

## OPTIMUM CHEMICAL BALANCE WEIGHING DESIGNS WITH DIAGONAL VARIANCE - COVARIANCE MATRIX OF ERRORS

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### Abstract

In this paper we study the estimation problem of individual measurements (weights) of objects in a model of chemical balance weighing design with diagonal variance - covariance matrix of errors under the restriction  $k_1 + k_2 < p$ , where  $k_1$  and  $k_2$  represent the number of objects placed on the right and left pans, respectively. We want all variances of estimated measurements to be equal and attaining their lower bound. We give a necessary and sufficient condition under which this lower bound is attained by the variance of each of the estimated measurements. To construct the design matrix  $\mathbf{X}$  of the considered optimum chemical balance weighing design we use the incidence matrices of balanced bipartite weighing designs.

**Keywords:** balanced bipartite weighing design, chemical balance weighing design.

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### 1. INTRODUCTION

The results of  $n$  weighing operations aimed at determining the individual weights of  $p$  objects with a balance corrected for bias will fit into the linear model

$$(1.1) \quad \mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e},$$

where  $\mathbf{y}$  is an  $n \times 1$  random column vector of the observed weights, the design matrix  $\mathbf{X}$  belongs to the class of  $n \times p$  matrices of elements equal to  $-1, 0$  or  $1$  and in which the maximum number of elements equal to  $-1$  and  $1$  is in each column equal to  $m$ , i.e.,  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$ ,  $\mathbf{w}$  is an  $p \times 1$  column vector representing unknown weights of objects and  $\mathbf{e}$  is an  $n \times 1$  random column vector of errors such that  $E(\mathbf{e}) = \mathbf{0}_n$  and  $E(\mathbf{e}\mathbf{e}') = \sigma^2\mathbf{G}$ , where  $\mathbf{0}_n$  is an  $n \times 1$  column vector of zeros,  $\mathbf{G}$  is an  $n \times n$  positive definite diagonal matrix of known elements,  $E(\cdot)$  stands for the expectation of  $(\cdot)$  and  $(\cdot)'$  is used for the transpose of  $(\cdot)$ .

The normal equations for estimating  $\mathbf{w}$  are of the form

$$(1.2) \quad \mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y},$$

where  $\hat{\mathbf{w}}$  is the vector of unknown measurements estimated by the least squares method.

A chemical balance weighing design is said to be singular or nonsingular, depending on whether the matrix  $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$  is singular or nonsingular, respectively. It is obvious that because of the assumption connected with the matrix  $\mathbf{G}$ , the matrix  $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$  is nonsingular if and only if the matrix  $\mathbf{X}'\mathbf{X}$  is such, i.e., the matrix  $\mathbf{X}$  is of full column rank ( $= p$ ).

Now, if  $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$  is nonsingular, the least squares estimator of  $\mathbf{w}$  is given by

$$(1.3) \quad \hat{\mathbf{w}} = (\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$$

and the variance - covariance matrix of  $\hat{\mathbf{w}}$  is of the form

$$(1.4) \quad \text{Var}(\hat{\mathbf{w}}) = \sigma^2(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}.$$

Some problems connected with chemical balance weighing designs have been studied in Raghavarao (1971) and Banerjee (1975). In the case  $\mathbf{G} = \mathbf{I}_n$ , Hotelling (1944) has shown that for a chemical balance weighing design the minimum attainable variance for each of the estimated weights is  $\sigma^2/n$ .

He has shown that each of the variance of the estimated weights attains the lower bound if and only if  $\mathbf{X}'\mathbf{X} = n\mathbf{I}_p$ . The design satisfying this condition is said to be an optimum chemical balance weighing design. This condition implies that elements of the matrix  $\mathbf{X}$  of the optimum chemical balance weighing design are equal only to  $-1$  or  $1$ . In this case, several methods of construction of optimum chemical balance weighing designs are available in the literature.

In the situation when not all objects are included in each weighing operation, Ceranka and Graczyk (2001) have shown that for a chemical balance weighing design the minimum attainable variance for each of the estimated weights is  $\sigma^2/m$ , where  $m$  is the maximum number of elements equal to  $-1$  and  $1$  in each column of the design matrix  $\mathbf{X}$ . They have also proved that each of the variance of the estimated weights attains the lower bound if and only if  $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ . This design is also said to be an optimum chemical balance weighing design. Several methods of construction of the design matrix  $\mathbf{X}$  with elements  $-1$ ,  $0$  and  $1$  are given in Swamy (1982), Ceranka, Katulska and Mizera (1998), Ceranka and Katulska (1999), Ceranka and Graczyk (2001).

