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SOME CONSTRUCTIONS OF NESTED BALANCED EQUIREPLICATE BLOCK DESIGNS

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Abstract

Various methods of constructing nested ternary and quaternary efficiency balanced and variance balanced designs are proposed by applying some repetitions of treatments in all possible pairs of treatments. In these designs sub-blocks and super-blocks may form different p-ary designs, where sub-blocks have higher efficiency as compared to super-blocks, i.e., any two elementary treatment contrasts in the sub-blocks can be measured with higher efficiency than any two elementary contrasts in the super-block structure. A comparison is shown in Table 1.

Keywords: Balanced incomplete block (BIB) design; efficiency balanced (EB) design; variance balanced (VB) design; efficiency balanced ternary (EBT); variance balanced ternary (VBT); efficiency balanced quaternary (EBQ); variance balanced quaternary (VBQ).

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1. Introduction

A block design is said to form a nested structure, when there are two sources of variability and one source is nested within the other. Preece [15] introduced a class of nested balanced incomplete block designs in which there are two systems of blocks, the second system nested within the first (each block from the first system called super-blocks containing some blocks called sub-blocks from the second). These designs are quite useful for experimental situations in which one nuisance factor is needed within the blocking factor. In the light of Preece [15], we will introduce some nested ternary and quaternary designs, which have both properties being efficiency balanced (EB) and variance balanced (VB).

Ghosh, Shah and Kageyama [9] introduced VB and EB ternary block designs. Katulska [12] provided necessary and sufficient conditions for the existence of variance balanced ternary (VBT) and efficiency balanced ternary (EBT) designs. Ghosh, Joshi and Kageyama [7] gave VBT designs. Colbourn and Colbourn [5], Jimbo and Kuriki [10] and Longyear [13] gave some nested block designs. A good survey on nested designs has been given by Morgan [14]. But it appears that the present paper is the first to strictly show the existence of ternary and quaternary designs having both properties of EB, VB and nestedness.

We provide here five methods of constructing nested ternary or quaternary VB, EB designs by applying some repetitions of treatments in all possible pairs of treatments. The sub-blocks and super-blocks may form different VB and EB p-ary block designs, i.e., VB and EB designs nested in some other VB and EB designs, where sub-blocks have higher efficiency as compared to super-blocks. These designs can be used in clinical trials. When we consider any p-ary design, the efficiency of any elementary contrast is very poor. Hence we can partition the super-blocks into sub-blocks such that if we consider any two treatment contrasts in sub-blocks it should be estimated through higher efficiency than any two elementary contrasts in super-blocks. Though VB or EB designs are introduced from a statistical point of view, this paper may be of combinatorial interest. For notations and formulas on terminologies we refer to Dey [6] and Raghavarao [16].

2. Definitions

A block design $\mathcal{D}$ with parameters v (a number of treatments), b (a number of blocks), $\boldsymbol{r} = (r_1, r_2, ..., r_v)'$ and $\boldsymbol{k} = (k_1, k_2, ..., k_b)'$ is described by the

$v \times b$ incidence matrix $\boldsymbol{N} = (n_{ij})$, where the ith treatment occurs n_{ij} times in the jth block for $i = 1, 2, ..., v$ and $j = 1, 2, ..., b$. Here $r_i = \sum_{j=1}^{b} n_{ij}$ (a replication of the ith treatment) and $k_j = \sum_{i=1}^{v} n_{ij}$ (the jth block size).

(i) A block design is called EB if $\boldsymbol{r}^{-\delta}\boldsymbol{N}\boldsymbol{k}^{-\delta}\boldsymbol{N}' = \mu\boldsymbol{I}_v + [(1-\mu)/n]\boldsymbol{1}_v\boldsymbol{r}'$ for some constant μ, where $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ (see, e.g., [3, 4, 11]).

(ii) A block design is called VB if $\boldsymbol{r}^{\delta} - \boldsymbol{N}\boldsymbol{k}^{-\delta}\boldsymbol{N}' = \psi[\boldsymbol{I}_v - (1/v)\boldsymbol{1}_v\boldsymbol{1}_v']$ for some constant ψ (see, e.g., [3, 4, 11]).

(iii) A block design is said to be (combinatorial or pairwise) balanced if $\sum_{j=1}^{b} n_{ij}n_{i'j} = \Lambda$ (a constant) for all i, i', $i \neq i'$ (see, e.g., [3, 4, 6, 16]).

(iv) A balanced block design is said to be: (a) binary if $n_{ij} = 0$ or 1 only, for all i, j, and it has parameters $v, b, \boldsymbol{r}, \boldsymbol{k}, \Lambda$ $(= \lambda$, say) [in this case, when $\boldsymbol{r} = r\boldsymbol{1}_v$ and $\boldsymbol{k} = k\boldsymbol{1}_b$, it is called a balanced incomplete block (BIB) design with parameters v, b, r, k, λ] (cf. [16]); (b) ternary if $n_{ij} = 0$, 1 or 2, for all i, j, and it has parameters $v, b, \rho_1, \rho_2, \boldsymbol{r}, \boldsymbol{k}, \Lambda$ (cf. [2]); (c) generalized binary if $n_{ij} = 0$ or x, for all i, j and some positive integer $x(> 1)$, and it has parameters $v, b, \boldsymbol{r}, \boldsymbol{k}, \Lambda$ (cf. [1]); (d) generalized ternary if $n_{ij} = 0, x$ or y, for all i, j and some positive integers $x, y \neq (1, 2)$ or (2,1), and it has parameters $v, b, \delta_1, \delta_2, \boldsymbol{r}, \boldsymbol{k}, \Lambda$; (e) quaternary if $n_{ij} = 0, 1, 2$ or 3, for all i, j, and it has parameters $v, b, \rho_1, \rho_2, \rho_3, \boldsymbol{r}, \boldsymbol{k}, \Lambda$. Similarly, a generalized quaternary design can be defined.

Throughout this paper, $\boldsymbol{1}_v$ is a $v \times 1$ column vector with ones, $\boldsymbol{I}_v$ is the identity matrix of order v, $\boldsymbol{J}_v = \boldsymbol{1}_v\boldsymbol{1}_v'$, $\boldsymbol{r}^{\delta}$ represents the diagonal matrix of components of $\boldsymbol{r}$, $\boldsymbol{k}^{\delta}$ represents the diagonal matrix of components of $\boldsymbol{k}$. Also $\boldsymbol{r}^{-\delta}$ and $\boldsymbol{k}^{-\delta}$ are inverse matrices of $\boldsymbol{r}^{\delta}$ and $\boldsymbol{k}^{\delta}$, respectively. Furthermore, $v^*, b^*, \boldsymbol{r}^*, \boldsymbol{k}^*, \Lambda^*$ are the parameters of a block design with the incidence matrix $\boldsymbol{N}^*$ formed by sub-blocks, while $v_*, b_*, \boldsymbol{r}_*, \boldsymbol{k}_*, \Lambda_*$ are the parameters of a block design with the incidence matrix $\boldsymbol{N}_*$ formed by super-blocks, μ represents the loss of information, i.e., $1 - \mu$ represents an efficiency of the design, $(1/\psi)$ represents a variance of any normalized contrast in the intra-block analysis (see [3, 7, 8, 9, 11]), and ρ_1, ρ_2, ρ_3, δ_1, δ_2 are the numbers of times $1, 2, 3, x$ and y occur in the incidence matrix, respectively.

