AN ALTERNATIVE APPROACH TO BONUS MALUS

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Abstract

Under the assumptions of an open portfolio, i.e., considering that a policyholder can transfer his policy to another insurance company and the continuous arrival of new policyholders into a portfolio which can be placed into any of the bonus classes and not only in the "starting class", we developed a model (Stochastic Vortices Model) to estimate the *Long Run Distribution* for a Bonus Malus System. These hypothesis render the model quite representative of the reality.

With the obtained *Long Run Distribution*, a few optimal bonus scales were calculated, such as Norberg's (1979), Borgan, Hoem's and Norberg's (1981), Gilde and Sundt's (1989) and Andrade e Silva's (1991).

To compare our results, since this was the first application of the model, we used the Classic Model for Bonus Malus and the Open Model developed by Centeno and Andrade e Silva (2001).

The results of the Stochastic Vortices and the Open Model are highly similar and quite different from those of the Classic Model. Besides this the distribution of policyholders in the various bonus classes was derived assuming that the entrances followed adequate stochastic models.

Keywords: bonus malus, stochastic vortices, long run distribution, optimal bonus scales.

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1. Introduction

The Bonus Malus Systems play a fundamental role in automobile insurance. Since automobile insurance holds a significant part of the non-life business of many companies, and considering the enormous and still growing competitiveness of the market, the Bonus Malus System should be efficient, penalizing to bad drivers and simultaneously competitive. Due to market competitiveness, it is well known that, at least in Portugal, many policyholders transfer their policy to another insurance company, seeking lower premiums or higher discounts.

The study of Bonus Malus Systems in automobile insurance aims at finding rating systems that adjust the premium paid by the insured according to his driving experience. Unfortunately, in Portugal, the transfer of information between insurers is not efficient, which allows the rotation of policyholders which, after a claim participation during an annuity, leave their insurer and buy another policy at a competitor declaring that this will be his first insurance policy, thus managing to escape the penalization of the premium: he can be treated as a free claim policyholder by his next insurer. Consequently, the policyholders in the aggravated classes tend to transfer their policy to another insurer.

Another aspect taken into consideration is the fact that every year there are new policyholders, not all of whom are placed in the pre-defined "starting class". In many cases, usually due to comercial goals, discounts are given to new policyholders. In other cases, when the insurance company requests the Tariff Certificate, the new policyholders are in a aggravated class. So, we find that it should not be assumed, as it is in classic models, that all policyholders start in the same class.

Assuming that the Portuguese situation may apply to other countries, we tried to develop a model that took all these aspects into consideration. The Stochastic Vortices Model is such an alternative approach to the usual model for Bonus Malus Systems, since it allows the subscription and the annulment of policies in the portfolio and, in that way, renders it more realistic.

Stochastic Vortices offer useful models in a great variety of situations in which the irreducible Markov chains of discrete parameter fail. These situations correspond to open populations in which there are entrances and departures. When the population is divided into sub-populations and the transition probabilities between sub-populations are invariant, these sub-populations can be considered as the transient states of a homogenous

Markov chain. As we shall see, under quite general assumptions in the entrances, the limit state probabilities are obtained. The fact that the population is taken as open, renders this kind of models much more realistic than classic models based on a closed population occupying the states of an irreducible Markov chain. The limit state probabilities correspond to the long run distribution.

Using a Portuguese data set, we applied the model, obtaining the correspondent long run distribution. Afterwards, we estimated some optimal bonus scales, such as Norberg's (1979), Borgan, Hoem and Norberg's (1981), Gilde and Sundt's (1989) and Andrade e Silva's (1991).

Since this was the first application of the model, we compared our results with the Classic Model (Closed Model) for Bonus Malus Systems and also with the Open Model developed by Centeno and Andrade e Silva (2001). The results of the Stochastic Vortices Model and the Open Model are highly similar.

After presenting the model and the data in Sections 2 and 3, Section 4 is devoted to results.

2. Model presentation

2.1. Stochastic vortices

Our model applies to populations divided into sub-populations which correspond to the transient states of homogeneous Markov chains. Every element entering the population will, after a finite time span, go into a final absorbing state and cease to belong to the population.

State probabilities will be the probabilities of a randomly chosen element in the population, belonging to the different sub-populations. As we shall see, under quite general conditions, these state probabilities will converge to limit state probabilities. Thus, in our model, the long run distribution will be constituted by the limit state probabilities. As to the study of the performance of the Bonus Malus Systems (BMS) this long run distribution may be used in just the same way as when an alternative model is used.