Katulska (1989) has shown that the minimum attainable variance for each of the estimated weights for a chemical balance weighing design with a positive definite diagonal variance - covariance matrix of errors is  $\sigma^2/tr(\mathbf{G}^{-1})$ . In the same paper she has shown that each of the variance of the estimated weights attains the minimum if and only if  $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = tr(\mathbf{G}^{-1})\mathbf{I}_p$ . This design is said to be an optimum chemical balance weighing design. This condition implies that the elements of the matrix  $\mathbf{X}$  of the optimum chemical balance weighing design are equal to  $-1$  and  $1$ , only.

In the present paper, we investigate the necessary and sufficient condition under which the minimum variance is attained for estimated weights in a chemical balance weighing design with a positive definite diagonal variance - covariance matrix of errors if the design matrix  $\mathbf{X}$  has the elements  $-1$ ,  $0$  or  $1$ . We give some methods of construction of an optimum chemical balance weighing design under the restriction on the number of objects placed on either of the pans. The methods utilize the incidence matrices of balanced bipartite weighing designs.

## 2. VARIANCE LIMIT OF ESTIMATED WEIGHTS

Let  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  be the matrix of rank  $p$  and let  $\mathbf{c}$  denote any  $p \times 1$  column vector. Then, from Section 1e.1(ii)(b) in Rao (1973), we have.

**Lemma 2.1.** *For a positive definite  $n \times n$  diagonal matrix  $\mathbf{G}$ , an  $n \times p$  matrix  $\mathbf{X}$  of rank  $p$  and a  $p \times 1$  column vector  $\mathbf{c}$ , the inequality*

$$(2.1) \quad \mathbf{c}'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}\mathbf{c} \geq \frac{(\mathbf{c}'\mathbf{c})^2}{\mathbf{c}'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})\mathbf{c}}$$

*holds, with the equality attained if and only if  $\mathbf{c}$  is an eigenvector of  $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ .*

We assume that matrix  $\mathbf{G}$  is given in the form

$$(2.2) \quad \mathbf{G} = \begin{bmatrix} \frac{1}{a_1}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}'_{n_2} & \cdots & \mathbf{0}_{n_1}\mathbf{0}'_{n_t} \\ \mathbf{0}_{n_2}\mathbf{0}'_{n_1} & \frac{1}{a_2}\mathbf{I}_{n_2} & \cdots & \mathbf{0}_{n_2}\mathbf{0}'_{n_t} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{n_t}\mathbf{0}'_{n_1} & \mathbf{0}_{n_t}\mathbf{0}'_{n_2} & \cdots & \frac{1}{a_t}\mathbf{I}_{n_t} \end{bmatrix},$$

where  $\sum_{h=1}^t n_h = n$ ,  $a_h > 0$  and  $\mathbf{I}_{n_h}$  is the  $n_h \times n_h$  identity matrix,  $h = 1, 2, \dots, t$ . In other words, we determine the optimality in the class  $\Phi_{n \times p, m}(-1, 0, 1)$  under the given restrictions for  $a_1, a_2, \dots, a_t$  and  $n_1, n_2, \dots, n_t$ . This structure of the variance-covariance matrix of errors may be useful in the following situation. Suppose that there are  $t$  kinds of chemical balances of different precision,  $a_1, a_2, \dots, a_t$ , one with a usual precision, the other with higher and one with the highest precision. Let  $n_1, n_2, \dots, n_t$  be the numbers of times in which the respective balances are used.

Suppose further that the matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  is partitioned correspondingly to the matrix  $\mathbf{G}$ , i.e.,

$$(2.3) \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_t \end{bmatrix}.$$

Let us denote by  $m_{hj}$  the number of elements  $-1$  and  $1$  in the  $j$ th column of  $\mathbf{X}_h = (x_{uj}^h)$ ,  $u = 1, 2, \dots, n_h$ ,  $j = 1, 2, \dots, p$ ,  $h = 1, 2, \dots, t$ .

**Theorem 2.1.** For any nonsingular chemical balance weighing design with the design matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given in the form (2.3) and with the variance-covariance matrix  $\sigma^2 \mathbf{G}$ , where the matrix  $\mathbf{G}$  is given in the form (2.2), the variance of  $\hat{w}_j$  for any  $j, j = 1, 2, \dots, p$ , cannot be less than  $\sigma^2/q$ , where  $q = \sum_{h=1}^t a_h m_h$  and  $m_h = \max\{m_{h1}, m_{h2}, \dots, m_{hp}\}$ .