The quantities μ and ψ play a key role to show the usefulness of block designs from a statistical point of view (cf. [3, 4, 11]). All of the designs constructed here are proper and equireplicate, i.e., its block sizes are all equal and its replication numbers are all equal. Note (see Kageyama [11]) that in an equireplicate block design with $\boldsymbol{r} = r\mathbf{1}_v$, a relation $1 - \mu = \psi/r$ holds.

3. Method of construction I

Consider a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ, and take any pair of treatments, say (θ, ϕ), from this design. Then make two blocks as follows.

(i) Repeat the treatment θ two times while the remaining $k-1$ treatments in the same block which contains the pair (θ, ϕ) remain as it is.

(ii) Repeat the treatment ϕ two times while the remaining $k-1$ treatments in the same block which contains the pair (θ, ϕ) remain as it is.

Repeating the same procedure for all $\binom{v}{2}$ pairs of treatments and considering the above two blocks obtained as two sub-blocks nested in a super-block, a nested EBT as well as VBT design can be constructed. The sub-blocks form a ternary design $\mathcal{D}^*$ with parameters $v^*, b^*, \rho_1, \rho_2, \boldsymbol{r}^*, \boldsymbol{k}^*, \Lambda^*$, while the super-blocks form a generalized ternary design $\mathcal{D}_*$ with parameters $v_*, b_*, \delta_1, \delta_2, \boldsymbol{r}_*, \boldsymbol{k}_*, \Lambda_*$.

Theorem 3.1. *The existence of a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ implies the existence of a nested EB as well as VB design. The sub-blocks form an EBT as well as VBT design with parameters*

$$v^* = v,\ b^* = v(v-1)\lambda,\ \rho_1 = r(k-1)^2,\ \rho_2 = r(k-1),$$

$$\boldsymbol{r}^* = r(k^2-1)\mathbf{1}_v,\ \boldsymbol{k}^* = (k+1)\mathbf{1}_{b^*},\ \Lambda^* = \lambda(k+2)(k-1),$$

$$\mu^* = \frac{k^*(v^*-k^*)+2v^*}{(k^*)^2(v^*-1)},\ \psi^* = \frac{\Lambda^* v^*}{k^*},$$

where $k^* = k+1$, while the super-blocks form a generalized EBT as well as VBT design with parameters

$$v_* = v,\ b_* = \binom{v}{2}\lambda,\ \delta_1 = r\binom{k-1}{2},\ \delta_2 = r(k-1),$$

$$\boldsymbol{r}_* = r(k^2-1)\mathbf{1}_v,\ \boldsymbol{k}_* = 2(k+1)\mathbf{1}_{b_*},\ \Lambda_* = \lambda(2k^2+2k-3),$$

$$\mu_* = \frac{k_*(2v_* - k_*) + 6v_*}{(k_*)^2(v_*-1)},\ \psi_* = \frac{\Lambda_* v_*}{k_*},$$

where $k_* = 2(k+1)$.

Proof. In the BIB design $\mathcal{D}$, the v treatments form $\binom{v}{2}$ pairs and every such pair occurs λ times. Under the present method of construction, we obtain two sub-blocks corresponding to each pair. Hence in the design $\mathcal{D}^*$, the parameters $v^* = v, b^* = 2\binom{v}{2}\lambda, \boldsymbol{k}^* = (k+1)\mathbf{1}_{b^*}$ are obvious from the construction. In the super-blocks, we merge the treatments of two sub-blocks and, therefore, in the design $\mathcal{D}_*$, the parameters $v_* = v, b_* = \binom{v}{2}\lambda, \boldsymbol{k}_* = 2(k+1)\mathbf{1}_{b_*}$ are obvious from the construction.

Any treatment, say θ, appears in r blocks of $\mathcal{D}$. Under the construction, the $k-1$ pairs with θ in the same block contribute $3r(k-1)$ times in $\mathcal{D}^*$. While these $k-1$ treatments other than θ make $\binom{k-1}{2}$ pairs and with these pairs the treatment θ occurs $2r\binom{k-1}{2}$ times in $\mathcal{D}^*$. Hence $\boldsymbol{r}^* = \boldsymbol{r}_* = r(k^2-1)\mathbf{1}_v$.

For the calculation of Λ in $\mathcal{D}^*$ and $\mathcal{D}_*$, consider a block of $\mathcal{D}$ where any pair (θ,ϕ) occurs. Under the present method of construction corresponding to the following pairs of the blocks of $\mathcal{D}$, we get (θ,ϕ) pairs in $\mathcal{D}^*$ and $\mathcal{D}_*$:

(i) Corresponding to a (θ,ϕ) pair;

(ii) Corresponding to $k-2$ (θ,δ) pairs in the block;

(iii) Corresponding to $k-2$ (ϕ,δ) pairs in the block;

(iv) Corresponding to $\binom{k-2}{2}$ (δ,ε) pairs in the block.

Under (i), the (θ,ϕ) pair occurs 4 times in $\mathcal{D}^*$ while the pair occurs 9 times in $\mathcal{D}_*$. Under each of (ii) and (iii), (θ,ϕ) pair occurs $3(k-2)$ times in $\mathcal{D}^*$ while $6(k-2)$ times in $\mathcal{D}_*$. Under (iv), (θ,ϕ) pair occurs $2\binom{k-2}{2}$ times in $\mathcal{D}^*$ while $4\binom{k-2}{2}$ times in $\mathcal{D}_*$.

Since each pair occurs λ times in $\mathcal{D}$, it follows that

$$\Lambda^* = \lambda[4 + 6(k-2) + (k-2)(k-3)] = \lambda(k-1)(k+2),$$

$$\Lambda_* = \lambda[9 + 12(k-2) + 2(k-2)(k-3)] = \lambda(2k^2 + 2k - 3).$$

The calculations of the efficiency and variance can be done as follows. For the sub-block structure, it holds that

$$\boldsymbol{N}^*(\boldsymbol{N}^*)' = r(k-1)(k+3)\boldsymbol{I}_v + \lambda(k-1)(k+2)(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

$$(\boldsymbol{r}^*)^{-\delta}\boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)'$$

$$= [(k+3)/(k+1)^2]\boldsymbol{I}_v + \{(k-1)(k+2)/[(k+1)^2(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

which yield $\mu^* = [k^*(v^* - k^*) + 2v^*]/[(k^*)^2(v^* - 1)]$. On the other hand, it holds that

$$(\boldsymbol{r}^*)^{\delta} - \boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)'$$

$$= [r(k-1)^2(k+2)/(k+1)]\boldsymbol{I}_v - [\lambda(k-1)(k+2)/(k+1)](\boldsymbol{J}_v - \boldsymbol{I}_v),$$

which yields $\psi^* = \Lambda^* v^*/k^*$.

