda^2 T^2 + (4m^2 - 1)\lambda T + m^2(2m+1) \right)}$$

$$\times \left(T^{m-1}(\lambda T + m)p(T) + \lambda^2 \int_0^T t^m p(t)\, dt - m(m-1) \int_0^T t^{m-2} p(t)\, dt \right)$$

and

$$(27) \quad \mathsf{Var}(\hat{\alpha}) = \frac{2(4m^2 - 1)\lambda\sigma^2}{T^{2m-1}\left((2m-1)\lambda^2 T^2 + (4m^2 - 1)\lambda T + m^2(2m+1)\right)}.$$

4.   Optimal trend estimation: discrete sampling

Apart from special cases such as (14) the best estimator $\hat{\alpha}$ depends upon an integral in the form of $\int_0^T \phi(t)\,p(t)\,dt$ provided $f \in H(R)$ is expressible due to (8). This integral can always be obtained as mean square limit of Riemann-Stieltjes sums but the resulting approximation may be inadequate because the rate of convergence would be poor.

Consider an arbitrary trend function f - which is not necessarily contained in $H(R)$ - and let $\Delta = \{t_1, t_2, \ldots, t_n\}$ be a set of prescribed sampling points with $0 \le t_1 < t_2 < \ldots < t_n \le T$. Define the column vectors f_Δ and p_Δ by

$$f_\Delta = (f(t_1), f(t_2), \ldots, f(t_n))', \ p_\Delta = (p(t_1), p(t_2), \ldots, p(t_n))'.$$

(x' denotes the transpose of the vector x. The uninteresting case $f_\Delta = 0$ is excluded.) Suppose also that R_Δ^{-1} is the inverse of the matrix R_Δ where the entry in the k-th row and j-th column of R_Δ is given by $R(t_k, t_j)$ for $1 \le k, j \le n$. Then the statistic

$$(28) \qquad \hat{\alpha}_\Delta = c_\Delta'\, p_\Delta \quad \text{with} \quad c_\Delta = \frac{R_\Delta^{-1}\, f_\Delta}{f_\Delta'\, R_\Delta^{-1}\, f_\Delta}$$

is linear in p_Δ with $\mathsf{E}(\hat{\alpha}_\Delta) = \alpha$, and among all linear unbiased estimators, $\hat{\alpha}_\Delta$ is of course optimal having the minimum variance

$$(29) \qquad \mathsf{Var}(\hat{\alpha}_\Delta) = \frac{1}{f_\Delta'\, R_\Delta^{-1}\, f_\Delta}.$$

R_Δ^{-1} is well-known for several covariance kernels; in particular, explicit formulae are available in case of Wiener and Ornstein-Uhlenbeck processes. Equation (29) holds for arbitrary trend functions. If f lies in the kernel reproducing Hilbert space $H(R)$ then the effect of discretization can be studied

by comparing (29) with corresponding equations for $\mathsf{Var}(\hat{\alpha})$ such as (10) or the relation $\mathsf{Var}(\hat{\alpha}) = \langle f, f \rangle^{-1}$, see the considerations of the third Section.

The evaluation of (28), (29) is now to be discussed in the case of Wiener processes. (For Ornstein-Uhlenbeck processes, the calculations can be carried out analogously and are therefore omitted.) If F is a standard Wiener process one can assume $t_1 > 0$, and the inverse of R_Δ is a triangular matrix:

$$
R_\Delta^{-1} = \frac{1}{\sigma^2}
\begin{pmatrix}
a_1 & c_1 & & & & \\
b_2 & a_2 & c_2 & & 0 & \\
 & b_3 & a_3 & c_3 & & \\
 & & \ddots & \ddots & \ddots & \\
0 & & & b_{n-1} & a_{n-1} & c_{n-1} \\
 & & & & b_n & a_n
\end{pmatrix}.
$$

Setting $\delta_k = t_k - t_{k-1}$ for $1 \le k \le n$ (and $t_0 = 0$), the nonzero entries of R_Δ^{-1} satisfy the relations

$$
(a_1, c_1) = \left(\delta_1^{-1} + \delta_2^{-1}, -\delta_2^{-1} \right),
$$

$$
(b_k, a_k, c_k) = \left(-\delta_k^{-1}, \delta_k^{-1} + \delta_{k+1}^{-1}, -\delta_{k+1}^{-1} \right), \quad 2 \le k \le n-1,
$$

$$
(b_n, a_n) = \left(-\delta_n^{-1}, \delta_n^{-1} \right).
$$

Using the abbreviations $f_k = f(t_k), p_k = p(t_k)$ for $1 \le k \le n$ (as well as $f_0 = p_0 = 0$), equations (28) and (29) are found to be

$$
(30) \qquad \hat{\alpha}_\Delta = \left(\sum_{k=1}^{n} \frac{(f_k - f_{k-1})(p_k - p_{k-1})}{\delta_k} \right) \cdot \left(\sum_{k=1}^{n} \frac{(f_k - f_{k-1})^2}{\delta_k} \right)^{-1}
$$

and

$$
(31) \qquad \mathsf{Var}(\hat{\alpha}_\Delta) = \sigma^2 \cdot \left(\sum_{k=1}^{n} \frac{(f_k - f_{k-1})^2}{\delta_k} \right)^{-1}.
$$