**Proof.** Let  $\mathbf{c}_j, j = 1, 2, \dots, p$ , be the vector equal to the  $j$ th column of  $\mathbf{I}_p$ . Then it follows that

$$\hat{w}_j = \mathbf{c}'_j \hat{\mathbf{w}}$$

and

$$\text{Var}(\hat{w}_j) = \sigma^2 \mathbf{c}'_j (\mathbf{X}' \mathbf{G}^{-1} \mathbf{X})^{-1} \mathbf{c}_j.$$

As the matrix  $\mathbf{X}$  is of full column rank, from Lemma 2.1 we have

$$\begin{aligned} \text{Var}(\hat{w}_j) &\geq \sigma^2 \frac{(\mathbf{c}'_j \mathbf{c}_j)^2}{\mathbf{c}'_j (\mathbf{X}' \mathbf{G}^{-1} \mathbf{X}) \mathbf{c}_j} = \frac{\sigma^2}{\sum_{h=1}^t a_h \sum_{u=1}^{n_h} (x_{uj}^h)^2} = \\ (2.4) \quad &= \frac{\sigma^2}{\sum_{h=1}^t a_h m_{hj}} \geq \frac{\sigma^2}{\sum_{h=1}^t a_h m_h} = \frac{\sigma^2}{q}, \end{aligned}$$

because  $x_{uj}^h = -1, 1$  or  $0$  only, and the number of elements equal to  $-1$  and  $1$  in each column of  $\mathbf{X}_h$  is equal to  $m_{hj}$ , where  $u = 1, 2, \dots, n_h, j = 1, 2, \dots, p, h = 1, 2, \dots, t$ . Hence, the Theorem is proved. ■

From  $\text{tr}(\mathbf{G}^{-1}) = \sum_{h=1}^t a_h n_h$ , it follows that if  $m_h = n_h$  for each  $h$ , then

$$\text{Var}(\hat{w}_j) \geq \frac{\sigma^2}{\text{tr}(\mathbf{G}^{-1})}.$$

This Theorem was originally proved by Katulska (1989) for the case  $m_h = n_h, h = 1, 2, \dots, t$ .

**Definition 2.1.** Any nonsingular chemical balance weighing design is said to be an optimum chemical balance weighing design if it estimates each of the weights with minimum variance.

In other words, any nonsingular chemical balance weighing design with the variance-covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where the matrix  $\mathbf{G}$  is given

in the form (2.2), is said to be optimal for the estimated individual measurements (weights) of objects if the variances of their estimators attain the lower bound given by Theorem 2.1, i.e., if

$$\text{Var}(\hat{w}_j) = \frac{\sigma^2}{q}, \quad j = 1, 2, \dots, p, \quad \text{where} \quad q = \sum_{h=1}^t a_h m_h.$$

Now, we consider the necessary and sufficient condition under which the minimum variance for the estimated weights is attained.

**Theorem 2.2.** *For any positive definite diagonal matrix  $\mathbf{G}$  given in the form (2.2) and any  $n \times p$  matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given in the form (2.3), under a nonsingular chemical balance weighing design, each of the variance of the estimated weights attains the minimum if and only if*

$$(2.5) \quad \mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = q\mathbf{I}_p.$$

**Proof.** To prove the necessity part, we observe that the equality in (2.4) holds for any  $j = 1, 2, \dots, p$  if and only if

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{c}_j = \mu_j \mathbf{c}_j, \quad \mu_j > 0$$

and

$$\mathbf{c}'_j \mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{c}_j = \sum_{h=1}^t a_h m_h, \quad j = 1, 2, \dots, p.$$

These conditions imply that  $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \text{diag}\{\mu_1, \mu_2, \dots, \mu_p\}$  and  $\mu_1 = \mu_2 = \dots = \mu_p = \sum_{h=1}^t a_h m_h = q$ . Then we have the condition (2.5). The proof of the sufficiency part is obvious. ■

### 3. BALANCED BIPARTITE WEIGHING DESIGNS

A balanced bipartite weighing design (see Huang (1976) and Swamy (1982)) with parameters  $v, k_1, k_2, \lambda_1$  is an arrangement of  $v$  elements into  $b$  blocks  $B_i = \{B_i^1 : B_i^2\}$  each with  $k = k_1 + k_2$  distinct elements, the number of elements in  $B_i^j$  being  $k_j, j = 1, 2, i = 1, 2, \dots, b$ , such that each element occurs in  $r$  blocks, each pair of distinct elements is linked in exactly  $\lambda_1$  blocks and  $l$ -linked in exactly  $\lambda_2$  blocks. If  $B$  is a block with subsets  $B^1$  and  $B^2$  such

that  $B = \{B^1 : B^2\}$ , where  $B^1 = \{a_1^1, a_2^1, \dots, a_{k_1}^1\}$ ,  $B^2 = \{a_1^2, a_2^2, \dots, a_{k_2}^2\}$ , then two elements in  $B$  are said to be linked or l-linked in  $B$ , if and only if they belong to different subsets or the same subset of  $B$ , respectively.

