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ESTIMATION IN UNIVERSAL MODELS WITH RESTRICTIONS *

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Abstract

In modelling a measurement experiment some singularities can occur even if the experiment is quite standard and simple. Such an experiment is described in the paper as a motivation example. It is presented in the papar how to solve these situations under special restrictions on model parameters. The estimability of model parameters is studied and unbiased estimators are given in explicit forms.

Keywords: universal linear model with restrictions, unbiased estimator.

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Motivation

Let a railway arc be under reconstruction. For the reconstruction it is necessary to know the radius of the arc. Since usually the center of the arc is unknown, the radius cannot be directly observed. An experiment for the determination of the radius can be done, e.g., in the following way. Firstly, points X_i, $i = 1, \ldots, 4$, are chosen elsewhere on the arc. Then other points Z_1, Z_2, Z_3 are chosen around the arc such that all distances Z_iX_j and Z_iZ_k, $i, k = 1, 2, 3$, $i \neq k$, $j = 1, \ldots, 4$, can be observed. Finally, points X_i, Z_j are put into proper coordinates system (the map), see Figure 1. For the sake

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of simplicity, let each distance be measured just once with the accuracy σ. The problem is to determine the radius and coordinates of the center of the arc subject to coordinates of points Z_1, Z_2, Z_3, if it is possible.

1. Introduction

Let a linear regression model be under consideration. There are situations when the mean value parameter must satisfy some linear restrictions. Here two typical situations can occur. Either the restrictions involve components of the mean value parameter only (the type I), or they involve other unknown parameters (the type II).

Generally, no assumptions on the rank of design and covariance matrices in a linear regression model or of matrices in restrictions are given. In such cases, some linear functions of the mean value parameter can be unbiasedly estimated only. Which functions are unbiasedly estimable and how they can be estimated in the universal model without restrictions or with restrictions of the type I has been studied, e.g., in [1, 2, 4].

The aim of this paper is to find the class of all unbiasedly estimable functions and to determine explicit expressions of the best linear unbiased estimators of these functions in the universal linear model with restrictions of the type II.

2. Notations and auxiliary statements

Let $\mathbf{A}$ be an $m \times n$ matrix. Let $\mathcal{M}(\mathbf{A}) = \{\mathbf{A}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^n\} \subset \mathbb{R}^m$ and $Ker(\mathbf{A}) = \{\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^n, \mathbf{A}\boldsymbol{u} = \mathbf{0}\} \subset \mathbb{R}^n$ denote the column space and the null space of the matrix $\mathbf{A}$, respectively. Let $\mathbf{W}$ be an $m \times m$ symmetric positive semidefinite matrix such that $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{W})$. Then $\mathbf{P}_A^W = \mathbf{A}(\mathbf{A}'\mathbf{W}\mathbf{A})^-\mathbf{A}'\mathbf{W}$ denotes a projector on $\mathcal{M}(\mathbf{A})$ in the $\mathbf{W}$-seminorm. The symbol $\mathbf{M}_A^W$ means $\mathbf{I}-\mathbf{P}_A^W$. If $\mathbf{W} = \mathbf{I}$ (identity matrix), symbols $\mathbf{P}_A$ and $\mathbf{M}_A$ are used. The $\mathbf{W}$-seminorm of $\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{R}^m$, is given by $\|\boldsymbol{x}\|_W = \sqrt{\boldsymbol{x}'\mathbf{W}\boldsymbol{x}}$. Symbols $\mathbf{A}^-$ and $\mathbf{A}^+$ mean the g-inverse and the Moore-Penrose inverse of the matrix $\mathbf{A}$, respectively.

Let $\mathbf{N}$ be an $n \times n$ symmetric positive semidefinite matrix. The symbol $\mathbf{A}^-_{m(N)}$ denotes the minimum $\mathbf{N}$-seminorm g-inverse of the matrix $\mathbf{A}$, i.e., the matrix $\mathbf{A}^-_{m(N)}$ satisfies equations

$$\mathbf{A}\mathbf{A}^-_{m(N)}\mathbf{A} = \mathbf{A}, \quad \mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A} = \mathbf{A}'\left(\mathbf{A}^-_{m(N)}\right)'\mathbf{N}. \tag{1}$$

One of representations of the matrix $\mathbf{A}^-_{m(N)}$ is

$$\mathbf{A}^-_{m(N)} = \begin{cases} \mathbf{N}^-\mathbf{A}'(\mathbf{A}\mathbf{N}^-\mathbf{A}')^- & \text{if } \mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{N}), \\ (\mathbf{N}+\mathbf{A}'\mathbf{A})^-\mathbf{A}'[\mathbf{A}(\mathbf{N}+\mathbf{A}'\mathbf{A})^-\mathbf{A}']^- & \text{otherwise.} \end{cases}$$

For more detail cf. [4].

3. UNIVERSAL MODEL WITH RESTRICTIONS

Let the vector parameter $\boldsymbol{\beta}$ consist of two parts, i.e., $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$. Let the part $\boldsymbol{\beta}_1$ be indirectly measured only, i.e., we have a linear regression model $\boldsymbol{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}_1, \boldsymbol{\Sigma})$. In addition, let auxiliary information on the vector of regression coefficients $\boldsymbol{\beta}_1$ and another unknown vector $\boldsymbol{\beta}_2$ be given. Hence, in this situation the parametric space is not the whole Euclidean space but its subset only.

The universal linear model with restrictions is considered in the form

$$\boldsymbol{Y} \sim_n \left(\mathbf{X}\boldsymbol{\beta}_1, \boldsymbol{\Sigma}\right), \quad \mathbf{B}\boldsymbol{\beta}_1 + \mathbf{C}\boldsymbol{\beta}_2 + \boldsymbol{b} = \mathbf{0}, \tag{2}$$

where $\boldsymbol{Y}$ is an n-dimensional random vector, $\mathbf{X}\boldsymbol{\beta}_1$ is the mean value of $\boldsymbol{Y}$ and $\boldsymbol{\Sigma}$ its covariance matrix. $\mathbf{X}$, $\mathbf{B}$ and $\mathbf{C}$ are given matrices with the dimension $n \times k_1$, $q \times k_1$ and $q \times k_2$, respectively and $\boldsymbol{\Sigma}$ is a given $n \times n$ symmetric positive semidefinite matrix. Since no assumption on ranks of matrices $\mathbf{B}$ and $\mathbf{C}$ is considered it must be assumed that a given q-dimensional vector $\boldsymbol{b}$ satisfies $\boldsymbol{b} \in \mathcal{M}(\mathbf{B}, \mathbf{C})$.

The model (2) can be written also in the form

$$\begin{gathered} \begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix} \sim_{n+q} \left[\begin{pmatrix} \mathbf{X}, & \mathbf{0} \\ \mathbf{B}, & \mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix} \right], \\ \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \mathcal{V} = \left\{ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} : \boldsymbol{u} \in \mathbb{R}^{k_1},\ \boldsymbol{v} \in \mathbb{R}^{k_2},\ \boldsymbol{b} + \mathbf{B}\boldsymbol{u} + \mathbf{C}\boldsymbol{v} = \mathbf{0} \right\}. \end{gathered} \tag{3}$$

The notation

$$\mathbf{F}\boldsymbol{\beta} = \begin{pmatrix} \mathbf{X}, & \mathbf{0} \\ \mathbf{B}, & \mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}$$

will be used in what follows.