We point out that:

- the state probabilities will be proportional to the mean values of the dimension of the sub-populations;

- in the application to BMS we start by considering transition probabilities which depend on a parameter λ. This parameter will

have a structural distribution F. Thus, we will have to obtain first a λ dependent long run distribution and then decondition it to get the unconditional long run distribution which we use in the performance analysis.

2.1.1. Transition Matrices

As said above, besides s transient states, as much as the sub-populations, there will be a final absorbing state. In the applications to BMS we are led to consider the transient states as constituting a communication class, but our treatment does not require this assumption. Moreover, parameter λ will refer to the distribution of the number of claims. In the portfolio we are going to study, this parameter is unidimensional but the final deconditioning can also be carried out for a vector of parameters. We also point out that transition steps will correspond to years, and that the origin of time ($t = 0$) will be the beginning of the year in which the portfolio has begun.

Let $K_{1,\lambda}$ be the $s \times s$ matrix of one step transition between transient states. The full one step transition matrix will be

$$(1) \qquad\qquad P_{T,\lambda} = \begin{pmatrix} K_{1,\lambda} & \vec{q}_{1,\lambda}^{\,s} \\ \vec{0}'^{s} & 1 \end{pmatrix}$$

the last line corresponding to the absorbing state. The components of $\vec{q}_{1,\lambda}^{\,s}$ are the probabilities for the policyholders in the s classes quitting after one year.

Note that, if we add the elements of a row of $K_{1,\lambda}$ with the correspondent component of $\vec{q}_{1,\lambda}^{\,s}$, the sum will have to equal 1. In fact, at the end of an annuity, the policyholder either remains in the portfolio, occupying the class foreseen in the transition rules or annuls his policy, leaving the Company.

We now establish

Lemma 1. *The n steps transition matrix will be:*

$$P_{T,\lambda}^{(n)} = \begin{pmatrix} K_{n,\lambda} & \vec{q}_{n,\lambda}^{\,s} \\ \vec{0}'^{s} & 1 \end{pmatrix}$$

with:

$$K_{n,\lambda} = K_{1,\lambda}^{n},$$

$$\vec{q}_{n,\lambda}^{\,s} = \sum_{j=0}^{n-1} K_{1,\lambda}^{j} \cdot \vec{q}_{1,\lambda}^{\,s}.$$

Proof. Since the Markov chain is homogeneous we will have $P_{T,\lambda}^{(n)} = P_{T,\lambda}^{n}$ and the thesis is easily established through mathematical induction once it is observed that

$$K_{n,\lambda} = K_{1,\lambda} \cdot K_{n-1,\lambda} = K_{1,\lambda} \cdot K_{1,\lambda}^{n-1},$$

$$\vec{q}_{n,\lambda}^{\,s} = K_{1,\lambda} \cdot \vec{q}_{n-1,\lambda}^{\,s} + \vec{q}_{1,\lambda}^{\,s} = K_{1,\lambda} \cdot \sum_{j=0}^{n-2} K_{1,\lambda}^{j} \cdot \vec{q}_{1,\lambda}^{\,s} + \vec{q}_{1,\lambda}^{\,s} = \sum_{j=0}^{n-1} K_{1,\lambda}^{j} \cdot \vec{q}_{1,\lambda}^{\,s}.$$

$\blacksquare$

Thus the transition probabilities in n steps between the sub-populations will be the elements of the $K_{1,\lambda}^{n}$ matrix.

If the probabilities of a new policyholder being placed in the s classes are the components of the row vector $\vec{p}_{0,\lambda}^{\,\prime s}$, the corresponding probabilities after n years will be the components of the vector

$$(2) \qquad \vec{p}_{n,\lambda}^{\,\prime s} = \vec{p}_{0,\lambda}^{\,\prime s} \cdot K_{1,\lambda}^{n}.$$