Let  $\mathbf{N}$  be the incidence matrix of such a design. The parameters are not independent and they are related by the following identities

$$\begin{aligned}
 vr &= bk, \\
 b &= \frac{\lambda_1 v(v-1)}{2k_1 k_2}, \\
 \lambda_2 &= \frac{\lambda_1 [k_1(k_1-1) + k_2(k_2-1)]}{2k_1 k_2}, \\
 r &= \frac{\lambda_1 k(v-1)}{2k_1 k_2},
 \end{aligned}
 \tag{3.1}$$

$$\mathbf{N}\mathbf{N}' = (r - \lambda_1 - \lambda_2)\mathbf{I}_v + (\lambda_1 + \lambda_2)\mathbf{1}_v\mathbf{1}_v'.$$

In the next part of the paper balanced bipartite weighing design with parameters  $v, k_1, k_2, \lambda_1$  will be written as  $v, b, r, k_1, k_2, \lambda_1, \lambda_2$ .

#### 4. CONSTRUCTION OF THE DESIGN MATRIX

Let  $\mathbf{N}_h$  be the incidence matrix of a balanced bipartite weighing design with the parameters  $v, b_h, r_h, k_{1h}, k_{2h}, \lambda_{1h}, \lambda_{2h}, h = 1, 2, \dots, t$ . From  $\mathbf{N}'_h$  we obtain another matrix  $\mathbf{N}^*_h$  by replacing  $k_{1h}$  unities of each row which correspond to the elements belonging to the first subblock  $B^1$ , by  $-1$ , i.e.,

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}^*_1 \\ \mathbf{N}^*_2 \\ \vdots \\ \mathbf{N}^*_t \end{bmatrix}.
 \tag{4.1}$$

Thus, each row of  $\mathbf{N}_h^*$  will contain  $k_{1h}$  elements equal to  $-1$ ,  $k_{2h}$  elements equal to  $1$  and  $v - k_{1h} - k_{2h}$  elements equal to  $0$ . Clearly, such a design implies that in each weighing, exactly  $k_h = k_{1h} + k_{2h}$ ,  $h = 1, 2, \dots, t$  objects are weighed,  $k_{1h}$  of them on the right pan and  $k_{2h}$  on the left. Also, each object is weighed  $m = \sum_{h=1}^t r_h$  times in the  $n = b = \sum_{h=1}^t b_h$  weighing operations.

**Lemma 4.1.** *The chemical balance weighing design with the matrix  $\mathbf{X}$  given in the form (4.1) is nonsingular if and only if*

$$(4.2) \quad k_{1h} \neq k_{2h}$$

for at least one  $h, h = 1, \dots, t$ .

**Proof.** Because the matrix  $\mathbf{G}$  is an  $n \times n$  positive definite diagonal matrix, the matrix  $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$  is nonsingular if and only if the matrix  $\mathbf{X}'\mathbf{X}$  is nonsingular, i.e., the matrix  $\mathbf{X}$  is of full column rank. Thus for the design matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given in the form (4.1) we have

$$(4.3) \quad \begin{aligned} \mathbf{X}'\mathbf{X} &= \left[ \sum_{h=1}^t (r_h - \lambda_{2h} + \lambda_{1h}) \right] \mathbf{I}_v \\ &+ \left[ \sum_{h=1}^t (\lambda_{2h} - \lambda_{1h}) \right] \mathbf{1}_v \mathbf{1}_v' \end{aligned}$$

and

$$(4.4) \quad \begin{aligned} &\det(\mathbf{X}'\mathbf{X}) \\ &= \left[ \sum_{h=1}^t (r_h - \lambda_{2h} + \lambda_{1h}) \right]^{v-1} \sum_{h=1}^t [r_h + (v-1)(\lambda_{2h} - \lambda_{1h})]. \end{aligned}$$



The determinant (4.4) equals 0 if and only if

$$(4.5) \quad \sum_{h=1}^t r_h = \sum_{h=1}^t (\lambda_{2h} - \lambda_{1h})$$

or

$$(4.6) \quad (1 - v) \sum_{h=1}^t (\lambda_{2h} - \lambda_{1h}) = \sum_{h=1}^t r_h.$$

Using (3.1) it can be shown that (4.5) implies

$$\frac{v}{2} \sum_{h=1}^t \frac{\lambda_{1h}(k_{1h} + k_{2h})}{k_{1h}k_{2h}} = \frac{1}{2} \sum_{h=1}^t \frac{\lambda_{1h}(k_{1h} - k_{2h})^2}{k_{1h}k_{2h}},$$

which is not satisfied, because  $v \geq k_{1h} + k_{2h}$ ,  $h = 1, 2, \dots, t$ . Using the relations (3.1) once again we can see that (4.6) implies

$$(k_{1h} - k_{2h})^2 = 0 \quad \text{for each } h = 1, 2, \dots, t.$$

The last expression does not hold if and only if  $k_{1h} \neq k_{2h}$  for at least one  $h, h = 1, 2, \dots, t$ . Thus, the Lemma is proved. ■

