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ON THE BAYES ESTIMATORS OF THE PARAMETERS OF INFLATED MODIFIED POWER SERIES DISTRIBUTIONS

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Abstract

In this paper, we study the class of inflated modified power series distributions (IMPSD) where inflation occurs at any of support points. This class includes among others the generalized Poisson, the generalized negative binomial and the lost games distributions. We derive the Bayes estimators of parameters for these distributions when a parameter of inflation is known. First, we take as the prior distribution the uniform, Beta and Gamma distribution. In the second part of this paper, the prior distribution is the generalized Pareto distribution.

Keywords and phrases: posterior distributions; posterior moments; Bayes estimator; inflated distribution; generalized Pareto distribution; generalized Poisson distribution; generalized negative binomial distribution; lost games distribution.

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1. Introduction

In this note, we consider a mixed population consisting of two groups of individuals. The individuals of the first group follow the simple distribution, while those of the second group are observed with a frequency significantly higher than can be expected on the basis of the simple distribution. These random phenomenae are well described by inflated probability distributions. We assume that these simple distributions can be generated by power series.

Pandey (1964–65) has described a situation with an inflated Poisson distribution dealing with the numbers of flowers of plants of Primula veris (cf. Cramer (1945)). He has shown that the excessive number of plants with eight flowers implies application of Poisson distribution inflated at the point 8 (not zero).

One can also describe such situations in the following gambler's ruin problem. Let us consider two gamblers, one of whom (gambler A) is infinitely rich and the other (gambler B) is starting with a monetary units. We observe now joint number of games lost by these two gamblers against an infinitely rich adversary (gambler C), without knowing who of them, the gambler A or the gambler B, takes the game. In this case, the observed number of games lost is inflated at the point a.

One can also find non-zero inflated distribution in the following queue M/M/1 problem. Let us consider two queues, one of which is infinitely long (queue A) and the other (queue B) starting with a customers. We observe the joint number of customers served in a busy period of these two queues, without knowing which of them is served. In this case, the observed number of served customers in a busy period is inflated at the point a.

Gupta, Gupta and Tripathi (1995) gave the maximum likelihood estimators and their asymptotic variance-covariance matrix of zero inflated modified power series distribution. Our aim is to give Bayes estimators of the parameter θ of non-zero inflated modified power series distribution represented by the following probability function

$$p(x;\theta) = \begin{cases} \beta + \alpha a(s)[g(\theta)]^s[f(\theta)]^{-1}, & x = s, \\ \alpha a(x)[g(\theta)]^x[f(\theta)]^{-1}, & x \neq s, \end{cases} \tag{1.1}$$

where $s \geq 0$, $0 < \alpha \leq 1$, $\beta = 1 - \alpha$, functions $f(\theta) = \sum_{x=0}^{\infty} a(x)[g(\theta)]^x$, $g(\theta)$ are positive, finite and differentiable, and $a(x) > 0$ are free of θ.

In Section 2, we consider the problem of estimating θ with the quadratic loss function and the prior distribution Gamma, Beta or uniform, while in Section 3, the prior distribution is generalized Pareto distribution. In these Sections, we assume that the parameter of inflation α is known.

In Section 4, we derive a posterior distribution and posterior moments of the parameter θ based on the likelihood function.

2. THE BAYES ESTIMATORS OF θ

Let us consider the problem of estimating θ with quadratic loss function based on the simple observation from (1.1). Then we have

Theorem 2.1. *Let X be a random variable with a modified power series distribution inflated at the point s.*

(i) *If the prior distribution of θ is the Beta distribution with parameters $a > 0$ and $b > 0$, then the posterior moments of θ are given by the formula*

$$E[\theta^r \mid x] = \begin{cases} \dfrac{\beta B(r+a,b) + \alpha a(s)\int_0^1 \theta^{r+a-1}(1-\theta)^{b-1}[g(\theta)]^s[f(\theta)]^{-1}d\theta}{\beta B(a,b) + \alpha a(s)\int_0^1 \theta^{a-1}(1-\theta)^{b-1}[g(\theta)]^s[f(\theta)]^{-1}d\theta}, & x = s, \\ \dfrac{\int_0^1 \theta^{r+a-1}(1-\theta)^{b-1}[g(\theta)]^x[f(\theta)]^{-1}d\theta}{\int_0^1 \theta^{a-1}(1-\theta)^{b-1}[g(\theta)]^x[f(\theta)]^{-1}d\theta}, & x \neq s, \end{cases}$$

where $B(a,b) = \int_0^1 \theta^{a-1}(1-\theta)^{b-1}d\theta.$

(ii) *If the prior distribution of θ is the Gamma distribution with parameters $p > 0$ and $b > 0$, then the posterior moments of θ are given by the formula*