Another form of the model (2) can be obtained as follows. A general solution of the equation $\boldsymbol{b} + \mathbf{B}\boldsymbol{\beta}_1 + \mathbf{C}\boldsymbol{\beta}_2 = \mathbf{0}$ is

$$(4) \qquad \begin{aligned} \boldsymbol{\beta}_1 &= \boldsymbol{\beta}_{1,0} + \mathbf{K}_B\boldsymbol{\gamma}, \\ \boldsymbol{\beta}_2 &= \boldsymbol{\beta}_{2,0} + \mathbf{K}_C\boldsymbol{\gamma}, \quad \boldsymbol{\gamma} \in \mathbb{R}^{k_1+k_2-\mathrm{rank}(B,C)}, \end{aligned}$$

where $(\boldsymbol{\beta}'_{1,0}, \boldsymbol{\beta}'_{2,0})'$ is a partial solution of this equation and $\mathbf{K}_B$ and $\mathbf{K}_C$ are matrices of the type $k_1 \times [k_1 + k_2 - \mathrm{rank}(\mathbf{B}, \mathbf{C})]$ and $k_2 \times [k_1 + k_2 - \mathrm{rank}(\mathbf{B}, \mathbf{C})]$, respectively, with the property $Ker(\mathbf{B}, \mathbf{C}) = \mathcal{M}\begin{pmatrix} \mathbf{K}_B \\ \mathbf{K}_C \end{pmatrix}$. Thus the reparametrized version of the model (2) is

$$(5) \qquad \boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_{1,0} \sim_n (\mathbf{X}\mathbf{K}_B\boldsymbol{\gamma}, \boldsymbol{\Sigma}), \quad \boldsymbol{\gamma} \in \mathbb{R}^{k_1+k_2-\mathrm{rank}(B,C)}$$

and the original parameter $\boldsymbol{\beta}$ is given by (4). Note, that the model (5) is one without restrictions.

The linear manifold $\mathcal{V}$ of admissible values of parameter $\boldsymbol{\beta}$ is given for both, observable and not observable, parts of $\boldsymbol{\beta}$ simultaneously. How to determine linear manifolds for one part of $\boldsymbol{\beta}$ independently of the other one is given in the following lemma.

Lemma 3.1. *Let us denote*

$$\mathcal{V}_1 = \left\{ \boldsymbol{\beta}_1 : \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \mathcal{V} \right\},$$

$$\mathcal{V}_2 = \left\{ \boldsymbol{\beta}_2 : \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \mathcal{V} \right\}.$$

Then

$$\mathcal{V}_1 = \{ \boldsymbol{u} : \mathbf{M}_C\boldsymbol{b} + \mathbf{M}_C\mathbf{B}\boldsymbol{u} = \mathbf{0} \},$$

$$\mathcal{V}_2 = \{ \boldsymbol{v} : \mathbf{M}_B\boldsymbol{b} + \mathbf{M}_B\mathbf{C}\boldsymbol{v} = \mathbf{0} \}.$$

Proof. The inclusion $\mathcal{V}_1 \subset \{\boldsymbol{u} : \mathbf{M}_C \boldsymbol{b} + \mathbf{M}_C \mathbf{B} \boldsymbol{u} = \mathbf{0}\}$ is obvious. Let

$$\boldsymbol{\beta}_1 \in \{\boldsymbol{u} : \mathbf{M}_C \boldsymbol{b} + \mathbf{M}_C \mathbf{B} \boldsymbol{u} = \mathbf{0}\},$$

i.e.,

$$\boldsymbol{b} + \mathbf{B}\boldsymbol{\beta}_1 \in \mathcal{M}(\mathbf{C}) \quad \Leftrightarrow \quad \exists \boldsymbol{\beta}_2 \in \mathbb{R}^{k_2} : \quad \boldsymbol{b} + \mathbf{B}\boldsymbol{\beta}_1 + \mathbf{C}\boldsymbol{\beta}_2 = \mathbf{0}$$

and thus the opposite inclusion in the first statement is proved. The second statement can be proved similarly. ■

4. Unbiasedly estimable functions

Lemma 4.1. *In the universal model* (2) *a linear function*

$$h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \boldsymbol{h}_1' \boldsymbol{\beta}_1 + \boldsymbol{h}_2' \boldsymbol{\beta}_2, \quad (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')' \in \mathcal{V}, \; \boldsymbol{h}_1 \in \mathbb{R}^{k_1}, \; \boldsymbol{h}_2 \in \mathbb{R}^{k_2},$$

is linearly unbiasedly estimable if and only if

$$\boldsymbol{h} = \begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \end{pmatrix} \in \mathcal{M} \begin{pmatrix} \mathbf{X}', & \mathbf{B}' \\ \mathbf{0}, & \mathbf{C}' \end{pmatrix}.$$

Proof. The function $h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is linearly unbiasedly estimable if and only if there exists a statistic $\mathbf{L}_1' \boldsymbol{Y} + \mathbf{L}_2'(-\boldsymbol{b})$, $\mathbf{L}_1 \in \mathbb{R}^n$, $\mathbf{L}_2 \in \mathbb{R}^q$, with properties

$$E\left[\mathbf{L}_1' \boldsymbol{Y} + \mathbf{L}_2'(-\boldsymbol{b})\right] = \boldsymbol{h}_1' \boldsymbol{\beta}_1 + \boldsymbol{h}_2' \boldsymbol{\beta}_2 \quad \forall \boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')' \in \mathcal{V},$$

i.e.,

$$\mathbf{L}_1' \mathbf{X} \boldsymbol{\beta}_1 + \mathbf{L}_2'(-\boldsymbol{b}) = \boldsymbol{h}_1' \boldsymbol{\beta}_1 + \boldsymbol{h}_2' \boldsymbol{\beta}_2 \quad \forall \boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')' \in \mathcal{V},$$

what is equivalent to

$$\exists \boldsymbol{k} \in \mathbb{R}^q : \quad \mathbf{L}_1' \mathbf{X} - \boldsymbol{h}_1' = \boldsymbol{k}' \mathbf{B}, \; -\boldsymbol{h}_2' = \boldsymbol{k}' \mathbf{C}, \; \mathbf{L}_2' = -\boldsymbol{k}'$$

and thus

$$\begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \end{pmatrix} \in \mathcal{M} \begin{pmatrix} \mathbf{X}', & \mathbf{B}' \\ \mathbf{0}, & \mathbf{C}' \end{pmatrix}.$$

■

Corollary 4.2. *In the universal model* (2) *the vector function*

$$\begin{pmatrix} \mathbf{X}, & \mathbf{0} \\ \mathbf{B}, & \mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \mathcal{V}$$

is always linearly unbiasedly estimable.