**Theorem 4.1.** *The nonsingular chemical balance weighing design with the matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given by (4.1) and with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where  $\mathbf{G}$  is of the form (2.2), is optimal if and only if*

$$(4.7) \quad \sum_{h=1}^t a_h (\lambda_{2h} - \lambda_{1h}) = 0.$$

**Proof.** For the design matrix  $\mathbf{X}$  given by (4.1) and the variance - covariance matrix of errors  $\sigma^2\mathbf{G}$ , where  $\mathbf{G}$  is of the form (2.2), we have

$$\mathbf{X}\mathbf{G}^{-1}\mathbf{X} = \left[ \sum_{h=1}^t a_h(r_h - \lambda_{2h} + \lambda_{1h}) \right] \mathbf{I}_v + \left[ \sum_{h=1}^t a_h(\lambda_{2h} - \lambda_{1h}) \right] \mathbf{1}_v \mathbf{1}'_v.$$

From Theorem 2.2 it follows that a chemical balance weighing design is optimal if and only if the condition (2.5) holds. Hence the Theorem.  $\blacksquare$

If the chemical balance weighing design given by the matrix  $\mathbf{X}$  of the form (4.1) and with the variance-covariance matrix of errors  $\sigma^2\mathbf{G}$ , where the matrix  $\mathbf{G}$  is given by (2.2), is optimal, then

$$\text{Var}(\hat{w}_j) = \frac{\sigma^2}{q}, \quad j = 1, 2, \dots, p.$$

where  $q = \sum_{h=1}^t a_h r_h$ .

We assume, that  $\lambda_{2h} - \lambda_{1h} = 0$  for each  $h, h = 1, 2, \dots, t$ . Then from Theorem 4.1 we have

**Corollary 4.1.** *If  $\lambda_{2h} - \lambda_{1h} = 0, h = 1, 2, \dots, t$ , then the chemical balance weighing design given by the matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  of the form (4.1) and with the variance - covariance matrix of errors  $\sigma^2\mathbf{G}$ , where  $\mathbf{G}$  is of the form (2.2), is optimal for any  $a_h, a_h > 0, h = 1, 2, \dots, t$ .*

**Corollary 4.2.** *The existence of balanced bipartite weighing designs with the parameters  $v, b_h, r_h, k_{1h}, k_{2h}, \lambda_{1h}, \lambda_{2h}$  for which the condition  $\lambda_{2h} - \lambda_{1h} = 0$  holds,  $h = 1, 2, \dots, t$ , implies the existence of an optimum chemical balance weighing design with the matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  of the form (4.1) and with the variance - covariance matrix of errors  $\sigma^2\mathbf{G}$ , where  $\mathbf{G}$  is given by (2.2).*

Ceranka and Katulska (1999) gave series of balanced bipartite weighing designs which satisfy the condition  $\lambda_{2h} - \lambda_{1h} = 0, h = 1, 2, \dots, t$ . These designs lead to the optimum chemical balance weighing with the matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  of the form (4.1) with the variance - covariance matrix of errors  $\sigma^2\mathbf{G}$ , where  $\mathbf{G}$  is given by (2.2).

There is a large number of combinations between parameters of balanced bipartite weighing designs for which the condition (4.7) holds. Thus for given  $p = v$  and  $n = \sum_{h=1}^t b_h$  we have many possibilities of construction

of the design matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  of an optimum chemical balance weighing design with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where  $\mathbf{G}$  is of the form (2.2).

It is obvious also that we have many interesting possibilities of choosing the matrix  $\mathbf{G}$ . The constructions of an optimum chemical balance weighing design with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$  for each form of the matrix  $\mathbf{G}$  must be investigated separately. In the next section, we will present construction of the design matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  of an optimum chemical balance weighing design for the simplest form of  $\mathbf{G} \neq \mathbf{I}_n$ .

#### 5. CONSTRUCTION OF THE DESIGN MATRIX USING TWO BALANCED BIPARTITE WEIGHING DESIGNS

We assume, that the matrix  $\mathbf{G}$  is given in the form

$$(5.1) \quad \mathbf{G} = \begin{bmatrix} \frac{1}{a} \mathbf{I}_{n_1} & \mathbf{0}_{n_1} \mathbf{0}'_{n_2} \\ \mathbf{0}_{n_2} \mathbf{0}'_{n_1} & \mathbf{I}_{n_2} \end{bmatrix}, a > 0.$$

Suppose further that the matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  based on two balanced bipartite weighing designs is partitioned in accordance with the matrix  $\mathbf{G}$ , i.e.,

$$(5.2) \quad \mathbf{X} = \begin{bmatrix} \mathbf{N}_1^* \\ \mathbf{N}_2^* \end{bmatrix}.$$

Then, from Theorem 4.1, the matrix  $\mathbf{X}$  given by (5.2) is the matrix of the optimum chemical balance weighing design with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where the matrix  $\mathbf{G}$  is given by (5.1), if and only if

$$(5.3) \quad a(\lambda_{21} - \lambda_{11}) + (\lambda_{22} - \lambda_{12}) = 0.$$

We can notice that if parameters of two balanced bipartite weighing designs satisfy the condition (5.3) for a given  $a$ , then the chemical balance weighing design with the matrix  $\mathbf{X}$  given in the form (5.2) and with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where the matrix  $\mathbf{G}$  is given by (5.1),

is optimal. We have formulated Theorems following from Huang (1976). Parameters of these balanced bipartite weighing designs satisfy the condition (5.3).