$$E[\theta^r \mid x] = \begin{cases} \dfrac{\beta\Gamma(p+r) + \alpha a(s)b^{p+r}\int_0^\infty \theta^{p+r-1}e^{-\theta b}[g(\theta)]^s[f(\theta)]^{-1}d\theta}{\beta b^r\Gamma(p) + \alpha a(s)b^{p+r}\int_0^\infty \theta^{p-1}e^{-\theta b}[g(\theta)]^s[f(\theta)]^{-1}d\theta}, & x = s, \\ \dfrac{\int_0^\infty \theta^{p+r-1}e^{-\theta b}[g(\theta)]^x[f(\theta)]^{-1}d\theta}{\int_0^\infty \theta^{p-1}e^{-\theta b}[g(\theta)]^x[f(\theta)]^{-1}d\theta}, & x \neq s, \end{cases}$$

where $\Gamma(p) = \int_0^\infty \theta^{p-1}e^{-\theta}d\theta.$

Proof. Let the prior distribution of θ be the Beta distribution with parameters $a > 0$ and $b > 0$, given as

$$\phi(\theta) = \begin{cases} \dfrac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)}, & 0 < \theta < 1, \\ 0, & \theta \in (-\infty, 0 > \cup < 1, \infty). \end{cases}$$

In this case

$$q(x;\theta) = \begin{cases} \Big(\beta + \alpha a(s)[g(\theta)]^s[f(\theta)]^{-1}\Big)\, \dfrac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)}, & x = s,\ 0 < \theta < 1, \\ \alpha a(x)[g(\theta)]^x[f(\theta)]^{-1} \dfrac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)}, & x \neq s,\ 0 < \theta < 1. \end{cases}$$

Hence the probability function of θ given x has the form

$$h(\theta \mid x) = \begin{cases} \dfrac{[\beta + \alpha a(s)[g(\theta)]^s[f(\theta)]^{-1}]\theta^{a-1}(1-\theta)^{b-1}}{\beta B(a,b) + \alpha a(s)\int_0^1 \theta^{a-1}(1-\theta)^{b-1}[g(\theta)]^s[f(\theta)]^{-1}d\theta}, & x = s, \\ \dfrac{[g(\theta)]^x[f(\theta)]^{-1}\theta^{a-1}(1-\theta)^{b-1}}{\int_0^1 \theta^{a-1}(1-\theta)^{b-1}[g(\theta)]^x[f(\theta)]^{-1}d\theta}, & x \neq s \end{cases}$$

and the posterior moments are given by (i).

To obtain the posterior moments from (ii) we observe that if the prior distribution of θ is gamma distribution with the probability function

$$\phi(\theta) = \begin{cases} \dfrac{b^p}{\Gamma(p)}\theta^{p-1}e^{-b\theta}, & \theta > 0, \\ 0, & \theta \le 0, \end{cases}$$

then the probability function of θ given x is

$$h(\theta \mid x) = \begin{cases} \dfrac{[\beta + \alpha a(s)[g(\theta)]^s[f(\theta)]^{-1}]b^p\theta^{p-1}e^{-b\theta}}{\beta\Gamma(p) + \alpha a(s)b^p \int_0^\infty \theta^{p-1}e^{-b\theta}[g(\theta)]^s[f(\theta)]^{-1}d\theta}, & x = s, \\ \dfrac{[g(\theta)]^x[f(\theta)]^{-1}\theta^{p-1}e^{-b\theta}}{\int_0^\infty \theta^{p-1}e^{-b\theta}[g(\theta)]^x[f(\theta)]^{-1}d\theta}, & x \neq s. \end{cases}$$

This ends the proof.

It is interesting to note that putting in (i) $a = 1$ and $b = 1$ we get the following result.

Corollary 2.1. *If the prior distribution of* θ *is the uniform distribution on* $(0, 1)$, *then the posterior moments of* θ *are given by the formula*

$$E[\theta^r \mid x] = \begin{cases} \dfrac{\beta + (r+1)\alpha a(s) \int_0^1 \theta^r [g(\theta)]^s [f(\theta)]^{-1} d\theta}{(r+1)\left[\beta + \alpha a(s) \int_0^1 [g(\theta)]^s [f(\theta)]^{-1} d\theta\right]}, & x = s, \\ \dfrac{\int_0^1 \theta^r [g(\theta)]^x [f(\theta)]^{-1} d\theta}{\int_0^1 [g(\theta)]^x [f(\theta)]^{-1} d\theta}, & x \neq s. \end{cases}$$

One can see that putting in the above formulae $r = 1$, we obtain the Bayes estimators $\delta(x) = E[\theta \mid x]$ of parameter θ of IMPSD, when the parameter of inflation α is known.

Now we give some examples.