2.1.2. Limit state probabilities

Under very general conditions we have (Healy, 1986)

$$(3) \qquad K_{1,\lambda} = \sum_{l=1}^{s} \eta_{l,\lambda} \, \vec{\alpha}_{l,\lambda}^{\,s} \, \vec{\beta}_{l,\lambda}^{\,\prime s}$$

where the $\eta_{l,\lambda}$ $[\vec{\alpha}_{l,\lambda}^{\,s} \; ; \; \vec{\beta}_{l,\lambda}^{\,\prime s}];$ $l=1,\dots,s$ are the eigenvalues [left and right eigenvectors] of $K_{1,\lambda}$, with

$$(4) \qquad \vec{\beta}_{l,\lambda}^{\,\prime s} \cdot \vec{\alpha}_{h,\lambda}^{\,s} = 0 \; ; \; l \neq h$$

so that

$$(5) \qquad K_{n,\lambda} = K_{1,\lambda}^{n} = \sum_{l=1}^{s} \eta_{l,\lambda}^{n} \, \vec{\alpha}_{l,\lambda}^{\,s} \, \vec{\beta}_{l,\lambda}^{\,\prime s}.$$

Now (Parzen, 1965), the transition probabilities between transient states tend to zero with the number of steps, thus

(6) $$|\eta_{l,\lambda}| < 1 \quad ; l = 1, \ldots, s.$$

2.1.3. Entrances on the system

2.1.3.1 Assymptotic Model

Let θ_i be the mean number of admissions at the i-th year. To lighten the computation we assume that the admissions are at the beginning of each year. We now get

Lemma 2. *The mean value for the number of policyholders in the different classes will, at the end of n years, be the components of*

(7) $$\vec{v}_{n,\lambda}^{\prime s} = \vec{p}_{0,\lambda}^{\prime s} \sum_{i=0}^{n} \theta_{n-i} \, K_{1,\lambda}^{i}.$$

Proof. The mean values we are looking for are the sum of the mean values for the number of policyholders in the different classes that were admitted at years $n - i$, $i = 0, \ldots, n$. According to expression (2) these partial mean values will be the components of the row vectors

$$\theta_{n-i} \cdot \vec{p}_{i,\lambda}^{\prime s} = \theta_{n-i} \cdot \vec{p}_{0,\lambda}^{\prime s} \cdot K_{1,\lambda}^{i} \quad ; \; i = 0, \ldots, n$$

and the thesis is established. ∎

We now introduce an assumption about the growth of the portfolio. Due to the high competitiveness of the market (at least in Portugal) we put

(8) $$\theta_i = \kappa \, (1 - e^{-\beta i}), \quad \kappa, \beta > 0$$

so that the admissions will tend to a limit κ. We point out that, due to (6), limit state probabilities exist even under more aggressive growth.

Let us establish

Proposition 1. *When* (8) *holds, the mean values for the number of policyholders in the different classes will be, after n years, the components of*

$$(9) \qquad \vec{v}_{n,\lambda}^{\,\prime s} = \kappa \, \vec{p}_{0,\lambda}^{\,\prime s} \left[\sum_{i=0}^{n} K_{1,\lambda}^{i} - e^{-\beta n} \sum_{i=0}^{n} \left(e^{\beta} K_{1,\lambda} \right)^{i} \right].$$

Moreover

$$(10) \qquad \vec{v}_{\infty,\lambda}^{\,\prime s} = \lim_{n \to \infty} \vec{v}_{n,\lambda}^{\,\prime s} = \kappa \, \vec{p}_{0,\lambda}^{\,\prime s} \sum_{l=1}^{s} \frac{1}{1 - \eta_{l,\lambda}} \, \vec{\alpha}_{l,\lambda}^{s} \, \vec{\beta}_{l,\lambda}^{\,\prime s}.$$

Proof. The first part of the thesis follows from expressions (7) and (8). Going over to the second part of the thesis we have, according to (5) and (6),

$$\sum_{i=0}^{\infty} K_{1,\lambda}^{i} = \sum_{i=0}^{\infty} \sum_{l=1}^{s} \eta_{l,\lambda}^{i} \, \vec{\alpha}_{l,\lambda}^{s} \vec{\beta}_{l,\lambda}^{\,\prime s}$$

$$= \sum_{l=1}^{s} \vec{\alpha}_{l,\lambda}^{s} \vec{\beta}_{l,\lambda}^{\,\prime s} \sum_{i=0}^{\infty} \eta_{l,\lambda}^{i} = \sum_{l=1}^{s} \vec{\alpha}_{l,\lambda}^{s} \vec{\beta}_{l,\lambda}^{\,\prime s} \frac{1}{1 - \eta_{l,\lambda}}$$