**Theorem 5.1.** *For a given  $a = 2$  the existence of the balanced bipartite weighing designs with the parameters*

- (i)  $v = 4s + 1, b_1 = s(4s + 1), r_1 = 5s, k_{11} = 1, k_{21} = 4, \lambda_{11} = 2, \lambda_{21} = 3$   
and  $v = 4s + 1, b_2 = 2s(4s + 1), r_2 = 16s, k_{12} = 3, k_{22} = 5, \lambda_{12} = 15, \lambda_{22} = 13, s = 2, 3, \dots,$
- (ii)  $v = 10s + 1, b_1 = s(10s + 1), r_1 = 6s, k_{11} = 1, k_{21} = 5, \lambda_{11} = 1, \lambda_{21} = 2$   
and  $v = 10s + 1, b_2 = 5s(10s + 1), r_2 = 40s, k_{12} = 3, k_{22} = 5, \lambda_{12} = 15, \lambda_{22} = 13, s = 1, 2, \dots,$
- (iii)  $v = 2s + 1, b_1 = s(2s + 1), r_1 = 7s, k_{11} = 2, k_{21} = 5, \lambda_{11} = 10, \lambda_{21} = 11$   
and  $v = 2s + 1, b_2 = s(2s + 1), r_2 = 8s, k_{12} = 3, k_{22} = 5, \lambda_{12} = 15, \lambda_{22} = 13, s = 4, 5, \dots,$
- (iv)  $v = 2s, b_1 = s(2s - 1), r_1 = 3(2s - 1), k_{11} = 2, k_{21} = 4, \lambda_{11} = 8, \lambda_{21} = 7$   
and  $v = 2s, b_2 = 2s(2s - 1), r_2 = 7(2s - 1), k_{12} = 2, k_{22} = 5, \lambda_{12} = 20, \lambda_{22} = 22, s = 5, 6, \dots,$
- (v)  $v = 4s + 1, b_1 = 2s(4s + 1), r_1 = 6s, k_{11} = 1, k_{21} = 2, \lambda_{11} = 2, \lambda_{21} = 1$   
and  $v = 4s + 1, b_2 = s(4s + 1), r_2 = 8s, k_{12} = 2, k_{22} = 6, \lambda_{12} = 6, \lambda_{22} = 8, s = 2, 3, \dots,$
- (vi)  $v = 4s + 1, b_1 = s(4s + 1), r_1 = 4s, k_{11} = 2, k_{21} = 2, \lambda_{11} = 2, \lambda_{21} = 1$   
and  $v = 4s + 1, b_2 = s(4s + 1), r_2 = 8s, k_{12} = 2, k_{22} = 6, \lambda_{12} = 6, \lambda_{22} = 8, s = 2, 3, \dots,$
- (vii)  $v = 4s + 1, b_1 = s(4s + 1), r_1 = 5s, k_{11} = 2, k_{21} = 3, \lambda_{11} = 3, \lambda_{21} = 2$   
and  $v = 4s + 1, b_2 = s(4s + 1), r_2 = 8s, k_{12} = 2, k_{22} = 6, \lambda_{12} = 6, \lambda_{22} = 8, s = 2, 3, \dots,$
- (viii)  $v = 4s + 1, b_1 = 2s(4s + 1), r_1 = 12s, k_{11} = 2, k_{21} = 4, \lambda_{11} = 8, \lambda_{21} = 7$   
and  $v = 4s + 1, b_2 = s(4s + 1), r_2 = 8s, k_{12} = 2, k_{22} = 6, \lambda_{12} = 6, \lambda_{22} = 8, s = 2, 3, \dots,$

implies the existence of an optimum chemical balance weighing design with the matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given by (5.2) and with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where the matrix  $\mathbf{G}$  is of the form (5.1).

**Proof.** It is easy to prove that parameters of balanced bipartite weighing designs satisfy the condition (5.3) for a given  $a = 2$ . ■

**Theorem 5.2.** For a given  $a = s(s - 3)/2$  the existence of the balanced bipartite weighing designs with the parameters  $v, b_1 = v(v - 1)/2, r_1 = 3(v - 1)/2, k_{11} = 1, k_{21} = 2, \lambda_{11} = 2, \lambda_{21} = 1$  and  $v, b_2 = b_1, r_2 = (v - 1)(s + 1)/2, k_{12} = 1, k_{22} = s, \lambda_{12} = s, \lambda_{22} = s(s - 1)/2, v$  is odd,  $s \geq 4$  implies the existence of an optimum chemical balance design with the design matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given by (5.2) and with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where the matrix  $\mathbf{G}$  is of the form (5.1).