For the super-block structure, it follows that

$$\boldsymbol{N}_*\boldsymbol{N}_*' = r(k-1)(2k+5)\boldsymbol{I}_v + \lambda(2k^2 + 2k - 3)(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

$$(\boldsymbol{r}_*)^{-\delta}\boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)'$$

$$= \{(2k+5)/[2(k+1)^2]\}\boldsymbol{I}_v + \{(2k^2 + 2k - 3)/[2(k+1)^2(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

which show $\mu_* = [k_*(2v_* - k_*) + 6v_*]/[(k_*)^2(v_* - 1)]$. On the other hand, it holds that $(\boldsymbol{r}_*)^{\delta} - \boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)' = \{r(k-1)(2k^2+2k-3)/[2(k+1)]\}\boldsymbol{I}_v - \{\lambda(2k^2+2k-3)/[2(k+1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v)$, showing $\psi_* = \Lambda_* v_*/k_*$. This completes the proof. ■

Example 3.1. Consider an unreduced BIB design with parameters $v = b = 4, r = k = 3$, $\lambda = 2$, having blocks (2, 3, 4), (1, 3, 4), (1, 2, 4), (1, 2, 3). By Theorem 3.1 a nested EB as well as VB design is obtained. The sub-blocks form an EBT as well as VBT design with parameters $v^* = 4, b^* = 24, \rho_1 = 12, \rho_2 = 6, \boldsymbol{r}^* = 24\mathbf{1}_4, \boldsymbol{k}^* = 4\mathbf{1}_{24}$, $\Lambda^* = 20$. On the other hand, the super-blocks form a generalized EBT as well as VBT design with parameters $v_* = 4, b_* = 12, \delta_1 = 3, \delta_2 = 6, \boldsymbol{r}_* = 24\mathbf{1}_4, \boldsymbol{k}_* = 8\mathbf{1}_{12}$, $\Lambda^* = 42$. The nested structure of the blocks of the design is given by

[(1,1,2,3), (1,2,2,3)], [(1,1,2,4), (1,2,2,4)], [(1,1,2,3), (1,2,3,3)],

[(1,1,3,4), (1,3,3,4)], [(1,1,2,4), (1,2,4,4)], [(1,1,3,4), (1,3,4,4)],

[(1,2,2,3), (1,2,3,3)], [(2,2,3,4), (2,3,3,4)], [(1,2,2,4), (1,2,4,4)],

[(2,2,3,4), (2,3,4,4)], [(1,3,3,4), (1,3,4,4)], [(2,3,3,4), (2,3,4,4)],

with $\mu^* = 1/6, \psi^* = 20, \mu_* = 1/8$ and $\psi_* = 21$. Here [ ] shows a super-block.

Remark. For $k = 2$ the super-blocks form a generalized binary VB as well as EB design.

4. Method of construction II

Consider a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ, and take any pair of treatments, say (θ, ϕ), from this design. Then make two blocks from a block which contain the pair (θ, ϕ) as follows.

(i) Repeat the treatment θ two times and eliminate the treatment ϕ while the other $k-2$ remaining treatments in the same block remain as it is.

(ii) Repeat the treatment ϕ two times and eliminate the treatment θ while the other $k-2$ remaining treatments in the same block remain as it is.

Repeating the same procedure for all $\binom{v}{2}$ pairs of treatments and considering the above two blocks obtained as two sub-blocks nested in a super-block, a nested EB as well as VB design can be constructed. The sub-blocks form a ternary design $\mathcal{D}^*$ with parameters v^*, b^*, ρ_1, ρ_2, $\boldsymbol{r}^*$, $\boldsymbol{k}^*$, Λ^*, while the super-blocks form a generalized binary design $\mathcal{D}_*$ with parameters v_*, b_*, $\boldsymbol{r}_*$, $\boldsymbol{k}_*$, Λ_*.

Theorem 4.1. *The existence of a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ implies the existence of a nested EB as well as VB design. The sub-blocks form an EBT as well as VBT design with parameters*

$$v^* = v,\ b^* = v(v-1)\lambda,\ \rho_1 = r(k-1)(k-2),\ \rho_2 = r(k-1),$$

$$\boldsymbol{r}^* = rk(k-1)\mathbf{1}_v,\ \boldsymbol{k}^* = k\mathbf{1}_{b^*},\ \Lambda^* = \lambda(k-2)(k+1),$$

$$\mu^* = \frac{v^*(k^*+2)-(k^*)^2}{(k^*)^2(v^*-1)},\ \psi^* = \frac{\Lambda^* v^*}{k^*},$$

where $k^* = k$, while the super-blocks form a generalized binary EB as well as VB design with parameters

$$v_* = v,\ b_* = \binom{v}{2}\lambda,\ \boldsymbol{r}_* = rk(k-1)\mathbf{1}_v,\ \boldsymbol{k}_* = 2k\mathbf{1}_{b^*},\ \Lambda_* = 2\lambda k(k-1),$$

$$\mu_* = \frac{2v_* - k_*}{k_*(v_*-1)},\ \psi_* = \frac{\Lambda_* v_*}{k_*},$$

where $k_* = 2k$.

Proof. In the BIB design $\mathcal{D}$, the v treatments form $\binom{v}{2}$ pairs and every such pair occurs λ times. Under the method of construction mentioned above, we obtain two sub-blocks corresponding to each pair. Hence in the design $\mathcal{D}^*$, the parameters $v^* = v, b^* = 2\binom{v}{2}\lambda, \boldsymbol{k}^* = k\mathbf{1}_{b^*}$ are obvious from the construction. In the super-blocks, we are merging the treatments of two sub-blocks and, therefore, in the design $\mathcal{D}_*$, the parameters $v_* = v, b_* = \binom{v}{2}\lambda$, $\boldsymbol{k}_* = 2k\mathbf{1}_{b_*}$ are obvious from the construction.

Any treatment, say θ, appears in r blocks of $\mathcal{D}$. Under the construction, the $k-1$ pairs with θ in the same block contribute $2r(k-1)$ times in $\mathcal{D}^*$.

While these $k-1$ treatments other than θ make $\binom{k-1}{2}$ pairs and with these pairs θ occurs $2r\binom{k-1}{2}$ times in $\mathcal{D}^*$. Hence $\boldsymbol{r}^* = \boldsymbol{r}_* = rk(k-1)\mathbf{1}_v$.

For the calculation of Λ in $\mathcal{D}^*$ and $\mathcal{D}_*$, consider a block of $\mathcal{D}$ where any pair (θ, ϕ) occurs. Under the present method of construction corresponding to the following pairs of the blocks of $\mathcal{D}$, we get (θ, ϕ) pairs in $\mathcal{D}^*$ and $\mathcal{D}_*$:

(i) Corresponding to a (θ, ϕ) pair;

(ii) Corresponding to $k-2$ (θ, δ) pairs in the block, where $\delta \neq \phi$;

(iii) Corresponding to $k-2$ (ϕ, δ) pairs in the block, where $\delta \neq \theta$;

(iv) Corresponding to $\binom{k-2}{2}$ (δ, ε) pairs in the block, where $\delta, \varepsilon \notin \{\theta, \phi\}$.

Under (i), no (θ, ϕ) pair occurs in $\mathcal{D}^*$ while the pair occurs 4 times in $\mathcal{D}_*$. Under each of (ii) and (iii), (θ, ϕ) pair occurs $2(k-2)$ times in $\mathcal{D}^*$ while $4(k-2)$ times in $\mathcal{D}_*$. Under (iv), (θ, ϕ) pair occurs $2\binom{k-2}{2}$ times in $\mathcal{D}^*$ while $4\binom{k-2}{2}$ times in $\mathcal{D}_*$.