and according to

$$\sum_{i=0}^{n} e^{\beta(i-n)} \eta_{l,\lambda}^{i} = e^{-\beta n} \frac{1 - e^{\beta(n+1)} \, \eta_{l,\lambda}^{n+1}}{1 - e^{\beta} \, \eta_{l,\lambda}}$$

$$= \frac{e^{-\beta n} - e^{\beta} \, \eta_{l,\lambda}^{n+1}}{1 - e^{\beta} \, \eta_{l,\lambda}} \longrightarrow 0 \ \text{ as } \ n \to \infty \text{ for } l = 1, \ldots, s,$$

we will also get

$$\sum_{i=0}^{n} e^{\beta(i-n)} K_{1,\lambda}^{i}$$

$$(11)$$

$$= \sum_{l=1}^{s} \vec{\alpha}_{l,\lambda}^{s} \vec{\beta}_{l,\lambda}^{\,\prime s} \sum_{i=0}^{n} e^{\beta(i-n)} \, \eta_{l,\lambda}^{i} \longrightarrow 0 \ \text{ as } \ n \to \infty \text{ for } l = 1, \ldots, s$$

and the thesis is established.   ■

When $\vec{v}_{\infty,\lambda}^{'s}$ is defined, to obtain the limit state probabilities we have only to divide the components of $\vec{v}_{\infty,\lambda}^{'s}$ by their sum. Let $\vec{\pi}_{T,\lambda}^{'s}$ be the row vector of these limit probabilities which clearly do not depend on κ, this is, on the upper bound for admissions.

To obtain the unconditional long run distribution we must decondition, getting

$$(12) \qquad \pi_T(j) = \int_0^\infty \pi_{T,\lambda}(j) dF(\lambda) \quad, j = 1, \ldots, s,$$

where the $\pi_{T,\lambda}(j) \; [\pi_T(j)] \,, j = 1, \ldots, s$, are the conditional [unconditional] limit state probabilities.

In the next point we will assume that the entrances in the system are made according to the Poisson distribution, instead of being estimated by a certain fit.

2.1.3.2. Stochastic Model

Let us assume that the entrances in the system in **one** year will be Poisson distributed with mean value θ. If k is the number of admissions in one year, the number of policyholders in the different classes, after n years will be multinomial distributed with parameters

$$k, \; p_{n,\lambda}(1), \; p_{n,\lambda}(2), \; \ldots, p_{n,\lambda}(s+1)$$

and have a moment generating function (m.g.f.) given by:

$$\varphi(u_1, \ldots, u_{s+1}) = \left(\sum_{j=1}^{s+1} p_{n,\lambda}(j)\, e^{u_j} \right)^k.$$

Since we assumed that the number of entrances in the system is aleatory, we have to decondition it, so the m.g.f. will now be

$$\varphi(u_1, \ldots, u_{s+1}|\theta) = \prod_{j=1}^{s+1} e^{\theta \cdot p_{n,\lambda}(j)\, (e^{u_j}-1)}$$

with $\vec{p}_{0,\lambda}^{'s+1} \cdot P_{n,\lambda}$, this is, the number of policyholders in the different classes will be independent and Poisson distributed with mean values $\theta \cdot p_{n,\lambda}(j)$, $j = 1, \ldots, s+1$.

Let us assume that until the present, entrances are made in a stable way, with X_n the number of policyholders admitted to the system in the n-th year. The policyholders in the various bonus classes will be the sum of those who entered through the years, and did not leave the system.

Assuming that X_n is Poisson distributed with expected value θ_n and independence between the entries in different years, we will have

$$\varphi(u_1, \ldots, u_{s+1} | \theta_1, \ldots, \theta_n) = \prod_{j=1}^{s+1} e^{\sum_{w=0}^{n} \theta_w \, p_{w,\lambda}(j) \, (e^{u_j} - 1)}.$$

We now see that, after n years, the policyholders in each bonus class are independent Poisson distributed variables, with mean value $\sum_{w=0}^{n} \theta_w p_{w,\lambda}(j)$.