Lemma 4.3. *In the universal model* (2) *a linear function* $h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \boldsymbol{h}_1'\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_1 \in \mathcal{V}_1$, $\boldsymbol{h}_1 \in \mathbb{R}^{k_1}$, *is linearly unbiasedly estimable if and only if*

$$\boldsymbol{h}_1 \in \mathcal{M}\left(\mathbf{X}', \mathbf{B}'\mathbf{M}_C\right).$$

Proof. With respect to Lemma 4.1 a function $h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \boldsymbol{h}_1'\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_1 \in \mathcal{V}_1$, $\boldsymbol{h}_1 \in \mathbb{R}^{k_1}$, is linearly unbiasedly estimable if and only if

$$\begin{pmatrix} \boldsymbol{h}_1 \\ \mathbf{0} \end{pmatrix} \in \mathcal{M} \begin{pmatrix} \mathbf{X}', & \mathbf{B}' \\ \mathbf{0}, & \mathbf{C}' \end{pmatrix}.$$

It means

$$\exists \boldsymbol{k}_1 \in \mathbb{R}^{k_1}, \exists \boldsymbol{k}_2 \in \mathbb{R}^q : \quad \boldsymbol{h}_1 = \mathbf{X}'\boldsymbol{k}_1 + \mathbf{B}'\boldsymbol{k}_2, \ \mathbf{0} = \mathbf{C}'\boldsymbol{k}_2,$$

what is equivalent to

$$\boldsymbol{k}_2 = \mathbf{M}_C\boldsymbol{u}, \ \boldsymbol{h}_1 = \mathbf{X}'\boldsymbol{k}_1 + \mathbf{B}'\mathbf{M}_C\boldsymbol{u}, \quad \boldsymbol{u} \in \mathbb{R}^q, \text{ arbitrary.}$$

Hence $\boldsymbol{h}_1 \in \mathcal{M}(\mathbf{X}', \mathbf{B}'\mathbf{M}_C)$. ∎

Lemma 4.4. *In the universal model* (2) *a linear function* $h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \boldsymbol{h}_2'\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_2 \in \mathcal{V}_2$, $\boldsymbol{h}_2 \in \mathbb{R}^{k_2}$, *is linearly unbiasedly estimable if and only if*

$$\boldsymbol{h}_2 \in \mathcal{M}\left(\mathbf{C}'\mathbf{M}_{BM_{X'}}\right).$$

Proof. A function $h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \boldsymbol{h}_2'\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_2 \in \mathcal{V}_2$, $\boldsymbol{h}_2 \in \mathbb{R}^{k_2}$, is linearly unbiasedly estimable if and only if

$$\begin{pmatrix} \mathbf{0} \\ \boldsymbol{h}_2 \end{pmatrix} \in \mathcal{M} \begin{pmatrix} \mathbf{X}', & \mathbf{B}' \\ \mathbf{0}, & \mathbf{C}' \end{pmatrix}$$

$$\Leftrightarrow \quad \exists \boldsymbol{k}_1 \in \mathbb{R}^{k_1}, \exists \boldsymbol{k}_2 \in \mathbb{R}^q : \quad \mathbf{0} = \mathbf{X}'\boldsymbol{k}_1 + \mathbf{B}'\boldsymbol{k}_2, \ \boldsymbol{h}_2 = \mathbf{C}'\boldsymbol{k}_2.$$

The first condition $\mathbf{0} = \mathbf{X}'\boldsymbol{k}_1 + \mathbf{B}'\boldsymbol{k}_2$ is equivalent to

$$\begin{pmatrix} \boldsymbol{k}_1 \\ \boldsymbol{k}_2 \end{pmatrix} \in \mathcal{M}\left(\mathbf{M}_{\binom{X}{B}}\right)$$

$$= \mathcal{M}\begin{pmatrix} \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{X}', & -\mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{B}' \\ -\mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{X}', & \mathbf{I} - \mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{B}' \end{pmatrix}.$$

Since

$$\begin{aligned} \mathbf{M}_{BM_{X'}} &= \mathbf{I} - \mathbf{B}\mathbf{M}_{X'}(\mathbf{M}_{X'}\mathbf{B}'\mathbf{B}\mathbf{M}_{X'})^{+}\mathbf{M}_{X'}\mathbf{B}' \\ &= \mathbf{I} - \mathbf{B}\Big[\mathbf{M}_{X'}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})\mathbf{M}_{X'}\Big]^{+}\mathbf{B}' \\ &= \mathbf{I} - \mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{B}' \\ &\quad + \mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{X}'[\mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{X}']^{-}\mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{B}', \end{aligned}$$

using relations

$$\mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{W}),\ \mathbf{W} \text{ p.s.d.} \quad \Rightarrow \quad \mathcal{M}(\mathbf{A}) = \mathcal{M}(\mathbf{A}\mathbf{W}\mathbf{A}')$$

and

$$\mathcal{M}(\mathbf{A}_1, \mathbf{A}_2) = \mathcal{M}(\mathbf{A}_1 + \mathbf{A}_2) \text{ for } \mathbf{A}_1, \mathbf{A}_2 \text{ p.s.d.}$$

we obtain

$$\mathcal{M}\left([-\mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{X}', \mathbf{I} - \mathbf{B}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{B}']\right) = \mathcal{M}\Big(\mathbf{M}_{BM_{X'}}\Big).$$

Analogously

$$\mathcal{M}\left([\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{X}', -\mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{B}'\mathbf{B})^{-}\mathbf{B}']\right) = \mathcal{M}\Big(\mathbf{M}_{XM_{B'}}\Big).$$

Thus

$$\mathbf{X}'\boldsymbol{k}_1 + \mathbf{B}'\boldsymbol{k}_2 = \mathbf{0} \quad \Leftrightarrow \quad \boldsymbol{k}_1 \in \mathcal{M}\Big(\mathbf{M}_{XM_{B'}}\Big),\ \boldsymbol{k}_2 \in \mathcal{M}\Big(\mathbf{M}_{BM_{X'}}\Big).$$

Now, if the second condition $\boldsymbol{h}_2 = \mathbf{C}'\boldsymbol{k}_2$ is also taken into account, we obtain the statement. ■

5. Explicit expression of unbiased estimators

Lemma 5.1. *One version of the minimum* $\begin{pmatrix} \boldsymbol{\Sigma}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix}$*- seminorm g-inverse of the matrix* $\begin{pmatrix} \mathbf{X}', & \mathbf{B}' \\ \mathbf{0}, & \mathbf{C}' \end{pmatrix}$ *is given by the relation*

$$\left[\begin{pmatrix} \mathbf{X}', & \mathbf{B}' \\ \mathbf{0}, & \mathbf{C}' \end{pmatrix}^-_{m\begin{pmatrix} \Sigma, & 0 \\ 0, & 0 \end{pmatrix}} \right]' = \begin{pmatrix} \mathbf{A}_{11}, & \mathbf{A}_{12} \\ \mathbf{A}_{21}, & \mathbf{A}_{22} \end{pmatrix},$$

where

$$\mathbf{A}_{11} = \mathbf{M}_{B'M_C} \left[\left(\mathbf{M}_{B'M_C} \mathbf{X}' \right)^-_{m(\Sigma)} \right]',$$

$$\mathbf{A}_{12} = \mathbf{W}^+ \mathbf{B}' \left(\mathbf{M}_C \mathbf{B} \mathbf{W}^+ \mathbf{B}' \mathbf{M}_C \right)^+,$$

$$\mathbf{A}_{21} = - \left[\left(\mathbf{C}' \right)^-_{m(BW^+B')} \right]' \mathbf{B} \mathbf{W}^+ \mathbf{X}' \left(\boldsymbol{\Sigma} + \mathbf{X} \mathbf{M}_{B'M_C} \mathbf{X}' \right)^+,$$

$$\mathbf{A}_{22} = \left[\left(\mathbf{C}' \right)^-_{m(BW^+B')} \right]',$$

$$\mathbf{W} = \mathbf{X}' \left(\boldsymbol{\Sigma} + \mathbf{X} \mathbf{M}_{B'M_C} \mathbf{X}' \right)^+ \mathbf{X} + \mathbf{B}' \mathbf{M}_C \mathbf{B}.$$