Since each pair occurs λ times in $\mathcal{D}$, it holds that

$$\Lambda^* = \lambda[4(k-2) + (k-2)(k-3)] = \lambda(k-2)(k+1),$$

$$\Lambda_* = \lambda[4 + 8(k-2) + 2(k-2)(k-3)] = 2\lambda k(k-1).$$

The efficiency and variance for the designs can be calculated as follows. For the sub-block structure, it follows that

$$\boldsymbol{N}^*(\boldsymbol{N}^*)' = r(k-1)(k+2)\boldsymbol{I}_v + \lambda(k+1)(k-2)(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

$$(\boldsymbol{r}^*)^{-\delta}\boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)' = [(k+2)/k^2]\boldsymbol{I}_v + \{(k+1)(k-2)/[k^2(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

which yield $\mu^* = [v^*(k^*+2) - (k^*)^2]/[(k^*)^2(v^*-1)]$, while it holds that

$$(\boldsymbol{r}^*)^{\delta} - \boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)' = [r(k^2-1)(k-2)/k]\boldsymbol{I}_v - [\lambda(k+1)(k-2)/k](\boldsymbol{J}_v - \boldsymbol{I}_v),$$

which shows that $\psi^* = \Lambda^* v^*/k$. For the super-block structure, it follows that

$$\boldsymbol{N}_*(\boldsymbol{N}_*)' = 2rk(k-1)\boldsymbol{I}_v + 2\lambda k(k-1)(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

$$(\boldsymbol{r}_*)^{-\delta}\boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)' = (1/k)\boldsymbol{I}_v + \{(k-1)/[k(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

showing $\mu_* = (2v_* - k_*)/[k_*(v_* - 1)]$, while

$$(\boldsymbol{r}_*)^{\delta} - \boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)' = r(k-1)^2\boldsymbol{I}_v - \lambda(k-1)(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

giving $\psi_* = \Lambda_* v_*/(2k)$. This completes the proof. ■

Example 4.1. For the BIB design given in Example 3.1, Theorem 4.1 provides a nested EB as well as VB design. The sub-blocks form an EBT as well as VBT design with parameters $v^* = 4, b^* = 24, \rho_1 = 6, \rho_2 = 6, \boldsymbol{r}^* = 18\boldsymbol{1}_4, \boldsymbol{k}^* = \boldsymbol{3}\boldsymbol{1}_{24}$, $\Lambda^* = 8$, while the super-blocks form a generalized binary EB as well as VB design with parameters $v_* = 4, b_* = 12, \boldsymbol{r}_* = 18\boldsymbol{1}_4, \boldsymbol{k}_* = \boldsymbol{6}\boldsymbol{1}_{12}$, $\Lambda^* = 24$. The nested structure of the blocks of the design is given by

[(1,1,3), (2,2,3)], [(1,1,4), (2,2,4)], [(1,1,2), (2,3,3)],

[(1,1,4), (3,3,4)], [(1,1,2), (2,4,4)], [(1,1,3), (3,4,4)],

[(1,2,2), (1,3,3)], [(2,2,4), (3,3,4)], [(1,2,2), (1,4,4)],

[(2,2,3), (3,4,4)], [(1,3,3), (1,4,4)], [(2,3,3), (2,4,4)],

with $\mu^* = 11/27, \psi^* = 32/3, \mu_* = 1/9$ and $\psi_* = 16$.

5. Method of construction III

Consider a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ, and take any pair of treatments, say (θ, ϕ), from this design. Then make three blocks from a block which contains the pair (θ, ϕ) as follows.

(i) Repeat the treatment θ two times and eliminate the treatment ϕ while the other $k-2$ remaining treatments in the same block remain as it is.

(ii) Repeat the treatment ϕ two times and eliminate the treatment θ while the other $k-2$ remaining treatments in the same block remain as it is.

(iii) All the treatments including (θ, ϕ) in the block remain as it is.

Repeating the same procedure for all $\binom{v}{2}$ pairs of treatments and considering the above three blocks obtained as three sub-blocks nested in a super-block, a nested EB as well as VB design can be constructed. The sub-blocks form a ternary design $\mathcal{D}^*$ with parameters v^*, b^*, ρ_1, ρ_2, $\boldsymbol{r}^*$, $\boldsymbol{k}^*$, Λ^*, while the super-blocks form a generalized binary design $\mathcal{D}_*$ with parameters v_*, b_*, $\boldsymbol{r}_*$, $\boldsymbol{k}_*$, Λ_*.

Theorem 5.1. *The existence of a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ implies the existence of a nested EB as well as VB design. The sub-blocks form an EBT as well as VBT design with parameters*

$$v^* = v,\ b^* = 3\binom{v}{2}\lambda,\ \rho_1 = \frac{r(k-1)(3k-4)}{2},\ \rho_2 = r(k-1),$$

$$\boldsymbol{r}^* = \frac{3rk(k-1)}{2}\mathbf{1}_v,\ \boldsymbol{k}^* = k\mathbf{1}_{b^*},\ \Lambda^* = \frac{\lambda(3k^2-3k-4)}{2},$$

$$\mu^* = \frac{4v^* + 3k^*(v^*-k^*)}{3(k^*)^2(v^*-1)},\ \psi^* = \frac{\Lambda^* v^*}{k^*},$$

where $k^* = k$, while the super-blocks form a generalized binary EB as well as VB design with parameters

$$v_* = v,\ b_* = \binom{v}{2}\lambda,\ \boldsymbol{r}_* = 3r\binom{k}{2}\mathbf{1}_v,\ \boldsymbol{k}_* = 3k\mathbf{1}_{b^*},\ \Lambda_* = 9\lambda\binom{k}{2},$$

$$\mu_* = \frac{3v_* - k_*}{k_*(v_*-1)},\ \psi_* = \frac{\Lambda_* v_*}{k_*},$$

where $k_* = 3k$.

Proof. In the BIB design $\mathcal{D}$, the v treatments form $\binom{v}{2}$ pairs and every such pair occurs λ times. Under the method of construction in this section, we obtain three sub-blocks corresponding to each pair. Hence in the design $\mathcal{D}^*$, the parameters $v^* = v, b^* = 3\binom{v}{2}\lambda, \boldsymbol{k}^* = k\mathbf{1}_{b^*}$ are obvious from the construction. In the super-blocks, we merge the treatments of three sub-blocks and, therefore, in the design $\mathcal{D}_*$, the parameters $v_* = v, b_* = \binom{v}{2}\lambda$, $\boldsymbol{k}_* = 3k\mathbf{1}_{b_*}$ are obvious from the construction.

Any treatment, say θ, appears in r blocks of $\mathcal{D}$. Under the construction, the $k-1$ pairs with θ in the same block contribute $3r(k-1)$ times in $\mathcal{D}^*$. While these $k-1$ treatments other than θ make $\binom{k-1}{2}$ pairs and with these pairs θ occurs $3r\binom{k-1}{2}$ times in $\mathcal{D}^*$. Hence $\boldsymbol{r}^* = \boldsymbol{r}_* = [3rk(k-1)/2]\mathbf{1}_v$.

For the calculation of Λ in $\mathcal{D}^*$ and $\mathcal{D}_*$, consider a block of $\mathcal{D}$ where any pair (θ, ϕ) occurs. Under the present method of construction corresponding to the following pairs of the blocks of $\mathcal{D}$, we get (θ, ϕ) pairs in $\mathcal{D}^*$ and $\mathcal{D}_*$:

(i) Corresponding to a (θ, ϕ) pair;

(ii) Corresponding to $k-2$ (θ, δ) pairs in the block, where $\delta \neq \phi$;

(iii) Corresponding to $k-2$ (ϕ, δ) pairs in the block, where $\delta \neq \theta$;

(iv) Corresponding to $\binom{k-2}{2}$ (δ, ε) pairs in the block, where $\delta, \varepsilon \notin \{\theta, \phi\}$.