Example 2.2. The generalized negative binomial probability function inflated at the point $s \geq 0$

$$p(x;\theta)=\begin{cases}1-\alpha+\alpha\dfrac{m\Gamma(m+\gamma x)[\theta(1-\theta)^{\gamma-1}]^x}{x!\Gamma(m+\gamma x-x+1)(1-\theta)^{-m}}, & x=s,\\ \alpha\dfrac{m\Gamma(m+\gamma x)[\theta(1-\theta)^{\gamma-1}]^x}{x!\Gamma(m+\gamma x-x+1)(1-\theta)^{-m}}, & x\neq s,\end{cases}$$

where $0<\theta<1, \mid \theta\gamma \mid< 1$, can be represented in the form (1.1) with

$$a(x)=\frac{m\Gamma(n+\gamma x)}{x!\Gamma(m+\gamma x-x+1)},\ f(\theta)=(1-\theta)^{-m},\ g(\theta)=\theta(1-\theta)^{\gamma-1}.$$

For this probability function we obtain the following results.

(a) If the prior distribution of θ is the uniform distribuiton, then the Bayes estimator of θ has the form

$$\delta(x)=\begin{cases}\dfrac{\beta\Gamma(\gamma s+k+3)+2\alpha k(s+1)\Gamma(k+\gamma s)}{2\beta\Gamma(\gamma s+k+3)+2\alpha k\Gamma(k+\gamma s)(\gamma s+k+2)}, & x=s,\\ \dfrac{x+1}{\gamma x+k+2}, & x\neq s.\end{cases}$$

(b) If the prior distribution of θ is the Beta distribution with the parameters $a>0,\ b>0$, then the Bayes estimator of θ has the form

$$\delta(x)=\begin{cases}\dfrac{\beta s!\Gamma(\gamma s-s+k+1)B(a+1,b)+\alpha k\Gamma(\gamma s+k)B(a+s+1,\gamma s-s+k+b)}{\beta k!\Gamma(\gamma s-s+k+1)B(a,b)+\alpha k\Gamma(\gamma s+k)B(a+s,\gamma s-s+k+b)}, & x=s,\\ \dfrac{x+a}{a+b+k+\gamma x}, & x\neq s.\end{cases}$$

Example 2.3. The generalized Poisson probability function inflated at the point s

$$p(x;\theta)=\begin{cases}1-\alpha+\alpha\theta^x(1+\gamma x)^{x-1}e^{-\theta(1+\gamma x)}/x!, & x=s,\\ \alpha\theta^x(1+\gamma x)^{x-1}e^{-\theta(1+\gamma x)}/x!, & x\neq s,\end{cases}$$

where $|\theta\gamma| < 1$ can be represented in the form (1.1) with

$$a(x) = \frac{(1+\gamma x)^{x-1}}{x!},\ f(\theta) = e^{\theta},\ g(\theta) = \theta e^{-\gamma\theta}.$$

If the prior distribution of θ is the Gamma distribution with the parameters $p > 0$ and $b > 0$, then the Bayes estimator of θ has the form

$$\delta(x) = \begin{cases} \dfrac{s!\beta\Gamma(p+1)(b+\gamma s+1)^{p+s+1} + \alpha(1+\gamma s)^{s-1} b^{p+1}\Gamma(p+s+1)}{s!\beta\Gamma(p)(b+\gamma s+1)^{p+s+1} + \alpha(1+\gamma s)^{s-1} b^{p}\Gamma(p+s)(b+\gamma s+1)}, & x = s, \\ \dfrac{x+p}{1+b+\gamma x}, & x \neq s. \end{cases}$$

In practice, we use n observations of a sample $X_1, X_2, ..., X_n$. To give the above estimators in this case we can apply the distribution of sum $Z_n = X_1 + X_2 + ... + X_n$. The probability function of Z_n is known when the inflation occurs at the first point of the support of X (cf. Gupta, Gupta and Tripathi (1995) for $s = 0$; Murat and Szynal (1998) for $s \geq 0$).

Theorem 2.4. *The distribution of Z_n is given by*

$$P[Z_n = z] = \begin{cases} \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \frac{a_k(ks)[g(\theta)]^{ks}}{[f(\theta)]^k}, & z = ns, \\ \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \frac{a_k(z-(n-k)s)[g(\theta)]^{z-(n-k)s}}{[f(\theta)]^k}, & z = ns+1, ns+2, ..., \end{cases}$$

where $a_0(0) = 1$, $a_1(x) = a(x)$, $a_k(x) = \sum_{y=s}^{x} a(y) a_{k-1}(x-y)$, $k = 2, 3, ..., n$, are coefficients of $[g(\theta)]^x$ in $[f(\theta)]^k$.