Proof. It is necessary to verify conditions (1), i.e., to prove equalities

$$\text{(i)} \quad \mathbf{X}' = \mathbf{X}' \left(\mathbf{M}_{B'M_C} \mathbf{X}' \right)^-_{m(\Sigma)} \mathbf{M}_{B'M_C} \mathbf{X}' + \mathbf{B}' \left(\mathbf{M}_C \mathbf{B} \mathbf{W}^+ \mathbf{B}' \mathbf{M}_C \right)^+ \mathbf{B} \mathbf{W}^+ \mathbf{X}',$$

$$\text{(ii)} \quad \mathbf{B}' = \mathbf{X}' \left(\mathbf{M}_{B'M_C} \mathbf{X}' \right)^-_{m(\Sigma)} \mathbf{M}_{B'M_C} \mathbf{B}' + \mathbf{B}' \left(\mathbf{M}_C \mathbf{B} \mathbf{W}^+ \mathbf{B}' \mathbf{M}_C \right)^+ \mathbf{B} \mathbf{W}^+ \mathbf{B}' - \mathbf{X}' \left(\boldsymbol{\Sigma} + \mathbf{X} \mathbf{M}_{B'M_C} \mathbf{X}' \right)^+ \mathbf{X} \mathbf{W}^+ \mathbf{B}' \left(\mathbf{C}' \right)^-_{m(BW^+B')} \mathbf{C}' + \mathbf{B}' \left(\mathbf{C}' \right)^-_{m(BW^+B')} \mathbf{C}',$$

(iii) $\mathbf{0} = \mathbf{C}' \left(\mathbf{M}_C \mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+ \mathbf{B}\mathbf{W}^+\mathbf{X}'$,

(iv) $\mathbf{C}' = \mathbf{C}' \left(\mathbf{M}_C \mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+ \mathbf{B}\mathbf{W}^+\mathbf{B}' + \mathbf{C}' \left(\mathbf{C}'\right)^-_{m(BW^+B')} \mathbf{C}'$,

(v) $\mathbf{\Sigma} \left(\mathbf{M}_{B'M_C}\mathbf{X}'\right)^-_{m(\Sigma)} \mathbf{M}_{B'M_C}\mathbf{X}' = \mathbf{X}\mathbf{M}_{B'M_C} \left[\left(\mathbf{M}_{B'M_C}\mathbf{X}'\right)^-_{m(\Sigma)}\right]' \mathbf{\Sigma}$,

(vi) $\mathbf{\Sigma} \left(\mathbf{M}_{B'M_C}\mathbf{X}'\right)^-_{m(\Sigma)} \mathbf{M}_{B'M_C}\mathbf{B}'$

$$-\mathbf{\Sigma} \left(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^+ \mathbf{X}\mathbf{W}^+\mathbf{B}' \left(\mathbf{C}'\right)^-_{m(BW^+B')} \mathbf{C}' = \mathbf{0}.$$

The expression $\mathbf{X}' \left(\mathbf{M}_{B'M_C}\mathbf{X}'\right)^-_{m(\Sigma)} \mathbf{M}_{B'M_C}\mathbf{X}'$ can be rewritten in the form

$$\begin{aligned}
&\mathbf{X}' \left(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^+ \mathbf{X}\mathbf{M}_{B'M_C} \\
&\quad \times \left[\mathbf{M}_{B'M_C}\mathbf{X}' \left(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^+ \mathbf{X}\mathbf{M}_{B'M_C}\right]^+ \mathbf{M}_{B'M_C}\mathbf{X}' \\
&\quad = \mathbf{W} \left[\mathbf{W}^+ - \mathbf{W}^+\mathbf{B}'\mathbf{M}_C \left(\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+ \mathbf{M}_C\mathbf{B}\mathbf{W}^+\right] \mathbf{X}' \\
&\quad = \mathbf{X}' - \mathbf{B}' \left(\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+ \mathbf{B}\mathbf{W}^+\mathbf{X}',
\end{aligned}$$

since

$$\mathbf{B}'\mathbf{M}_C\mathbf{B}\mathbf{M}_{B'M_C} = \mathbf{0}, \quad \mathbf{W}\mathbf{W}^+\mathbf{X}' = \mathbf{X}', \quad \mathbf{W}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C = \mathbf{B}'\mathbf{M}_C$$

and

$$\left(\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+ = \mathbf{M}_C \left(\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+.$$

Hence the equality (i) is proved.

Let $\mathbf{V} = \mathbf{B}\mathbf{W}^{+}\mathbf{B}' + \mathbf{C}\mathbf{C}'$. As far as the equality (ii) is concerned, the right-hand side equals to

$$\begin{aligned}
&\mathbf{W}\left[\mathbf{W}^{+} - \mathbf{W}^{+}\mathbf{B}'\mathbf{M}_{C}\left(\mathbf{M}_{C}\mathbf{B}\mathbf{W}^{+}\mathbf{B}'\mathbf{M}_{C}\right)^{+}\mathbf{M}_{C}\mathbf{B}\mathbf{W}^{+}\right]\mathbf{B}'\\
&\qquad +\mathbf{B}'\left(\mathbf{M}_{C}\mathbf{B}\mathbf{W}^{+}\mathbf{B}'\mathbf{M}_{C}\right)^{+}\mathbf{B}\mathbf{W}^{+}\mathbf{B}'\\
&\qquad -\left(\mathbf{W} - \mathbf{B}'\mathbf{M}_{C}\mathbf{B}\right)\mathbf{W}^{+}\mathbf{B}'\mathbf{V}^{+}\mathbf{C}\left(\mathbf{C}'\mathbf{V}^{+}\mathbf{C}\right)^{+}\mathbf{C}'\\
&\qquad +\mathbf{B}'\mathbf{V}^{+}\mathbf{C}\left(\mathbf{C}'\mathbf{V}^{+}\mathbf{C}\right)^{+}\mathbf{C}'\\
&= \mathbf{W}\mathbf{W}^{+}\mathbf{B}' + \left(\mathbf{I} - \mathbf{W}\mathbf{W}^{+}\right)\mathbf{B}'\mathbf{V}^{+}\mathbf{C}\left(\mathbf{C}'\mathbf{V}^{+}\mathbf{C}\right)^{+}\mathbf{C}'\\
&\qquad +\mathbf{B}'\mathbf{M}_{C}\mathbf{V}\mathbf{V}^{+}\mathbf{C}\left(\mathbf{C}'\mathbf{V}^{+}\mathbf{C}\right)^{+}\mathbf{C}'\\
&= \mathbf{W}\mathbf{W}^{+}\mathbf{B}' + \left(\mathbf{I} - \mathbf{W}\mathbf{W}^{+}\right)\mathbf{B}'\left(\mathbf{M}_{C} + \mathbf{P}_{C}\right)\mathbf{V}^{+}\mathbf{C}\left(\mathbf{C}'\mathbf{V}^{+}\mathbf{C}\right)^{+}\mathbf{C}'\\
&= \mathbf{W}\mathbf{W}^{+}\mathbf{B}'\left(\mathbf{M}_{C} + \mathbf{P}_{C}\right) + \left(\mathbf{I} - \mathbf{W}\mathbf{W}^{+}\right)\mathbf{B}'\mathbf{P}_{C}\\
&= \mathbf{W}\mathbf{W}^{+}\mathbf{B}'\mathbf{M}_{C} + \mathbf{B}'\mathbf{P}_{C} = \mathbf{B}'\left(\mathbf{M}_{C} + \mathbf{P}_{C}\right) = \mathbf{B}',
\end{aligned}$$

since

$$\mathbf{V}\mathbf{V}^{+}\mathbf{C} = \mathbf{C}, \quad \mathbf{P}_{C}\mathbf{V}^{+}\mathbf{C}\left(\mathbf{C}'\mathbf{V}^{+}\mathbf{C}\right)^{+}\mathbf{C}' = \mathbf{P}_{C}.$$