Under (i), (θ, ϕ) pair occurs once in $\mathcal{D}^*$ while the pair occurs 9 times in $\mathcal{D}_*$. Under each of (ii) and (iii), (θ, ϕ) pair occurs $3(k-2)$ times in $\mathcal{D}^*$ while $9(k-2)$ times in $\mathcal{D}_*$. Under (iv), (θ, ϕ) pair occurs $3\binom{k-2}{2}$ times in $\mathcal{D}^*$ while $9\binom{k-2}{2}$ times in $\mathcal{D}_*$.

Since each pair occurs λ times in $\mathcal{D}$, it holds that

$$\Lambda^* = \lambda[1 + 6(k-2) + 3\binom{k-2}{2}] = \lambda(3k^2 - 3k - 4)/2,$$

$$\Lambda_* = \lambda[9 + 18(k-2) + 9\binom{k-2}{2}] = 9\lambda k(k-1)/2.$$

The efficiency and variance can be calculated as follows. For the sub-block structure, it follows that

$$\boldsymbol{N}^*(\boldsymbol{N}^*)' = [r(k-1)(3k+4)/2]\boldsymbol{I}_v + [\lambda(3k^2 - 3k - 4)/2](\boldsymbol{J}_v - \boldsymbol{I}_v),$$

$$(\boldsymbol{r}^*)^{-\delta}\boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)'$$

$$= [(3k+4)/(3k^2)]\boldsymbol{I}_v + \{(3k^2-3k-4)/[3k^2(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

which show $\mu^* = [4v^* + 3k^*(v^* - k^*)]/[3(k^*)^2(v^*-1)]$, while it holds that

$$(\boldsymbol{r}^*)^{\delta} - \boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)'$$

$$= [r(k-1)(3k^2-3k-4)/(2k)]\boldsymbol{I}_v - [\lambda(3k^2-3k-4)/(2k)](\boldsymbol{J}_v - \boldsymbol{I}_v),$$

which yields that $\psi^* = \Lambda^* v^*/k^*$. For the super-block structure, it follows that

$$\boldsymbol{N}_*(\boldsymbol{N}_*)' = [9rk(k-1)/2]\boldsymbol{I}_v + [9\lambda k(k-1)/2](\boldsymbol{J}_v - \boldsymbol{I}_v),$$

$$(\boldsymbol{r}_*)^{-\delta}\boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)' = (1/k)\boldsymbol{I}_v + \{(k-1)/[k(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

showing $\mu_* = (3v_* - k_*)/[k_*(v_*-1)]$, while

$$(\boldsymbol{r}_*)^{\delta} - \boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)' = [3r(k-1)^2/2]\boldsymbol{I}_v - [3\lambda(k-1)/2](\boldsymbol{J}_v - \boldsymbol{I}_v),$$

giving $\psi_* = \Lambda_* v_*/k_*$. This completes the proof. ■

Example 5.1. For the BIB design given in Example 3.1, by Theorem 5.1 a nested EB as well as VB design is obtained. The sub-blocks form an EBT as well as VBT design with parameters $v^* = 4, b^* = 36, \rho_1 = 15, \rho_2 = 6, \boldsymbol{r}^* = 27\boldsymbol{1}_4, \boldsymbol{k}^* = 3\boldsymbol{1}_{36}$, $\Lambda^* = 14$. On the other hand, the super-blocks form a generalized binary EB as well as VB design with parameters $v_* = 4, b_* = 12, \boldsymbol{r}_* = 27\boldsymbol{1}_4, \boldsymbol{k}_* = 9\boldsymbol{1}_{12}$, $\Lambda^* = 54$. The nested structure of the blocks of the design is given by

[(1,1,3), (2,2,3), (1,2,3)], [(1,1,4), (2,2,4), (1,2,4)],

[(1,1,2), (2,3,3), (1,2,3)], [(1,1,4), (3,3,4), (1,3,4)],

[(1,1,2), (2,4,4), (1,2,4)], [(1,1,3), (3,4,4), (1,3,4)],

[(1,2,2), (1,3,3), (1,2,3)], [(2,2,4), (3,3,4), (2,3,4)],

[(1,2,2), (1,4,4), (1,2,4)], [(2,2,3), (3,4,4), (2,3,4)],

[(1,3,3), (1,4,4), (1,3,4)], [(2,3,3), (2,4,4), (2,3,4)],

with $\mu^* = 25/81, \psi^* = 56/3, \mu_* = 1/9$ and $\psi_* = 24$.

6. Method of construction IV

Consider a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ, and take any pair of treatments, say (θ, ϕ), from this design. Then make three blocks from a block which contains the pair (θ, ϕ) as follows.

(i) Repeat the treatment θ as well as ϕ two times while the other $k-2$ remaining treatments in the same block remain as it is.

(ii) Repeat the treatment θ three times while the other $k-1$ remaining treatments in the same block remain as it is.

(iii) Repeat the treatment ϕ three times while the other $k-1$ remaining treatments in the same block remain as it is.

Repeating the same procedure for all $\binom{v}{2}$ pairs of treatments and considering the above three blocks obtained as three sub-blocks nested in a super-block, a nested EB as well as VB design can be constructed. The sub-blocks form a quaternary design $\mathcal{D}^*$ with parameters $v^*, b^*, \rho_1, \rho_2, \rho_3, \boldsymbol{r}^*, \boldsymbol{k}^*, \Lambda^*$, while the super-blocks form a generalized ternary design $\mathcal{D}_*$ with parameters $v_*, b_*, \delta_1, \delta_2, \boldsymbol{r}_*, \boldsymbol{k}_*, \Lambda_*$.

Theorem 6.1. *The existence of a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ implies the existence of a nested EB as well as VB design. The sub-blocks form an EBQ as well as VBQ design with parameters*

$$v^* = v,\ b^* = 3\binom{v}{2}\lambda,\ \rho_1 = \frac{r(k-1)(3k-4)}{2},\ \rho_2 = \rho_3 = r(k-1),$$

$$\boldsymbol{r}^* = \frac{3r(k-1)(k+2)}{2}\mathbf{1}_v,\ \boldsymbol{k}^* = (k+2)\mathbf{1}_{b^*},\ \Lambda^* = \frac{\lambda(3k^2+9k-10)}{2},$$

$$\mu^* = \frac{3k^*(v^*-k^*)+16v^*}{3(k^*)^2(v^*-1)},\ \psi^* = \frac{\Lambda^* v^*}{k^*},$$

where $k^* = k+2$, while the super-blocks form a generalized EBT as well as VBT design with parameters

$$v_* = v,\ b_* = \binom{v}{2}\lambda,\ \delta_1 = \frac{r(k-1)(k-2)}{2},\ \delta_2 = r(k-1),\ \boldsymbol{r}_* = \frac{3r(k-1)(k+2)}{2}\mathbf{1}_v,$$

$$\boldsymbol{k}_* = 3(k+2)\mathbf{1}_{b^*},\ \Lambda_* = \frac{9\lambda(k^2+3k-2)}{2},$$

$$\mu_* = \frac{k_*(3v_*-k_*)+36v_*}{(k_*)^2(v_*-1)},\ \psi_* = \frac{\Lambda_* v_*}{k_*},$$

where $k_* = 3(k+2)$.