Considering an infinite horizon and focusing only on the transient states of the Markov chain, let us establish

Proposition 2. *The series* $\sum_{n=0}^{+\infty} p_{n,\lambda}(j)$, $j = 1, \ldots, s$, *are convergent.*

Proof. By (5), we know that $K_{n,\lambda} = \sum_{l=1}^{s} \eta_{l,\lambda}^{n} \, \vec{\alpha}_{l,\lambda}^{s} \, \vec{\beta}_{l,\lambda}^{'s}$, so

$$\vec{p}_{0,\lambda}^{'s} \cdot \sum_{n=0}^{+\infty} K_{1,\lambda}^{n} = \sum_{l=1}^{s} \frac{1}{1 - \eta_{l,\lambda}} \, (\vec{p}_{0,\lambda}^{'s} \cdot \vec{\alpha}_{l,\lambda}^{s}) \cdot \vec{\beta}_{l,\lambda}^{'s}.$$

It now suffices to point out that $\sum_{n=0}^{+\infty} p_{n,\lambda}(j)$ is the j-th element of $\vec{p}_{0,\lambda}^{'s} \cdot \sum_{n=0}^{+\infty} K_{1,\lambda}^{n}$, to see that it is convergent.

Corolary 1. *If* $\theta_n < u$, $n = 0, 1, \ldots$, *then the series* $\sum_{n=0}^{+\infty} p_{n,\lambda}(j) \, \theta_n$, $j = 1, \ldots, s$, *are convergent.*

So, with $\theta_j^* = \sum_{n=0}^{+\infty} \theta_n \, p_{n,\lambda}(j)$, $j = 1, \ldots, s$, restricted to $\theta_n < k$, $n = 0, 1, \ldots$, the number of policyholders in class j will have the Poisson distribution with mean value θ_j^* as limit distribution, since the corresponding series converges and so we know that this limit distribution is defined.

As the joint m.g.f. for the number of policyholders in the different classes is the product of marginal m.g.f. it is easy to see that the same will be true for characteristic functions. In that way we can see that the joint distribution of the number of policyholders in the different classes is the product of the marginal distributions so the number of policyholders in the different classes will be independent random variables (Cramer, 1957). It is easy to see that the limit joint distribution will be the product of the limit distributions for the number of policyholders in the different classes.

3. Relevant data

We will apply our model to the portfolio of a recently established Portuguese insurance company. The BMS of this insurer has $s = 20$ classes, the premium coefficients increasing with the class index. For each claim free year the index decreases by one. The first [each of the next] claim increases the class index by three [five]. For our purposes the relevant data were

a) **Claims Frequency**
   The claims frequency (in year 2000) was

   Table 1. Claims frequency - Year 2000

N. accidents	N. policyholders
0	41.484
1	2.998
2	318
3	29
4	7
5	2
Total	44.838

   To these data it was possible to adjust a mixed Poisson distribution, the structural distribution of the λ parameter being Gamma. The maximum likelihood estimators for the parameters of this distribution being $\hat{\alpha} = 0,5204150$ and $\hat{p} = 0,8612576$.

b) **Admission numbers (using an assymptotic model)**
   Since the portfolio was quite recent, having been established in the end of 1996, we assumed the values in bold in Table 2 in order to adjust a model of type $X_i = \kappa(1 - e^{-\beta i})$, $i = 0, 1, \ldots$ to the number of admissions.

   Table 2. Number of admissions per year

Year	1997	1998	1999	2000	**2001**	**2002**	**2003**	**2004**
N. of new policies	4280	10.646	16.506	22.451	**28.067**	**32.251**	**33.457**	**33.670**

In carrying out the adjustment we assumed values for κ and then used least squares to estimate β. The final pair $(\hat{\kappa}, \hat{\beta})$ chosen was the one for highest R^2. Thus, with $\hat{\kappa} = 40.000$ and $\hat{\beta} = 0,239947$ we obtained $R^2 = 0,987$ which can be considered an excellent fit.
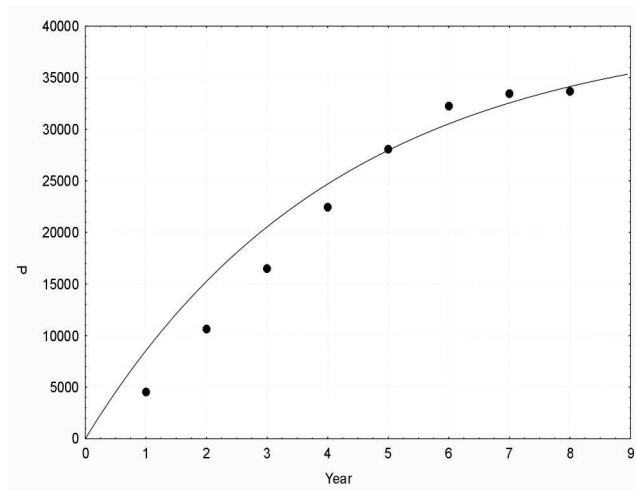
Figure 1 illustrates the fitness.