Equalities (iii) and (iv) are obvious since

$$\mathbf{C}'\left(\mathbf{M}_{C}\mathbf{B}\mathbf{W}^{+}\mathbf{B}'\mathbf{M}_{C}\right)^{+} = \mathbf{C}'\mathbf{M}_{C}\left(\mathbf{M}_{C}\mathbf{B}\mathbf{W}^{+}\mathbf{B}'\mathbf{M}_{C}\right)^{+} = \mathbf{0}.$$

The equality (v) follows from the definition of the minimum $\mathbf{\Sigma}$-seminorm g-inverse of the matrix $\mathbf{M}_{B'M_C}\mathbf{X}'$.

Finally, the left-hand side in the equality (vi) equals to

$$\begin{aligned}
&\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\mathbf{X}\\
&\quad\times\left(\left\{\mathbf{W}^{+}-\mathbf{W}^{+}\mathbf{B}'\left[\mathbf{V}^{+}-\mathbf{V}^{+}\mathbf{C}\left(\mathbf{C}'\mathbf{V}^{+}\mathbf{C}\right)^{+}\mathbf{C}'\mathbf{V}^{+}\right]\mathbf{B}\mathbf{W}^{+}\right\}\mathbf{B}'\right.\\
&\quad\left.-\mathbf{W}^{+}\mathbf{B}'\mathbf{V}^{+}\mathbf{C}\left(\mathbf{C}'\mathbf{V}^{+}\mathbf{C}\right)^{+}\mathbf{C}'\right)\\
&=\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\mathbf{X}\\
&\quad\times\Big[\mathbf{W}^{+}\mathbf{B}'-\mathbf{W}^{+}\mathbf{B}'\mathbf{V}^{+}\left(\mathbf{V}-\mathbf{C}\mathbf{C}'\right)\\
&\quad+\mathbf{W}^{+}\mathbf{B}'\mathbf{V}^{+}\mathbf{C}\left(\mathbf{C}'\mathbf{V}^{+}\mathbf{C}\right)^{+}\mathbf{C}'\left\{\mathbf{V}^{+}\left(\mathbf{V}-\mathbf{C}\mathbf{C}'\right)-\mathbf{I}\right\}\Big]=\mathbf{0}.
\end{aligned}$$

Thus the proof is finished. (See also [2].) ■

Theorem 5.2. *In the universal model* (2) *the BLUE (best linear unbiased estimator) of a vector function* $\mathbf{X}\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_1\in\mathcal{V}_1$, *is*

$$\widehat{\mathbf{X}\boldsymbol{\beta}_1}=\mathbf{X}\mathbf{M}_{B'M_C}\left[\left(\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{-}_{m(\Sigma)}\right]'\boldsymbol{Y}$$
$$-\mathbf{X}\mathbf{W}^{+}\mathbf{B}'\left(\mathbf{M}_C\mathbf{B}\mathbf{W}^{+}\mathbf{B}'\mathbf{M}_C\right)^{+}\boldsymbol{b}$$

with the covariance matrix

$$\operatorname{Var}\left(\widehat{\mathbf{X}\boldsymbol{\beta}_1}\right)=\mathbf{X}\left[\mathbf{M}_{B'M_C}\mathbf{X}'\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\mathbf{X}\mathbf{M}_{B'M_C}\right]^{-}\mathbf{X}'$$
$$-\mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'.$$

Proof. Let the universal model be considered in the form (3). Then the BLUE of the function

$$\mathbf{F}\boldsymbol{\beta}=\begin{pmatrix}\mathbf{X}, & \mathbf{0}\\ \mathbf{B}, & \mathbf{C}\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{pmatrix},\quad \boldsymbol{\beta}=\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{pmatrix}\in\mathcal{V},$$

i.e., of the mean value in the model under consideration, is (cf. [1], p. 86).

$$\widehat{\mathbf{F}\boldsymbol{\beta}} = \begin{pmatrix} \mathbf{X}, & \mathbf{0} \\ \mathbf{B}, & \mathbf{C} \end{pmatrix} \left[\begin{pmatrix} \mathbf{X}', & \mathbf{B}' \\ \mathbf{0}, & \mathbf{C}' \end{pmatrix}^{-}_{m\begin{pmatrix} \Sigma, & 0 \\ 0, & 0 \end{pmatrix}} \right]' \begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix}.$$

Hence the expression for $\widehat{\mathbf{X}\boldsymbol{\beta}_1}$ follows from Lemma 5.1 and the relation $\widehat{\mathbf{X}\boldsymbol{\beta}_1} = (\mathbf{I}, \mathbf{0})\widehat{\mathbf{F}\boldsymbol{\beta}}$. The covariance matrix can be derived straightforwardly by using the following equalities

$$\begin{aligned} \left(\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{-}_{m(\Sigma)} &= \left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\mathbf{X}\mathbf{M}_{B'M_C} \\ &\quad \times \left[\mathbf{M}_{B'M_C}\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\mathbf{X}\mathbf{M}_{B'M_C}\right]^{-}, \end{aligned}$$

$$\begin{aligned} &\mathbf{X}\mathbf{M}_{B'M_C}\left[\mathbf{M}_{B'M_C}\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\mathbf{X}\mathbf{M}_{B'M_C}\right]^{-} \\ &\quad \times \mathbf{M}_{B'M_C}\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\mathbf{X}\mathbf{M}_{B'M_C} = \mathbf{X}\mathbf{M}_{B'M_C}, \end{aligned}$$

$$\begin{aligned} &\mathbf{M}_{B'M_C}\left[\mathbf{M}_{B'M_C}\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\mathbf{X}\mathbf{M}_{B'M_C}\right]^{-} \\ &\quad = \left[\mathbf{M}_{B'M_C}\mathbf{X}'\left(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\mathbf{X}\mathbf{M}_{B'M_C}\right]^{-}. \end{aligned}$$

■

Theorem 5.3. *In the universal model* (2) *the class of BLUEs of all unbiasedly estimable linear functions of the parameter* $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_1 \in \mathcal{V}_1$, *is characterized by the statistic*

$$\begin{pmatrix} \mathbf{X}\mathbf{M}_{B'M_C}\left[\left(\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{-}_{m(\Sigma)}\right]'\boldsymbol{Y} - \mathbf{X}\mathbf{W}^{+}\mathbf{B}'\left(\mathbf{M}_C\mathbf{B}\mathbf{W}^{+}\mathbf{B}'\mathbf{M}_C\right)^{+}\boldsymbol{b} \\ -\mathbf{M}_C\boldsymbol{b} \end{pmatrix}, \tag{6}$$

i.e., any linear transformation of the vector (6) *is the BLUE of the mean value of this linear transformation.*