**Theorem 5.3.** For a given  $a = [-(u^2 - 5u + 2)]/(s^2 - 5s + 2)$  and

- (i)  $v$  is odd,  $u$  is a positive integer,  $u \geq 5, s = 2, 3, 4$  except the case  $u = 5$  and  $s = 4$ ,
- (ii)  $v$  is odd,  $u = 2, 3, 4, s$  is a positive integer,  $s \geq 5$  except the case  $u = 4$  and  $s = 5$ ,
- (iii)  $v$  is even,  $u \geq 6, s = 2, 4$ ,
- (iv)  $v$  is even,  $u = 2, 4, s \geq 6$  is even

the existence of the balanced bipartite weighing designs with the parameters  $v, b_1 = v(v - 1)/2, r_1 = (v - 1)(s + 2)/2, k_{11} = 2, k_{21} = s, \lambda_{11} = 2s, \lambda_{21} = (s^2 - s + 2)/2$  and  $v, b_2 = b_1, r_2 = (v - 1)(u + 2)/2, k_{12} = 2, k_{22} = u, \lambda_{12} = 2u, \lambda_{22} = (u^2 - u + 2)/2$  implies the existence of the optimum chemical balance design with the design matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given by (5.2) and with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where the matrix  $\mathbf{G}$  is of the form (5.1).

**Theorem 5.4.** For a given  $a = [-(u^2 - 7u + 6)]/(s^2 - 7s + 6)$  and

- (i)  $v$  is odd,  $u \geq 7, u$  is a positive integer,  $s = 2, 3, 4, 5$  except the case  $u = 7$  and  $s = 3, 4$ ,

- (ii)  $v$  is odd,  $u = 2, 3, 4, 5$ ,  $s$  is a positive integer,  $s \geq 7$  except the case  $u = 3, 4$  and  $s = 7$ ,
- (iii)  $v$  is a positive integer,  $u \geq 7$  is odd  $s = 3, 5$ ,
- (iv)  $v$  is a positive integer,  $u = 3, 5$ ,  $s \geq 7$  is odd

the existence of the balanced bipartite weighing designs with the parameters  $v, b_1 = v(v-1)/2$ ,  $r_1 = (v-1)(s+3)/2$ ,  $k_{11} = 3$ ,  $k_{21} = s$ ,  $\lambda_{11} = 3s$ ,  $\lambda_{21} = (s^2 - s + 6)/2$  and  $v, b_2 = b_1$ ,  $r_2 = (v-1)(u+3)/2$ ,  $k_{12} = 3$ ,  $k_{22} = u$ ,  $\lambda_{12} = 3u$ ,  $\lambda_{22} = (u^2 - u + 6)/2$  implies the existence of the optimum chemical balance design with the design matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given by (5.2) and with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where the matrix  $\mathbf{G}$  is of the form (5.1).

**Theorem 5.5.** For a given  $a = [-(u^2 - 5u + 2)]/(s^2 - 7s + 6)$  and

- (i)  $v$  is odd,  $u$  is a positive integer,  $u \geq 5$ ,  $s = 2, 3, 4, 5$ ,
- (ii)  $v$  is odd,  $u = 2, 3, 4, 5$ ,  $s$  is a positive integer,  $s \geq 7$ ,
- (iii)  $v$  is a positive integer,  $u \geq 6$  is even,  $s = 3, 5$ ,
- (iv)  $v$  is a positive integer,  $u = 2, 4$   $s \geq 7$  is odd

the existence of the balanced bipartite weighing designs with the parameters  $v, b_1 = v(v-1)/2$ ,  $r_1 = (v-1)(s+3)/2$ ,  $k_{11} = 3$ ,  $k_{21} = s$ ,  $\lambda_{11} = 3s$ ,  $\lambda_{21} = (s^2 - s + 6)/2$  and  $v, b_2 = b_1$ ,  $r_2 = (v-1)(u+2)/2$ ,  $k_{12} = 2$ ,  $k_{22} = u$ ,  $\lambda_{12} = 2u$ ,  $\lambda_{22} = (u^2 - u + 2)/2$  implies the existence of the optimum chemical balance design with the design matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given by (5.2) and with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where the matrix  $\mathbf{G}$  is of the form (5.1).