Figure 1. Entrances in the BMS

c) **Probabilities of new policies to be placed into each class**
   Despite the "standard entry class" being the 10th, many new policy-holders get, due to market competitiveness, a better rating. Using the available data we estimated the components of $\vec{p_0}^{\,s}$, considering it independent from λ. The results are presented in Table 3.

d) **Probabilities of annulment per class**
   Using the available data, considering it also independent from λ, we estimated the components of vector $\vec{q_1}^{\,'s}$. As we can see in Table 4, the highest probabilities are in higher classes, except for probability of Class 18 which is abnormally small.

Table 3. Probabilities of entrance
in each class

j	$p_0(j)$
1	0, 239402
2	0, 053668
3	0, 191427
4	0, 06955
5	0, 18862
6	0, 006072
7	0, 034191
8	0, 010409
9	0, 062468
10	0, 142443
11	0, 000552
12	0, 000363
13	0, 000252
14	0, 000237
15	0, 000205
16	0, 0000158
17	0, 0000315
18	0, 0000315
19	0
20	0, 0000631

Table 4. Probabilities of annulment
per class

j	$p_1(j)$
1	0, 038902
2	0, 049994
3	0, 05412
4	0, 121957
5	0, 110309
6	0, 125375
7	0, 108242
8	0, 113882
9	0, 148407
10	0, 203858
11	0, 204494
12	0, 276347
13	0, 153846
14	0, 262295
15	0, 265306
16	0, 421053
17	0, 447368
18	0, 142857
19	0, 5
20	0, 789474

4. Portfolio performance

We now assess the portfolio performance using our model as well as the Open Model of Centeno and Andrade e Silva (2001) and the Classic Closed Model.

4.1. Long run and weighted distribution

We now consider, besides the long run distribution, the weighted distribution required to apply the Borgan *et al.* (1981) premium scale. To obtain this,

we took a time horizon of 20 years, and following Centeno and Andrade e Silva (2001), used weights given by $w_n = \frac{w_{n-1}}{1+i}$ and $\sum_{n=1}^{20} w_n = 1$, with $i = 5\%$, where w_n represents the weight given to period n $(n = 1, 2, \ldots)$.

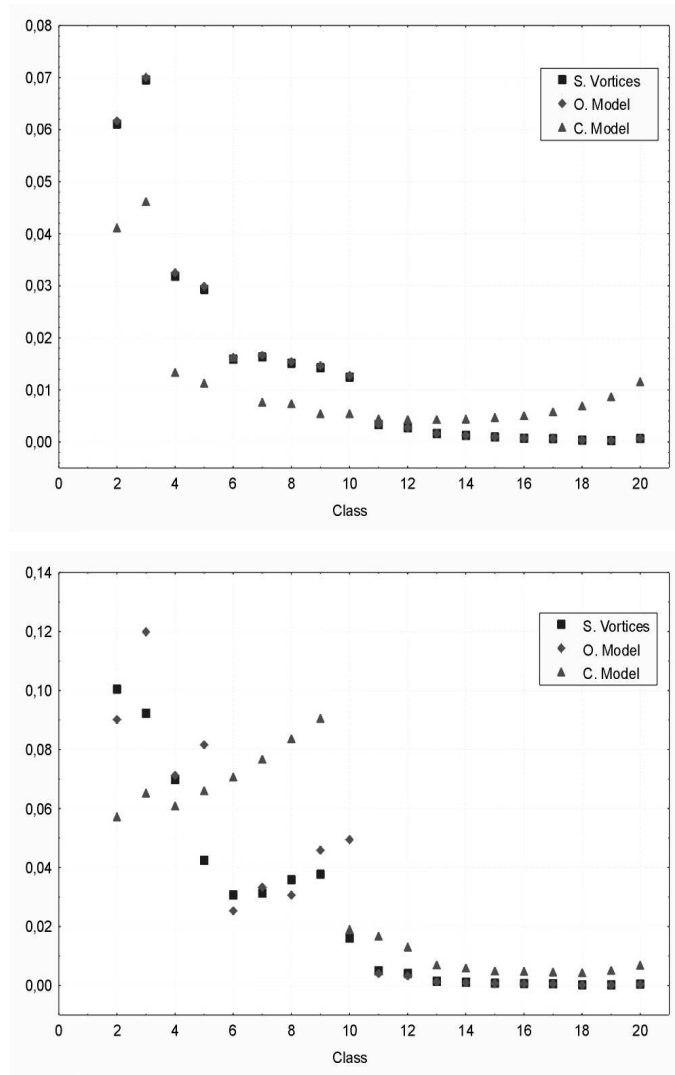The distributions are presented in Figure 2 with the exception of the