Proof. In the BIB design $\mathcal{D}$, the v treatments form $\binom{v}{2}$ pairs and every such pair occurs λ times. Under the method of construction mentioned above, we obtain three sub-blocks corresponding to each pair. Hence in the design $\mathcal{D}^*$, the parameters $v^* = v, b^* = 3\binom{v}{2}\lambda, \boldsymbol{k}^* = (k+2)\mathbf{1}_{b^*}$ are obvious from the construction. In the super-blocks, we merge the treatments of three sub-blocks and, therefore, in the design $\mathcal{D}_*$, the parameters $v_* = v, b_* = \binom{v}{2}\lambda, \boldsymbol{k}_* = 3(k+2)\mathbf{1}_{b_*}$ are obvious from the construction.

Any treatment, say θ, appears in r blocks of $\mathcal{D}$. Under the construction, the $k-1$ pairs with θ in the same block contribute $6r(k-1)$ times in $\mathcal{D}^*$. While these $k-1$ treatments other than θ make $\binom{k-1}{2}$ pairs and with these pairs θ occurs $3r\binom{k-1}{2}$ times in $\mathcal{D}^*$. Hence $\boldsymbol{r}^* = \boldsymbol{r}_* = [3r(k-1)(k+2)/2]\mathbf{1}_v$.

For the calculation of Λ in $\mathcal{D}^*$ and $\mathcal{D}_*$, consider a block of $\mathcal{D}$ where any pair (θ, ϕ) occurs. Under the present method of construction corresponding to the following pairs of the blocks of $\mathcal{D}$, we get (θ, ϕ) pairs in $\mathcal{D}^*$ and $\mathcal{D}_*$:

(i) Corresponding to a (θ, ϕ) pair;

(ii) Corresponding to $k-2$ (θ, δ) pairs in the block, where $\delta \neq \phi$;

(iii) Corresponding to $k-2$ (ϕ, δ) pairs in the block, where $\delta \neq \theta$;

(iv) Corresponding to $\binom{k-2}{2}$ (δ, ε) pairs in the block, where $\delta, \varepsilon \notin \{\theta, \phi\}$.

Under (i), (θ, ϕ) pair occurs 10 times in $\mathcal{D}^*$ while the pair occurs 36 times in $\mathcal{D}_*$. Under (ii), (θ, ϕ) pair occurs $6(k-2)$ times in $\mathcal{D}^*$ while $18(k-2)$ times in $\mathcal{D}_*$. Under (iii), (θ, ϕ) pair occurs $3(k-2)$ times in $\mathcal{D}^*$ while $9(k-2)$ times in $\mathcal{D}_*$. Under (iv), (θ, ϕ) pair occurs $3\binom{k-2}{2}$ times in $\mathcal{D}^*$ while $9\binom{k-2}{2}$ times in $\mathcal{D}_*$.

Since each pair occurs λ times in $\mathcal{D}$, it holds that

$$\Lambda^* = \lambda[10 + 12(k-2) + 3\binom{k-2}{2}] = \lambda(3k^2 + 9k - 10)/2,$$

$$\Lambda_* = \lambda[36 + 36(k-2) + 9\binom{k-2}{2}] = 9\lambda(k^2 + 3k - 2)/2.$$

The efficiency and variance can be calculated as follows. For the sub-block structure, it follows that $\boldsymbol{N}^*(\boldsymbol{N}^*)' = [r(k-1)(3k+22)/2]\boldsymbol{I}_v + [\lambda(3k^2+9k-10)/2](\boldsymbol{J}_v - \boldsymbol{I}_v)$ and $(\boldsymbol{r}^*)^{-\delta}\boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)' = \{(3k+22)/[3(k+2)^2]\}\boldsymbol{I}_v + \{(3k^2+9k-10)/[3(k+2)^2(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v)$, showing $\mu^* = [3k^*(v^* - k^*) + 16v^*]/[3(k^*)^2(v^*-1)]$, while it holds that $(\boldsymbol{r}^*)^{\delta} - \boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)' = \{r(k-1)(3k^2+9k-10)/[2(k+2)]\}\boldsymbol{I}_v - \{\lambda(3k^2+9k-10)/[2(k+2)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v)$, giving $\psi^* = \Lambda^* v^*/k^*$. For the super-block structure, it follows that $\boldsymbol{N}_*(\boldsymbol{N}_*)' = [9r(k+6)(k-1)/2]\boldsymbol{I}_v + [9\lambda(k^2+3k-2)/2](\boldsymbol{J}_v - \boldsymbol{I}_v)$ and $(\boldsymbol{r}_*)^{-\delta}\boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)' = [(k+6)/(k+2)^2]\boldsymbol{I}_v + \{(k^2+3k-2)/[(k-2)^2(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v)$, which yields that $\mu_* = [k_*(3v_* - k_*) + 36v_*]/[(k_*)^2(v_*-1)]$, while $(\boldsymbol{r}_*)^{\delta} - \boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)' = \{3r(k-1)(k^2+3k-2)/[2(k+2)]\boldsymbol{I}_v - \{9\lambda(k^2+3k-2)/[6(k+2)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v)$, which shows that $\psi_* = \Lambda_* v_*/k_*$. This completes the proof. ∎

Remark. For $k = 2$, super-blocks form a generalized binary EB as well as EB design.

Example 6.1. For the BIB design given in Example 3.1, Theorem 6.1 provides a nested EB as well as VB design. The sub-blocks form an EBQ as well as VBQ design with parameters $v^* = 4, b^* = 36, \rho_1 = 15, \rho_2 = \rho_3 = 6, \boldsymbol{r}^* = 45\mathbf{1}_4, \boldsymbol{k}^* = 5\mathbf{1}_{36}$, $\Lambda^* = 44$. On the other hand, the super-blocks form a generalized EBT as well as VBT design with parameters $v_* = 4, b_* = 12, \delta_1 = 3, \delta_2 = 6, \boldsymbol{r}_* = 45\mathbf{1}_4, \boldsymbol{k}_* = 15\mathbf{1}_{12}$, $\Lambda^* = 144$. The nested structure of the blocks of the design is given by

[(1,1,2,2,3), (1,1,1,3,4), (1,2,2,2,3)], [(1,1,2,2,4), (1,1,1,2,4), (1,2,2,2,4)],

[(1,1,2,3,3), (1,1,1,2,3), (1,2,3,3,3)], [(1,1,3,3,4), (1,1,1,3,4), (1,3,3,3,4)],

[(1,1,2,4,4), (1,1,1,2,4), (1,2,4,4,4)], [(1,1,3,4,4), (1,1,1,3,4), (1,3,4,4,4)],

[(1,2,2,3,3), (1,2,2,2,3), (1,2,3,3,3)], [(2,2,3,3,4), (2,2,2,3,4), (2,3,3,3,4)],

[(1,2,2,4,4), (1,2,2,2,4), (1,2,4,4,4)], [(2,2,3,4,4), (2,2,2,3,4), (2,3,4,4,4)],

[(1,3,3,4,4), (1,3,3,3,4), (1,3,4,4,4)], [(2,3,3,4,4), (2,3,3,3,4), (2,3,4,4,4)],

with $\mu^* = 49/225, \psi^* = 176/5, \mu_* = 11/75$ and $\psi_* = 192/5$.