We can notice that if the parameters  $v, b_1, r_1, k_{11}, k_{21}, \lambda_{11}, \lambda_{21}$  and  $v, b_2, r_2, k_{12}, k_{22}, \lambda_{12}, \lambda_{22}$  of two balanced bipartite weighing designs satisfy the condition (5.3) for a given  $a$ , then the parameters  $v^* = v, b_1^* = b_2, r_1^* = r_2, k_{11}^* = k_{12}, k_{21}^* = k_{22}, \lambda_{11}^* = \lambda_{12}, \lambda_{21}^* = \lambda_{22}$  and  $v^* = v, b_2^* = b_1, r_2^* = r_1, k_{12}^* = k_{11}, k_{22}^* = k_{21}, \lambda_{12}^* = \lambda_{11}, \lambda_{22}^* = \lambda_{21}$  of two balanced bipartite weighing designs satisfy the condition

$$a^*(\lambda_{21}^* - \lambda_{11}^*) + (\lambda_{22}^* - \lambda_{12}^*) = 0$$

for  $a^* = 1/a$  and the chemical balance weighing design with the design matrix  $\mathbf{X} \in \Phi_{n \times p, m}(-1, 0, 1)$  given by (5.2) and with the variance - covariance matrix of errors  $\sigma^2 \mathbf{G}$ , where  $\mathbf{G}$  is given by (5.1), is optimal for estimating individual weights of objects.

## 6. THE EXAMPLE

In the practical usage of the data theory, for example in astronomy, geodesy, a situation often occurs in which in the measure of the same object we get designs consisting of a several equal precise observations ( $n_h$ ,  $h = 1, 2, \dots, t$ ) linked into the groups ( $t$ ) (see Linnik (1962)). These different precision of measurements leads to the diagonal variance - covariance matrix of errors. On the other hand, in this issue the elements of the design matrix are equal to  $-1, 1$  or  $0$ . That is why we use the matrices of chemical balance weighing designs.

Let us consider the experiment in which using  $n = 42$  measurement operations we determine unknown measurements of  $p = 7$  objects and each of them is weighed at most  $m = 24$  times (because of destroying of the light objects or some other technical problems). For construction of the design matrix  $\mathbf{X}$  we use  $\mathbf{N}_1$  the incidence matrix of the balanced bipartite weighing design with the parameters  $v = 7$ ,  $b_1 = 21$ ,  $r_1 = 9$ ,  $k_{11} = 1$ ,  $k_{21} = 2$ ,  $\lambda_{11} = 2$ ,  $\lambda_{21} = 1$

$$\mathbf{N}_1 = \begin{bmatrix} 1_1 & 0 & 0 & 1_2 & 0 & 0 & 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 0 & 0 & 1_1 & 0 & 0 & 0 & 1_2 & 1_2 \\ 1_2 & 1_1 & 0 & 0 & 1_2 & 0 & 0 & 0 & 1_1 & 0 & 1_2 & 0 & 1_2 & 0 & 1_2 & 1_1 & 0 & 0 & 0 & 1_2 \\ 0 & 1_2 & 1_1 & 0 & 0 & 1_2 & 0 & 0 & 0 & 1_1 & 0 & 1_2 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 0 & 0 \\ 0 & 0 & 1_2 & 1_1 & 0 & 0 & 1_2 & 1_2 & 0 & 0 & 1_1 & 0 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_1 & 0 & 0 \\ 1_2 & 0 & 0 & 1_2 & 1_1 & 0 & 0 & 0 & 1_2 & 0 & 0 & 1_1 & 0 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_1 & 0 \\ 0 & 1_2 & 0 & 0 & 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 0 & 0 & 1_1 & 0 & 0 & 0 & 1_2 & 1_2 & 1_1 & 0 \\ 0 & 0 & 1_2 & 0 & 0 & 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 0 & 0 & 1_1 & 0 & 0 & 0 & 1_2 & 1_2 & 1_1 \end{bmatrix}$$

and  $\mathbf{N}_2$  the incidence matrix of the balanced bipartite weighing design with parameters  $v = 7$ ,  $b_2 = 21$ ,  $r_2 = 15$ ,  $k_{12} = 1$ ,  $k_{22} = 4$ ,  $\lambda_{12} = 4$ ,  $\lambda_{22} = 6$

$$\mathbf{N}_2 = \begin{bmatrix} 0 & 1_2 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 1_2 \\ 1_1 & 0 & 1_2 & 0 & 1_2 & 1_2 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 1_2 & 0 & 1_2 & 1_2 & 1_2 & 0 & 1_1 \\ 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 1_2 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 1_1 & 1_2 & 0 & 1_2 & 1_2 & 1_2 & 0 \\ 1_2 & 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 1_2 & 0 & 1_2 & 1_2 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 1_2 & 0 & 1_2 & 1_2 \\ 0 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 1_2 & 0 & 1_2 \\ 1_2 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 1_2 & 0 \end{bmatrix}$$

where  $1_1$  and  $1_2$  denote the object belonging to the first and second sub-block, respectively. Then we get the matrix  $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$ , determined by combinations between parameters, where

$$\mathbf{X}_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$



$$\mathbf{X}_2 = \begin{bmatrix} 0 & -1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & -1 \\ -1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

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