Proof. With respect to Lemma 4.3 a function $h(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \boldsymbol{h}_1'\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_1 \in \mathcal{V}_1$, is unbiasedly estimable if and only if $\boldsymbol{h}_1' = \boldsymbol{u}'\mathbf{X}+\boldsymbol{v}'\mathbf{M}_C\mathbf{B}$. Using Lemma 5.1, the BLUE of $\boldsymbol{h}_1'\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_1 \in \mathcal{V}_1$, is

$$\widehat{\boldsymbol{h}_1'\boldsymbol{\beta}_1} = \left(\boldsymbol{u}', \boldsymbol{v}'\mathbf{M}_C\right)\widehat{\mathbf{F}\boldsymbol{\beta}} = \left(\boldsymbol{u}'\mathbf{X} + \boldsymbol{v}'\mathbf{M}_C\mathbf{B}, \mathbf{0}\right)\begin{pmatrix} \mathbf{A}_{11}, & \mathbf{A}_{12} \\ \mathbf{A}_{21}, & \mathbf{A}_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix}$$

$$= \left(\boldsymbol{u}', \boldsymbol{v}'\right)$$

$$\times \begin{pmatrix} \mathbf{X}\mathbf{M}_{B'M_C}\left[(\mathbf{M}_{B'M_C}\mathbf{X}')^-_{m(\Sigma)}\right]'\boldsymbol{Y} - \mathbf{X}\mathbf{W}^+\mathbf{B}'\left(\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+\boldsymbol{b} \\ \mathbf{M}_C\mathbf{B}\left[(\mathbf{M}_{B'M_C}\mathbf{X}')^-_{m(\Sigma)}\right]'\boldsymbol{Y} - \mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\left(\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+\boldsymbol{b} \end{pmatrix}.$$

Since

$$\mathbf{M}_C\mathbf{B}\left[\mathbf{M}_{B'M_C}\mathbf{X}'\left(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^+\mathbf{X}\mathbf{M}_{B'M_C}\right]^-$$

$$= \mathbf{M}_C\mathbf{B}\mathbf{M}_{B'M_C}\left[\mathbf{M}_{B'M_C}\mathbf{X}'\left(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^+\mathbf{X}\mathbf{M}_{B'M_C}\right]^- = \mathbf{0},$$

it holds that

$$\mathbf{M}_C\mathbf{B}\left[(\mathbf{M}_{B'M_C}\mathbf{X}')^-_{m(\Sigma)}\right]'\boldsymbol{Y} = \mathbf{0}.$$

Further, from the relation

$$\mathbf{B}\boldsymbol{\beta}_1 + \mathbf{C}\boldsymbol{\beta}_2 + \boldsymbol{b} = \mathbf{0} \quad \Rightarrow \quad \mathbf{M}_C\boldsymbol{b} = -\mathbf{M}_C\mathbf{B}\boldsymbol{\beta}_1$$

it follows that

$$\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\left(\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+\boldsymbol{b}$$

$$= \mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\left(\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+\mathbf{M}_C\boldsymbol{b}$$

$$= -\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\left(\mathbf{M}_C\mathbf{B}\mathbf{W}^+\mathbf{B}'\mathbf{M}_C\right)^+\mathbf{M}_C\mathbf{B}\boldsymbol{\beta}_1$$

$$= -\mathbf{M}_C\mathbf{B}\boldsymbol{\beta}_1 = \mathbf{M}_C\boldsymbol{b}$$

and the proof is finished. ∎

Theorem 5.4. *In the universal model (2) the BLUE of the unbiasedly estimable function $\boldsymbol{h}_2'\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_2 \in \mathcal{V}_2$, $\boldsymbol{h}_2 \in \mathcal{M}(\mathbf{C}'\mathbf{M}_{BM_{X'}})$, is*

$$\widehat{\boldsymbol{h}_2'\boldsymbol{\beta}_2} = -\boldsymbol{h}_2'\left[(\mathbf{C}')^-_{m(BW^+B')}\right]'\left[\mathbf{B}\mathbf{W}^+\mathbf{X}'\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^+\boldsymbol{Y}+\boldsymbol{b}\right]$$

with the variance

$$\operatorname{Var}\left(\widehat{\boldsymbol{h}_2'\boldsymbol{\beta}_2}\right) = \boldsymbol{h}_2'\left[(\mathbf{C}')^-_{m(BW^+B')}\right]'\mathbf{B}\mathbf{W}^+\mathbf{X}'\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^+\boldsymbol{\Sigma}$$
$$\times\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^+\mathbf{X}\mathbf{W}^+\mathbf{B}'\left(\mathbf{C}'\right)^-_{m(BW^+B')}\boldsymbol{h}_2.$$

Proof. Let $\boldsymbol{h}_2 \in \mathcal{M}(\mathbf{C}'\mathbf{M}_{BM_{X'}})$, i.e., $\exists \boldsymbol{v} \in \mathbb{R}^q : \boldsymbol{h}_2' = \boldsymbol{v}'\mathbf{M}_{BM_{X'}}\mathbf{C}$. Using Lemma 5.1, the BLUE of $\boldsymbol{h}_2'\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_2 \in \mathcal{V}_2$, is

$$\begin{aligned}
\widehat{\boldsymbol{h}_2'\boldsymbol{\beta}_2} &= \left(\boldsymbol{u}', \boldsymbol{v}'\mathbf{M}_{BM_{X'}}\right)\widehat{\mathbf{F}\boldsymbol{\beta}} \\
&= \left(\boldsymbol{u}'\mathbf{X}+\boldsymbol{v}'\mathbf{M}_{BM_{X'}}\mathbf{B}, \boldsymbol{v}'\mathbf{M}_{BM_{X'}}\right)\begin{pmatrix}\mathbf{A}_{11}, & \mathbf{A}_{12}\\ \mathbf{A}_{21}, & \mathbf{A}_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{Y}\\ -\boldsymbol{b}\end{pmatrix} \\
&= \boldsymbol{h}_2'\mathbf{A}_{21}\boldsymbol{Y} - \boldsymbol{h}_2'\mathbf{A}_{22}\boldsymbol{b} \\
&= -\boldsymbol{h}_2'\left[(\mathbf{C}')^-_{m(BW^+B')}\right]'\left[\mathbf{B}\mathbf{W}^+\mathbf{X}'\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^+\boldsymbol{Y}+\boldsymbol{b}\right],
\end{aligned}$$

since vectors $\boldsymbol{u} \in \mathbb{R}^{k_1}$ and $\boldsymbol{v} \in \mathbb{R}^q$ are chosen such that

$$\boldsymbol{u}'\mathbf{X}+\boldsymbol{v}'\mathbf{M}_{BM_{X'}}\mathbf{B} = \mathbf{0}'$$

(cf. proof of Lemma 4.4).