Table 1 contains $\mathsf{Var}(\hat{\alpha}_\Delta)$ for several trend functions. For simplicity, equidistant sampling with step size $h > 0$ is considered, i.e., Δ is chosen to be $\Delta = \{h, 2h, 3h, \ldots, nh\}$ where n is an integer with $nh = T$. In the linear case, $\hat{\alpha}_\Delta$ is of course identical with $\hat{\alpha}$. A comparison between $\mathsf{Var}(\hat{\alpha}_\Delta)$ and $\mathsf{Var}(\hat{\alpha})$ for quadratic, cubic and cyclical trends - see (17) and (20), respectively - illustrates the size of the discretization error. As h tends to zero, one obtains $\lim_{h\to 0} \mathsf{Var}(\hat{\alpha}_\Delta) = \mathsf{Var}(\hat{\alpha}) > 0$. The concave trend function $f(t) = \sqrt{t}$, however, is no element of $H(R)$ which implies that the trend coefficient can be estimated consistently. For this peculiar trend function, Table 1 does not contain $\mathsf{Var}(\hat{\alpha}_\Delta)$ itself but an expression being asymptotically equivalent as h tends to zero. This expression ensures $\lim_{h\to 0} \mathsf{Var}(\hat{\alpha}_\Delta) = \lim_{h\to 0} 4\sigma^2/\log(T/h) = 0$, as expected.

Table 1. Variance of the best discrete estimator

Trend	Variance of $\hat{\alpha}_\Delta$
Linear: $f(t) = t$	σ^2/T
Quadratic: $f(t) = t^2$	$3\sigma^2/(4T^3 - h^2 T)$
Cubic: $f(t) = t^3$	$5\sigma^2/(9T^5 - 5h^2 T^3 + h^4 T)$
Cyclical: $f(t) = \sin \lambda t$	$2\sigma^2/\left(\frac{1-\cos\lambda h}{h^2} \cdot 2T + \frac{1-\cos\lambda h}{h\sin\lambda h} \cdot \sin 2\lambda T \right)$
Concave: $f(t) = \sqrt{t}$	$\cong 4\sigma^2/\log\left(\frac{T}{h}\right)$

5.   Extensions and complements

(i) Confidence intervals. The best linear unbiased estimator is given by (9) provided the trend function admits representation (8). Assumptions on finite-dimensional distributions of the processes involved are not required, as opposed to the maximum likelihood method which may entail expensive computations. If, in addition, F is known to be a Gaussian process then each estimator in the form of (9) can be verified to be normally distributed. Hence exact or approximate confidence intervals for α may be derived depending on whether the volatility is known or has been estimated (see the formulae for $\mathsf{Var}(\hat{\alpha})$ derived in Subsection 3.1 and Subsection 3.2).

(ii) Modified representations for the trend function. Trend functions satisfying (8) are necessarily contained in $H(R)$. On the other hand, there may be elements of $H(R)$ which are not expressible due to (8). Even if such a representation exists it can be difficult to find an appropriate function ϕ, coefficients c_k and time points s_k. For trend functions h and covariance kernels R which have at least $L \geq 0$ continuous derivatives on $[0, T]$ it is sometimes easier to verify generalizations of (8) such as

$$(32) \qquad h(t) = \int_0^T \sum_{l=0}^L \phi_l(s) \cdot \frac{\partial^l R}{(\partial s)^l}(s, t)\, ds + \sum_{k=1}^K \sum_{l=0}^L c_{lk} \cdot \frac{\partial^l R}{(\partial s)^l}(s_k, t),$$

where ϕ_l are continuous functions, c_{lk} are real coefficients, and s_k are points of $[0, T]$. (For example, suppose $T = 1$ and let F be an integrated standard Wiener process which can be interpreted as a 'smoothed' version of the original Wiener process. Then $N = \sigma F$ has covariance kernel

$$R(s, t) = \sigma^2 \cdot \int_0^{\min(s,t)} (s - u)(t - u)\, du,$$

and quadratic and cubic trend functions can be written in the form of

$$t^2 = \frac{2}{\sigma^2} \cdot \frac{\partial R}{\partial s}(1, t) \quad \text{and} \quad t^3 = \frac{6}{\sigma^2} \cdot \left(-R(1, t) + \frac{\partial R}{\partial s}(1, t) \right),$$

respectively.)