7. Method of construction V

Consider a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ, and take any pair of treatments, say (θ, ϕ), from this design. Then make four blocks from a block which contain the pair (θ, ϕ) as follows.

(i) Repeat the treatment θ twice while the other $k-1$ remaining treatments in the same block remain as it is.

(ii) Repeat the treatment ϕ twice while the other $k-1$ remaining treatments in the same block remain as it is.

(iii) Repeat the treatment θ three times and eliminate the treatment ϕ while the other $k-2$ remaining treatments in the same block remain as it is.

(iv) Repeat the treatment ϕ three times and eliminate the treatment θ while the other $k-2$ remaining treatments in the same block remain as it is.

Repeating the same procedure for all $\binom{v}{2}$ pairs of treatments and considering the above four blocks obtained as four sub-blocks nested in a super-block, a nested EB as well as VB design can be constructed. The sub-blocks form a quaternary design $\mathcal{D}^*$ with parameters $v^*, b^*, \rho_1, \rho_2, \rho_3, \boldsymbol{r}^*, \boldsymbol{k}^*, \Lambda^*$, while the super-blocks form a generalized ternary design $\mathcal{D}_*$ with parameters $v_*, b_*, \delta_1, \delta_2, \boldsymbol{r}_*, \boldsymbol{k}_*, \Lambda_*$.

Theorem 7.1. *The existence of a BIB design $\mathcal{D}$ with parameters v, b, r, k, λ implies the existence of a nested EB as well as VB design. The sub-blocks form an EBQ as well as VBQ design with parameters*

$$v^* = v,\ b^* = 4\binom{v}{2}\lambda,\ \rho_1 = r(k-1)(2k-3),\ \rho_2 = \rho_3 = r(k-1),$$

$$\boldsymbol{r}^* = 2r(k^2-1)\mathbf{1}_v,\ \boldsymbol{k}^* = (k+1)\mathbf{1}_{b^*},\ \Lambda^* = 2\lambda(k^2+k-4),$$

$$\mu^* = \frac{k^*(v^*-k^*)+4v^*}{(k^*)^2(v^*-1)},\ \psi^* = \frac{\Lambda^* v^*}{k^*},$$

where $k^* = k+1$, while the super-blocks form a generalized EBT as well as VBT design with parameters

$$v_* = v,\ b_* = \binom{v}{2}\lambda,\ \delta_1 = \frac{r(k-1)(k-2)}{2},\ \delta_2 = r(k-1),\ \boldsymbol{r}_* = 2r(k^2-1)\mathbf{1}_v,$$

$$\boldsymbol{k}_* = 4(k+1)\mathbf{1}_{b^*},\ \Lambda_* = 4\lambda(2k^2+2k-3),$$

$$\mu_* = \frac{k_*(4v_*-k_*)+24v_*}{(k_*)^2(v_*-1)},\ \psi_* = \frac{\Lambda_* v_*}{k_*},$$

where $k_* = 4(k+1)$.

Proof. In the BIB design $\mathcal{D}$, the v treatments form $\binom{v}{2}$ pairs and every such pair occurs λ times. Under the method of construction mentioned above, we obtain four sub-blocks corresponding to each pair. Hence in the design $\mathcal{D}^*$, the parameters $v^* = v, b^* = 4\binom{v}{2}\lambda, \boldsymbol{k}^* = (k+1)\mathbf{1}_{b^*}$ are obvious from the construction. In the super-blocks, we merge the treatments of four sub-blocks and, therefore, in the design $\mathcal{D}_*$, the parameters $v_* = v, b_* = \binom{v}{2}\lambda, \boldsymbol{k}_* = 4(k+1)\mathbf{1}_{b_*}$ are obvious from the construction.

Any treatment, say θ, appears in r blocks of $\mathcal{D}$. Under the construction, the $k-1$ pairs with θ in the same block contribute $6r(k-1)$ times in $\mathcal{D}^*$. While these $k-1$ treatments other than θ make $\binom{k-1}{2}$ pairs and with these pairs θ occurs $4r\binom{k-1}{2}$ times in $\mathcal{D}^*$. Hence $\boldsymbol{r}^* = \boldsymbol{r}_* = 2r(k^2-1)\mathbf{1}_v$.

For the calculation of Λ in $\mathcal{D}^*$ and $\mathcal{D}_*$, consider a block of $\mathcal{D}$ where any pair (θ, ϕ) occurs. Under the present method of construction corresponding to the following pairs of the blocks of $\mathcal{D}$, we get (θ, ϕ) pairs in $\mathcal{D}^*$ and $\mathcal{D}_*$:

(i) Corresponding to a (θ, ϕ) pair;

(ii) Corresponding to $k-2$ (θ, δ) pairs in the block, where $\delta \neq \phi$;

(iii) Corresponding to $k-2$ (ϕ, δ) pairs in the block, where $\delta \neq \theta$;

(iv) Corresponding to $\binom{k-2}{2}$ (δ, ε) pairs in the block, where $\delta, \varepsilon \notin \{\theta, \phi\}$.

Under (i), (θ, ϕ) pair occurs 4 times in $\mathcal{D}^*$ while the pair occurs 36 times in $\mathcal{D}_*$. Under each of (ii) and (iii), (θ, ϕ) pair occurs $6(k-2)$ times in $\mathcal{D}^*$ while $24(k-2)$ times in $\mathcal{D}_*$. Under (iv), (θ, ϕ) pair occurs $4\binom{k-2}{2}$ times in $\mathcal{D}^*$ while $16\binom{k-2}{2}$ times in $\mathcal{D}_*$.

Since each pair occurs λ times in $\mathcal{D}$, it holds that

$$\Lambda^* = \lambda[4 + 12(k-2) + 4\binom{k-2}{2}] = 2\lambda(k^2 + k - 4),$$

$$\Lambda_* = \lambda[36 + 48(k-2) + 16\binom{k-2}{2}] = 4\lambda(2k^2 + 2k - 3).$$

The efficiency and variance can be calculated as follows. For the sub-block structure, it follows that

$$\boldsymbol{N}^*(\boldsymbol{N}^*)' = 2r(k-1)(k+5)\boldsymbol{I}_v + 2\lambda(k^2+k-4)(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

$$(\boldsymbol{r}^*)^{-\delta}\boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)'$$
$$= [(k+5)/(k+1)^2]\boldsymbol{I}_v + \{(k^2+k-4)/[(k+1)^2(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

which show that $\mu^* = [k^*(v^*-k^*)+4v^*]/[(k^*)^2(v^*-1)]$, while it holds that $(\boldsymbol{r}^*)^{\delta} - \boldsymbol{N}^*(\boldsymbol{k}^*)^{-\delta}(\boldsymbol{N}^*)' = [2r(k-1)(k^2+k-4)/(k+1)]\boldsymbol{I}_v - [\lambda(k^2+k-4)/(k+1)](\boldsymbol{J}_v - \boldsymbol{I}_v)$, giving $\psi^* = \Lambda^* v^*/k^*$. For the super-block structure, it holds that

$$\boldsymbol{N}_*(\boldsymbol{N}_*)' = 4r(2k+5)(k-1)\boldsymbol{I}_v + 4\lambda(2k^2+2k-3)(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

$$(\boldsymbol{r}_*)^{-\delta}\boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)'$$
$$= \{(2k+5)/[2(k+1)^2]\}\boldsymbol{I}_v + \{(2k^2+2k-3)/[2(k+1)^2(v-1)]\}(\boldsymbol{J}_v - \boldsymbol{I}_v),$$

showing $\mu_* = [k_*(4v_*-k_*)+24v_*]/[(k_*)^2(v_*-1)]$, while

$$(\boldsymbol{r}_*)^{\delta} - \boldsymbol{N}_*(\boldsymbol{k}_*)^{-\delta}(\boldsymbol{N}_*)'$$
$$= [r(k-1)(2k^2+2k-3)/(k+1)]\boldsymbol{I}_v - [\lambda(2k^2+2k-3)/(k+1)](\boldsymbol{J}_v - \boldsymbol{I}_v),$$

giving $\psi_* = \Lambda_* v_*/k_*$. This completes the proof. ■

Remark. For $k = 2$, super-blocks form a generalized binary EB as well as EB design.