Figure 2. Long run distribution and weighted distribution.

results for Class 1 that are shown in Table 5. The results for this singling out being that otherwise there would not be enough graphic resolution to evaluate differences in the other classes.

Table 5. Long run and weighted distribution - Class 1

	S. Vortices	O. Model	C. Model
$\pi(1)$	$0,717041$	$0,721193$	$0,793887$
$p(1)$	$0,441117$	$0,527541$	$0,336097$

The results on the long run distribution, for our and Open Model are very similar and differ markedly from those of the closed portfolio model. This last model overevaluates the probabilities for the higher classes since it does not take into consideration that policyholders tend to leave when they attain higher maluses. For the lowest class the closed model differs less. This is certainly due to many admissions being placed directly in Class 1.

As a final remark we can point out to our and the Open Model being the more realistic ones.

4.2. Optimal bonus scales

For the BMS under study several optimal bonus scales, such as those of Norberg (1979), Borgan *et al.* (1981), Gilde and Sundt (1989) and Andrade e Silva (1991), were obtained. The results are shown in Figures 3 and 4.

Both our and the Open Model have quite similar optimal bonus scales, avoiding the very low or very high premiums that are obtained when using the closed model. Thus again the two first models are to be preferred to the last one.
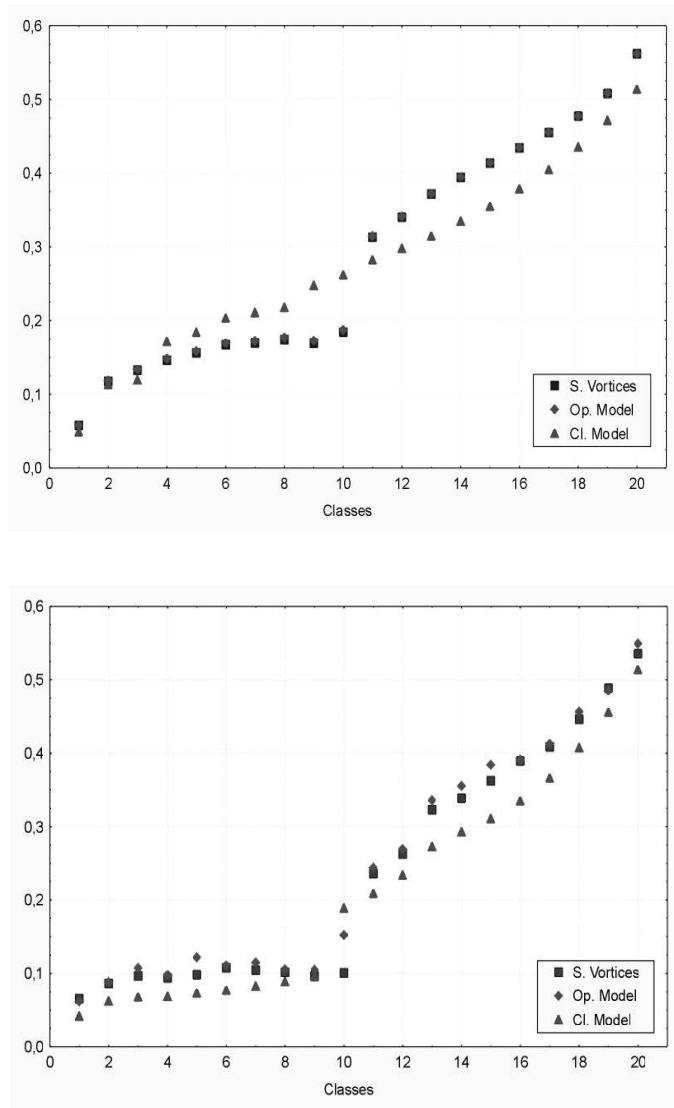
Figure 3. Optimal bonus scales of Norberg (1979) and Borgan *et al.* (1981).
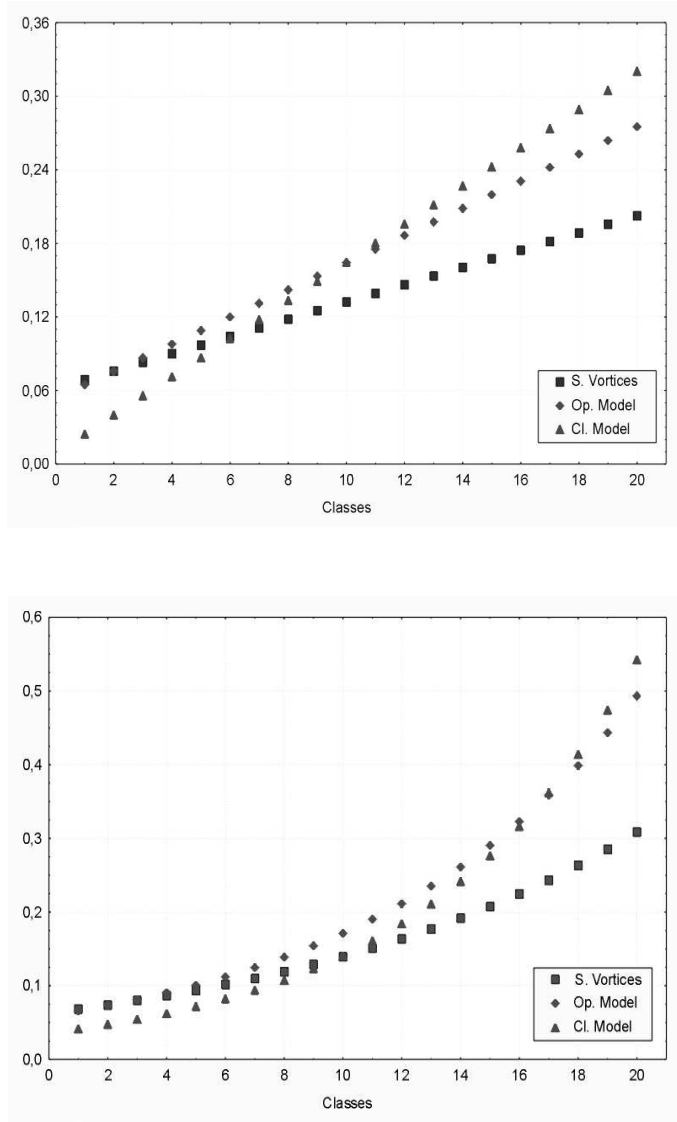
Figure 4. Optimal bonus scales of Gilde and Sundt (1989) and Andrade e Silva (1991).