It follows from (32) that the best linear unbiased estimator $\hat{\beta}$ and its variance are given by

$$(33) \qquad \hat{\beta} = \frac{\int_0^T \sum_{l=0}^L \phi_l(t) \cdot X^{(l)}(t)\, dt + \sum_{k=1}^K \sum_{l=0}^L c_{lk} \cdot X^{(l)}(s_k)}{\int_0^T \sum_{l=0}^L \phi_l(t) \cdot h^{(l)}(t)\, dt + \sum_{k=1}^K \sum_{l=0}^L c_{lk} \cdot h^{(l)}(s_k)}$$

and

$$(34) \qquad \mathsf{Var}(\hat{\beta}) = \left(\int_0^T \sum_{l=0}^{L} \phi_l(t) \cdot h^{(l)}(t)\, dt + \sum_{k=1}^{K} \sum_{l=0}^{L} c_{lk} \cdot h^{(l)}(s_k) \right)^{-1}.$$

Relations (33), (34) generalize (9), (10), and in view of correspondence (11) these relations can be used to treat a larger class of trend functions. ((34) is an immediate consequence of (32) and (33). The proof of (33) is similar to the proof of (9) and therefore omitted.)

(iii) Exogeneous shocks. The geometric asset price process described by (6) comprises trend and random fluctuations of the market. Frequently, the paths of asset price processes exhibit jumps being caused by unexpected exogeneous influences which are independent from the 'normal' fluctuations. Consider the modified model

$$(35) \qquad P(t) = P_0\, e^{\alpha\, f(t) + \sigma\, F(t) + J(t)}, \quad t \in [0, T],$$

where

$$(36) \qquad J(t) = \sum_{j=1}^{M(t)} Y_j$$

is a compound Poisson process, i.e., M is a Poisson process with parameter $\lambda > 0$ being independent from the i.i.d. random variables $Y_1, Y_2, \ldots$. If J and F are stochastically independent then the assumptions $\mathsf{E}(Y_j) = 0$ and $\mathsf{Var}(Y_j) = \tau^2$ with $0 < \tau^2 < \infty$ imply that the log-price process $p(t) = \log(P(t)/P_0)$ is again a special case of (7) with $N = \sigma\, F + J$ and

$$(37) \qquad R(s, t) = \mathsf{E}(N(s) \cdot N(t)) = \sigma^2\, \gamma(s, t) + \lambda \tau^2\, \min(s, t),$$

cf. relation (11). The compound Poisson process is widely used to describe unexpected shocks, and upon minor changes the considerations of Section 3 and Section 4 remain valid for the modified kernel given by (37).

The case $\gamma(s,t) = \min(s,t)$ is of particular interest. Here, (37) takes the form

$$R(s,t) = (\sigma^2 + \lambda\tau^2) \cdot \min(s,t)$$

whence it follows that all formulae and results regarding γ still hold provided the volatility σ is replaced by

$$\sigma_J = \sqrt{\sigma^2 + \lambda\tau^2}.$$

Especially, the formulae (12)–(20) as well as (30), (31) may be applied with σ_J instead of σ.

Though the presence of J would completely change both the finite-dimensional distributions and the sample path behaviour of the log-price process the optimality of the estimators is not affected. From an econometric point of view, this effect has the following interpretation: optimal trend estimation in the presence of exogeneous shocks is equivalent to optimal shock-free trend estimation with increased volatility. The quantity $\sqrt{\lambda}\,\tau$ plays the role of a 'virtual' volatility caused by J.

(iv) Vector-valued trend coefficients. Since the concept of kernel reproducing Hilbert spaces also works for vector-valued parameters, cf. Sacks and Ylvisaker (1968, 1970) as well as Su and Cambanis (1993), $\alpha \cdot f(t)$ may be replaced by $\sum_{i=1}^{m} \alpha_i \cdot f_i(t)$ where $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ denotes an m-dimensional trend coefficient and $(f_1(t), f_2(t), \ldots, f_m(t))$ is a prescribed m-dimensional trend function.

(v) Time-dependent stochastic volatility. Relations (8)–(10) have been applied in the special case $N = \sigma \cdot F$, see correspondence (11). However, N is only required to be an arbitrary mean square continuous process with vanishing mean values. Therefore, $\sigma = \sigma(t)$ is allowed to be a deterministic function of t because of $\mathsf{E}(F(t)) = 0$; in general, the volatility may be a stochastic process provided that $\mathsf{E}(\sigma(t) \cdot F(t)) = 0$ holds for all $t \in [0, T]$.