The rest of the proof is obvious. ■

Theorem 5.5. *In the universal model* (2) *the class of BLUEs of all unbiasedly estimable linear functions of the parameter* $\boldsymbol{\beta}_2$*,* $\boldsymbol{\beta}_2 \in \mathcal{V}_2$*, is characterized by the statistic*

$$(7) \quad -\mathbf{M}_{BM_{X'}}\mathbf{C}\left[(\mathbf{C}')^{-}_{m(BW^{+}B')}\right]'\left[\mathbf{B}\mathbf{W}^{+}\mathbf{X}'\left(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{B'M_C}\mathbf{X}'\right)^{+}\boldsymbol{Y}+\boldsymbol{b}\right],$$

i.e., any linear transformation of the vector (7) *is the BLUE of its mean value.*

Proof. It is an obvious consequence of Theorem 5.4. ■

6. Numerical demonstration

Let the following notation be used in the motivation example.

- $\boldsymbol{y} = (y_1, \ldots, y_{15})'$ ... a vector of observed distances Z_iX_j, Z_iZ_k, $i, k = 1, 2, 3,\ i \neq k,\ j = 1, \ldots, 4$,
- $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_8)'$ ... a vector of unknown coordinates of points $X_i = [\delta_{2i-1}, \delta_{2i}]',\ i = 1, \ldots, 4$,
- $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_6)'$ ... a vector of unknown coordinates of points $Z_i = [\gamma_{2i-1}, \gamma_{2i}]',\ i = 1, 2, 3$,
- $\boldsymbol{s} = (s_1, s_2)'$ ... a vector of unknown coordinates of the center $S = [s_1, s_2]'$,
- r ... the unknown radius

The mentioned process of measurement can be modelled by

$$\boldsymbol{Y} \sim_{15} \left(\boldsymbol{f}(\boldsymbol{\delta}, \boldsymbol{\gamma}), \sigma^2\mathbf{I}\right),$$

where, e.g.,

$$f_1(\boldsymbol{\delta}, \boldsymbol{\gamma}) = \sqrt{(\gamma_1 - \delta_1)^2 + (\gamma_2 - \delta_2)^2},$$

with restrictions (points $X_i, i = 1, \dots, 4$, must lie on the arc)

$$\boldsymbol{g}(\boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{s}, r) = \mathbf{0},$$

where

$$g_i(\boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{s}, r) = (\delta_{2i-1} - s_1)^2 + (\delta_{2i} - s_2)^2 - r^2, \quad i = 1, \dots, 4.$$

The linear version of the model can be written in the form

$$\boldsymbol{Y} - \boldsymbol{f}\left(\boldsymbol{\delta}^{(0)}, \boldsymbol{\gamma}^{(0)}\right) \sim_{15} \left(\mathbf{X}\Delta\boldsymbol{\beta}_1, \sigma^2\mathbf{I}\right), \quad \mathbf{B}\Delta\boldsymbol{\beta}_1 + \mathbf{C}\Delta\boldsymbol{\beta}_2 = \mathbf{0},$$

$$\Delta\boldsymbol{\beta}_1 = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_1^{(0)}, \quad \Delta\boldsymbol{\beta}_2 = \boldsymbol{\beta}_2 - \boldsymbol{\beta}_2^{(0)},$$

$$\boldsymbol{\beta}_1 = (\boldsymbol{\delta}', \boldsymbol{\gamma}')', \quad \boldsymbol{\beta}_2 = (s_1, s_2, r)',$$

where $\boldsymbol{\beta}_1^{(0)}$ and $\boldsymbol{\beta}_2^{(0)}$ are approximate values of vectors $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, respectively, and

$$\mathbf{X} = \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial \boldsymbol{u}'}\Big|_{u=\beta_1^{(0)}}, \quad \mathbf{B} = \frac{\partial \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{\beta}_2^{(0)})}{\partial \boldsymbol{u}'}\Big|_{u=\beta_1^{(0)}}, \quad \mathbf{C} = \frac{\partial \boldsymbol{g}(\boldsymbol{\beta}_1^{(0)}, \boldsymbol{v})}{\partial \boldsymbol{v}'}\Big|_{v=\beta_2^{(0)}}.$$

It should be mentioned that approximate values $\boldsymbol{\beta}_1^{(0)}$ and $\boldsymbol{\beta}_2^{(0)}$ must be chosen with sufficient accuracy in such a way that restrictions $\boldsymbol{g}(\boldsymbol{\delta}, \boldsymbol{g}, \boldsymbol{s}, r) = \mathbf{0}$ are satisfied (in more detail cf. [3]). In practice, it is suitable to choose approximate coordinates of all points Z and three from four points X in the first step, e.g., from a map with a sufficiently large scale. Then one can determine the approximate value of the radius r and the center S of the circle on which points X must lie. Finally, for the obtained circle coordinates of the last point X can be chosen. The graphical record of measurement results enables us to do it easily.

For the sake of simplicity, approximate values have been chosen as true values (in metres):

$$Z_1^{(0)} = \begin{pmatrix} 100 \\ 400 \end{pmatrix},\ Z_2^{(0)} = \begin{pmatrix} 800 \\ 650 \end{pmatrix},\ Z_3^{(0)} = \begin{pmatrix} 250 \\ 1100 \end{pmatrix},$$

$$X_1^{(0)} = \begin{pmatrix} 621.55983 \\ 896.57523 \end{pmatrix},\ X_2^{(0)} = \begin{pmatrix} 392.18187 \\ 845.72336 \end{pmatrix},$$

$$X_3^{(0)} = \begin{pmatrix} 250.00000 \\ 779.42286 \end{pmatrix},\ X_4^{(0)} = \begin{pmatrix} 121.49115 \\ 689.44000 \end{pmatrix},$$

$$S^{(0)} = \begin{pmatrix} 700 \\ 0 \end{pmatrix},\ r^{(0)} = 900.$$

In this case, terms in the linearized model are as follows:

$$\boldsymbol{f}\left(\boldsymbol{\delta}^{(0)}, \boldsymbol{\gamma}^{(0)}\right) = [\ 720.14694,\ 532.95362,\ 407.99719,\ 290.23677,\ 304.36859,\ 452.35303,\ 565.02237,\ 679.65416,\ 423.60164,\ 291.32850,\ 320.57714,\ 430.20232,\ 743.30344,\ 715.89105,\ 710.63352\ ]',$$

the design matrix $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$, where

$$\mathbf{X}_1 = \begin{pmatrix}
19.43540, & 18.50438, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 12.65635, & 19.30726, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 7.42613, & 18.78430, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 1.26149 \\
-10.22805, & 14.13349, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & -19.17466, & 9.20246, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & -23.13822, & 5.44475, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & -26.02625 \\
18.05302, & -9.88382, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 8.33015, & -14.89755, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 0, & -17.90467, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & -6.19579 \\
0, & 0, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 0
\end{pmatrix},$$

$$\mathbf{X}_2 = \begin{pmatrix}
0, & -19.43540, & -18.50438, & 0, & 0, & 0, & 0\\
0, & -12.65635, & -19.30726, & 0, & 0, & 0, & 0\\
0, & -7.42613, & -18.78430, & 0, & 0, & 0, & 0\\
16.98957, & -1.26149, & -16.98957, & 0, & 0, & 0, & 0\\
0, & 0, & 0, & 10.22805, & -14.13349, & 0, & 0\\
0, & 0, & 0, & 19.17466, & -9.20246, & 0, & 0\\
0, & 0, & 0, & 23.13822, & -5.44475, & 0, & 0\\
1.51284, & 0, & 0, & 26.02625, & -1.512840, & 0, & 0\\
0, & 0, & 0, & 0, & 0, & -18.05302, & 9.88382\\
0, & 0, & 0, & 0, & 0, & -8.33015, & 14.89755\\
0, & 0, & 0, & 0, & 0, & 0, & 17.90467\\
-19.79431, & 0, & 0, & 0, & 0, & 6.19579, & 19.79431\\
0, & -25.67527, & -9.16974, & 25.67527, & 9.16974, & 0, & 0\\
0, & -5.60619, & -26.16222, & 0, & 0, & 5.60619, & 26.16222\\
0, & 0, & 0, & 20.63193, & -16.88067, & -20.63193, & 16.88067
\end{pmatrix}.$$