Example 7.1. For the BIB design given in Example 3.1, by Theorem 7.1 a nested EB as well as VB design is obtained. The sub-blocks form an EBQ as well as VBQ design with parameters $v^* = 4, b^* = 48, \rho_1 = 18, \rho_2 = \rho_3 = 6, \boldsymbol{r}^* = 48\mathbf{1}_4, \boldsymbol{k}^* = 4\mathbf{1}_{48}$, $\Lambda^* = 32$. On the other hand, the super-blocks form a generalized EBT as well as VBT design with parameters $v_* = 4, b_* = 12, \delta_1 = 3, \delta_2 = 6, \boldsymbol{r}_* = 48\mathbf{1}_4, \boldsymbol{k}_* = 16\mathbf{1}_{12}$, $\Lambda^* = 168$. The nested structure of the blocks of the design is given by

[(1,1,2,3), (1,2,2,3), (1,1,1,3), (2,2,2,3)],

[(1,1,2,4), (1,2,2,4), (1,1,1,4), (2,2,2,4)],

[(1,1,2,3), (1,2,3,3), (1,1,1,2), (2,3,3,3)],

[(1,1,3,4), (1,3,3,4), (1,1,1,4), (3,3,3,4)],

[(1,1,2,4), (1,2,4,4), (1,1,1,2), (2,4,4,4)],

[(1,1,3,4), (1,3,4,4), (1,1,1,3), (3,4,4,4)],

[(1,2,2,3), (1,2,3,3), (1,2,2,2), (1,3,3,3)],

[(2,2,3,4), (2,3,3,4), (2,2,2,4), (3,3,3,4)],

[(1,2,2,4), (1,2,4,4), (1,2,2,2), (1,4,4,4)],

[(2,2,3,4), (2,3,4,4), (2,2,2,3), (3,4,4,4)],

[(1,3,3,4), (1,3,4,4), (1,3,3,3), (1,4,4,4)],

[(2,3,3,4), (2,3,4,4), (2,3,3,3), (2,4,4,4)],

with $\mu^* = 1/3, \psi^* = 32, \mu_* = 1/8$ and $\psi_* = 42$.

Finally, a comparison between sub-blocks and super-blocks is given in Table 1, in which $\boldsymbol{r}^* = r^*\mathbf{1}_{v^*}$, $\boldsymbol{k}^* = k^*\mathbf{1}_{b^*}$, $\boldsymbol{r}_* = r_*\mathbf{1}_{v_*}$ and $\boldsymbol{k}_* = k_*\mathbf{1}_{b_*}$.

No	v^*	b^*	ρ_1	ρ_2	ρ_3	r^*	k^*	Λ^*	μ^*	ψ^*	v_*	b_*	δ_1	δ_2	r_*	k_*	Λ_*	μ_*	ψ_*	Theor
1	4	12	3	3		9	3	4	0.407	5.33	4	6	0	3	9	6	9	0.333	6	3.1
2	4	18	3	3		9	2	1	0.778	2	4	6			9	6	9	0.333	6	5.1
3	4	18	3	3	3	18	4	10	0.444	10	4	6	0	3	18	12	36	0.333	12	6.1
4	4	24	6	6		18	3	8	0.407	10.7	4	12			18	6	24	0.111	16	4.1
5	4	24	3	3	3	18	3	4	0.704	5.33	4	6	0	3	18	12	36	0.333	12	7.1
6	4	24	12	6		24	4	20	0.167	20	4	12	3	6	24	8	42	0.125	21	3.1
7	4	36	15	6		27	3	14	0.309	18.7	4	12			27	9	54	0.111	24	5.1
8	5	20	4	4		12	3	4	0.444	6.67	5	10	0	4	12	6	9	0.375	7.5	3.1
9	5	30	4	4		12	2	1	0.792	2.5	5	10			12	6	9	0.375	7.5	5.1
10	5	30	4	4	4	24	4	10	0.479	12.5	5	10	0	4	24	12	36	0.375	15	6.1
11	5	40	4	4	4	24	3	4	0.722	6.67	5	10	0	4	24	12	36	0.375	15	7.1
12	6	30	5	5		15	3	4	0.467	8	6	15	0	5	15	6	9	0.4	9	3.1
13	6	45	5	5		15	2	1	0.8	3	6	15			15	6	9	0.4	9	5.1
14	6	45	5	5	5	30	4	10	0.5	15	6	15	0	5	30	12	36	0.4	18	6.1
15	6	60	6	10		30	3	8	0.467	16	6	30			30	6	24	0.2	24	4.1
16	6	60	5	5	5	30	3	4	0.733	8	6	15	0	5	30	12	36	0.4	18	7.1
17	7	42	6	6		18	3	4	0.481	9.33	7	21			18	6	12	0.222	14	4.1
18	7	42	6	6		18	3	4	0.481	9.33	7	21	0	6	18	6	9	0.417	10.5	3.1
19	7	63	6	6		18	2	1	0.806	3.5	7	21			18	6	9	0.417	10.5	5.1
20	7	42	12	6		24	4	10	0.271	17.5	7	21	3	6	24	8	21	0.234	18.4	3.1
21	7	63	15	6		27	3	7	0.395	16.3	7	21			27	9	27	0.222	21	5.1
22	8	56	7	7		21	3	4	0.492	10.7	8	28	0	7	21	6	9	0.429	12	3.1
23	8	84	7	7		21	2	1	0.81	4	8	28			21	6	9	0.429	12	5.1
24	9	72	6	8		24	3	4	0.5	12	9	36			24	6	12	0.25	18	4.1
25	9	72	8	8		24	3	4	0.5	12	9	36	0	8	24	6	9	0.438	13.5	3.1
26	9	108	8	8		24	2	1	0.813	4.5	9	36			24	6	9	0.438	13.5	5.1
27	10	90	9	9		27	3	4	0.506	13.3	10	45	0	9	27	6	9	0.444	15	3.1
28	10	135	9	9		27	2	1	0.815	5	10	45			27	6	9	0.444	15	5.1
29	11	110	10	10		30	3	4	0.511	14.7	11	55	0	10	30	6	9	0.45	16.5	3.1
30	11	165	10	10		30	2	1	0.817	5.5	11	55			30	6	9	0.45	16.5	5.1

Table 1. Comparison of sub- and super-structure of balanced designs with $r \leq 30$

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