(vi) Comparison with other estimators. The concept of kernel reproducing Hilbert spaces has turned out to be a device for optimal trend estimation in geometric asset price models. (As mentioned at the beginning of the third Section, the methodology is well-known, cf. relation (11), while the extensions discussed in 5 (ii) and 5 (iii) are new.) One may wish to

compare the resulting performance with the performance of alternatives such as estimators obtained by the method of estimating functions. Suppose, for instance, that α^* satisfies

$$(38) \qquad \sum_{k=1}^{n} \psi(Z_k, \alpha^*) = 0,$$

where $Z_1, Z_2, \ldots, Z_n$ are random variables depending upon the log-price process and ψ is an appropriate function which may depend upon the (nuisance) parameter σ^2. In connection with regression model (7) the literature is rather silent on existence and properties of α^*. For theorems on α^* would require assumptions on finite-dimensional distributions as well as certain regularity conditions, and verification of asymptotic properties of α^* such as consistency and asymptotic normality hinges on random variables $Z_1, \ldots, Z_n$ being independent and identically distributed (see, e.g., chapter 7 in Serfling (2002)). However, apart from special cases 'reasonable' observations Z_k from the log-price process are neither independent nor identically distributed. Note that the concept of kernel reproducing Hilbert spaces is distribution-free and depends only upon covariances. The results derived in the third Section also show that α^* is generally not best linear unbiased because the optimal estimator typically needs a whole trajectory (except for situations such as 3.1 (i)).

Nevertheless, optimality of α^* with respect to a fixed set of sampling points can be verified in case of Wiener processes. Consider assumptions and notation of the fourth Section. If α^* is chosen to be the maximum likelihood estimator of α with respect to the increments $Z_k = p_k - p_{k-1}$, then one finds that α^* already coincides with $\hat{\alpha}_\Delta$, cf. (30).

6.   Conclusions

(i) Suppose the asset price $P(t) = P_0 \exp(\alpha f(t) + \sigma F(t))$ is described by a generalized geometric process with trend function f and random fluctuation F. The problem of optimal estimation of the trend coefficient α is treated by applying the concept of kernel reproducing Hilbert spaces on the log-price process $p(t) = \log(P(t)/P_0)$. The kernel reproducing Hilbert space $H(R)$ associated with the covariance kernel R of σF is a set of real-valued functions

and has the following properties: if the trend function f is *not* contained in $H(R)$ then α can be estimated consistently; on the other hand, if f lies in $H(R)$ then consistent linear unbiased estimation of α is impossible but explicit representations for the best linear unbiased estimator and the minimum variance are available provided a mild additional condition is satisfied. From a heuristical point of view, $H(R)$ consists of all functions being similar to the structure of σF. Consequently, if the trend function f of a prescribed asset price is a member of $H(R)$ then a complete distinction between f and the random fluctuations of the market is impossible, and there is a residual error which cannot be reduced. (Since kernel reproducing Hilbert spaces are used in telecommunications this econometric interpretation has a physical analogue. Consider the problem of reconstructing a deterministic signal being contaminated by correlated random noise. If signal and noise coincide to a certain extent then even the recording of a whole sample path cannot completely recover the original signal.)

(ii) Optimal trend estimation in geometric asset price models essentially comprises the following steps, cf. correspondence (11): check the relation $f \in H(R)$ and verify a representation for f in the form of (8); then the optimal estimator and the minimum variance are given by (9) and (10), respectively. If necessary, perform an appropriate discretization or make use of the extensions given in Section 5. These extensions include generalizations of (8)–(10), handling of exogenous shocks leading to jumps, vector-valued trend coefficients and time-dependent volatilities.

(iii) Trend estimation is intimately connected with topics such as forecasting and efficient portfolio selection. In practice, one often finds that experienced stockbrokers and other authorities on these fields have no significant advantage over naive speculators, and sophisticated theories would not protect against misjudgements and gross mistakes. The results discussed in Section 3 yield a partial explanation: for standard random functions such as Wiener processes or Ornstein-Uhlenbeck processes, $H(R)$ turns out to be a 'large' space containing the majority of reasonable trend functions. This implies that consistent linear unbiased estimation of the trend coefficient can be achieved only in exceptional cases. Otherwise, it is possible to minimize the variance by applying the best estimator but one has to cope with a positive residual error whose dependence upon σ^2 and T is specified for several trend functions by (14), (17), (20) and (23), (25), (27), respectively.

The residual error may be viewed as a measure of the limiting precision which cannot be improved even by a perfect interpretation of the observed sample path.

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