Matrices in restrictions are $\mathbf{C}$ and

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11}, & \mathbf{0}_{2\times 2}, & \mathbf{0}_{2\times 6}\\ \mathbf{0}_{2\times 2}, & \mathbf{B}_{22}, & \mathbf{0}_{2\times 6} \end{pmatrix},$$

where

$$\mathbf{B}_{11} = \begin{pmatrix} -156.88034, & 1793.15046, & 0, & 0\\ 0, & 0, & -615.63626, & 1691.44672 \end{pmatrix},$$

$$\mathbf{B}_{22} = \begin{pmatrix} -900, & 1558.84573, & 0, & 0\\ 0, & 0, & -1157.01770, & 1378.88000 \end{pmatrix}$$

and

$$\mathbf{C} = \begin{pmatrix} 156.88034, & -1793.15046, & -1800\\ 615.63626, & -1691.44672, & -1800\\ 900.00000, & -1558.84573, & -1800\\ 1157.01770, & -1378.88000, & -1800 \end{pmatrix}.$$

At first, the problem of estimability will be discussed. The matrix $\mathbf{B}$ is of full row rank and $\mathbf{C}$ is of full column rank, however the design matrix $\mathbf{X}$ is singular, where $\mathrm{rank}(\mathbf{X}) = 11$. According to Lemma 4.4, the radius r and coordinates of the center S are estimable since the matrix $\mathbf{C}'\mathbf{M}_{BM_{X'}}$ is of full row rank, i.e., $\mathcal{M}\left(\mathbf{C}'\mathbf{M}_{BM_{X'}}\right) = \mathbb{R}^3$. According to Lemma 4.3, individual coordinates of points X and Z are not estimable, since the matrix $(\mathbf{X}', \mathbf{B}'\mathbf{M}_C)$ is singular $(\mathrm{rank}\,(\mathbf{X}', \mathbf{B}'\mathbf{M}_C) = 13)$. Only these functions $\boldsymbol{h}'\boldsymbol{\beta}_1$ of coordinates of points X and Z are estimable for which it holds that $\boldsymbol{h} \in \mathcal{M}\,(\mathbf{X}', \mathbf{B}'\mathbf{M}_C)$.

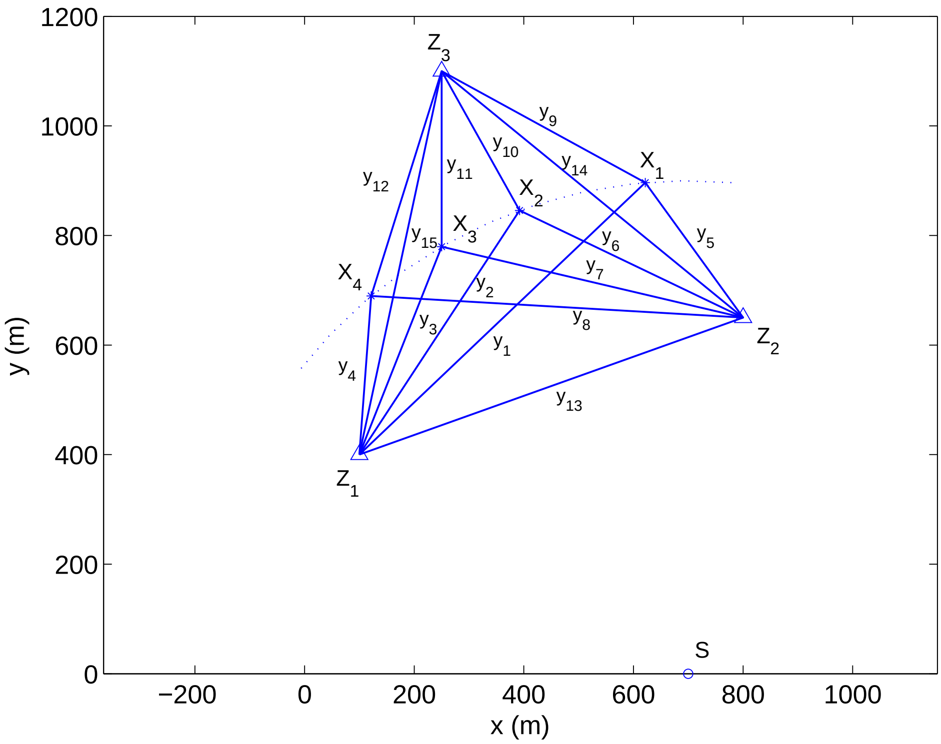

Figure 1. The plan of the experiment

Let the accuracy of measurement be $\sigma = 0.01$ m. Let the simulated data of observed distances (in metres) be

$$\boldsymbol{y} = [\,720.14261,\ 532.93696,\ 407.99845,\ 290.23965,\ 304.35712,\ 452.36493,\ 565.03426,\ 679.65378,\ 423.60491,\ 291.33025,\ 320.57527,\ 430.20958,\ 743.29755,\ 715.91289,\ 710.63216\,]'.$$

The BLUE of the parameter $\Delta\boldsymbol{\beta}_2 = (\Delta s_1, \Delta s_2, \Delta r)'$ is (cf. Theorem 5.4)

$$\widehat{\Delta r} = -0.00772, \quad \begin{pmatrix} \widehat{\Delta s_1} \\ \widehat{\Delta s_2} \end{pmatrix} = \begin{pmatrix} -0.00399 \\ 0.00674 \end{pmatrix},$$

i.e., the estimated radius r and coordinates of the center S are

$$\hat{r} = 899.99228, \quad \hat{S} = \begin{pmatrix} \hat{s}_1 \\ \hat{s}_2 \end{pmatrix} = \begin{pmatrix} 699.99601 \\ 0.00674 \end{pmatrix}.$$

The variance of the estimator of the radius is

$$\mathrm{Var}(\hat{r}) = 8.42069 \cdot 10^{-5}$$

and the covariance matrix of the center is

$$\mathrm{Var}\begin{pmatrix} \hat{s}_1 \\ \hat{s}_2 \end{pmatrix} = \begin{pmatrix} 1.24844 \cdot 10^{-5}, & -3.05357 \cdot 10^{-5} \\ -3.05357 \cdot 10^{-5}, & 7.50044 \cdot 10^{-5} \end{pmatrix}.$$

Thus standard errors are

$$\sqrt{\mathrm{Var}(\hat{r})} = 0.00918, \quad \sqrt{\mathrm{Var}(\hat{s}_1)} = 0.00353, \quad \sqrt{\mathrm{Var}(\hat{s}_2)} = 0.00866.$$

This example showed that a problem of engineering practice can lead to a singular regression model. Although the problem is standard, its solution needs a special approach.

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