REFERENCES

[1] J. Andrade e Silva, *Estruturas Tarifárias nos Ramos Reais da Indústria Segu-
    radora - Uma Aplicação ao sector automóvel em Portugal*, Instituto Superior
    de Economia e Gestão (In Portuguese) 1991.

[2] Ø. Borgan, J. Hoem and R. Norberg, *A non asymptotic criterion for the eval-
    uation of automobile bonus system*, Scandinavian Actuarial Journal (1981),
    165–178.

[3] L. Centeno and J. Andrade e Silva, *Bonus systems in open portfolio*, Insurance
    Mathematics e Economics (2001), 341–350.

[4] H. Cramer, *Mathematical Methods of Statistics*, Princeton University Press
    1957.

[5] V. Gilde and B. Sundt, *On bonus systems with credibility scales*, Scandinavian
    Actuarial Journal (1989), 13–22.

[6] G. Guerreiro, *Uma Abordagem Alternativa para Bonus Malus*, Instituto
    Superior de Economia e Gestão (In Portuguese) 2001.

[7] M. Healy, *Matrices for Statistics*, Oxford Science Publications 1986.

[8] J. Lemaire, *Bonus-Malus Systems in Automobile Insurance*, Kluwer Academic
    Publishers 1995.

[9] J. Mexia, *Vórtices Estocásticos de Parâmetro Discreto*, Comunication in the
    III Actuarial Colloquium FCT-UNL (In Portuguese) 1999.

[10] E. Parzen, *Stochastic Processes*, Holden Day, S. Francisco 1965.

[11] E. Seneta, *A Non-Negative Matrices and Markov Chains* Springer-Verlag
     1981.