Assume now that

$$(2.1) \qquad p(x;\theta) = \begin{cases} \beta + \alpha a(s)[g(\theta)]^s [f(\theta)]^{-1}, & x = s, \\ \alpha a(x)[g(\theta)]^x [f(\theta)]^{-1}, & x = s+1, s+2 \ ... \ . \end{cases}$$

Using Theorem 2.4 we get

Theorem 2.5. *Let $X_1, X_2, ..., X_n$ be a sample from* (2.1).

(i) *If the prior distribution of θ is the uniform distribution on* $(0, 1)$, *then the posterior moments of θ are given by the following formula*

$$E[\theta^r \,|\, \boldsymbol{x}] = \begin{cases} \dfrac{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(ks) \int\limits_0^1 \theta^r [g(\theta)]^{ks} [f(\theta)]^{-k} d\theta}{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(ks) \int\limits_0^1 [g(\theta)]^{ks} [f(\theta)]^{-k} d\theta}, & z = ns, \\ \dfrac{\sum\limits_{k=1}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(z-(n-k)s) \int\limits_0^1 \theta^r [g(\theta)]^{z-(n-k)s} [f(\theta)]^{-k} d\theta}{\sum\limits_{k=1}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(z-(n-k)s) \int\limits_0^1 [g(\theta)]^{z-(n-k)s} [f(\theta)]^{-k} d\theta}, & z > ns. \end{cases}$$

(ii) *If the prior distribution of θ is the Beta distribution with parameters $a > 0$ and $b > 0$, then the posterior moments of θ are given by the formula*

$$E[\theta^r | \boldsymbol{x}] = \begin{cases} \dfrac{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(ks) \int\limits_0^1 \theta^{a+r-1} (1-\theta)^{b-1} [g(\theta)]^{ks} [f(\theta)]^{-k} d\theta}{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(ks) \int\limits_0^1 \theta^{a-1} (1-\theta)^{b-1} [g(\theta)]^{ks} [f(\theta)]^{-k} d\theta}, & z = ns, \\ \dfrac{\sum\limits_{k=1}^{n} \binom{n}{k} \alpha^{n-k} \beta^{n-k} a_k(z-(n-k)s) \int\limits_0^1 \theta^{a+r-1} (1-\theta)^{b-1} [g(\theta)]^{z-(n-k)s} [f(\theta)]^{-k} d\theta}{\sum\limits_{k=1}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(z-(n-k)s) \int\limits_0^1 \theta^{a-1} (1-\theta)^{b-1} [g(\theta)]^{z-(n-k)s} [f(\theta)]^{-k} d\theta}, & z > ns. \end{cases}$$

(iii) *If the prior distribution of θ is the Gamma distribution with parameters $p > 0$ and $b > 0$, then the posterior moments of θ are given by the formula*

$$E[\theta^r \,|\, \boldsymbol{x}] = \begin{cases} \dfrac{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(ks) \int\limits_0^{\infty} \theta^{p+r-1} e^{-\theta b} [g(\theta)]^k s [f(\theta)]^{-k} \theta}{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(ks) \int\limits_0^{1} \theta^{p-1} e^{-\theta b} [g(\theta)]^k s [f(\theta)]^{-k} d\theta}, & z = ns, \\ \dfrac{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(z-(n-k)s) \int\limits_0^{1} \theta^{p+r-1} e^{-\theta b} [g(\theta)]^{z-(n-k)s} [f(\theta)]^{-k} d\theta}{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k(z-(n-k)s) \int\limits_0^{1} \theta^{p-1} e^{-\theta b} [g(\theta)]^{z-(n-k)s} [f(\theta)]^{-k} d\theta}, & z > ns. \end{cases}$$

From (i) – (iii), putting $r = 1$, we get the Bayes estimator $\delta(z) = E[\theta \mid \boldsymbol{x}]$ of parameter θ when the parameter of inflation α is known.

Let us consider some examples.

Example 2.6. If a sample $X_1, X_2, ..., X_n$ is from the inflated binomial distribution, then from Gupta, Gupta and Tripathi (1995) we have

$$a_k(z) = \frac{km\Gamma(\gamma z + km)}{\Gamma(z+1)\Gamma(\gamma z + km - x + 1)}.$$

(a) Assuming that the prior distribution of θ is the uniform distribution on $(0, 1)$ we get

$$\delta(z) = \begin{cases} \dfrac{s \sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \frac{k\Gamma(\gamma ks+km)\Gamma(m(n-k)+sk(\gamma-1))}{\Gamma(\gamma ks+km-ks+1)\Gamma(m(n-k)+\gamma ks+1)}}{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \frac{\Gamma(\gamma ks+km)\Gamma(m(n-k)+sk(\gamma-1))}{\Gamma(\gamma ks+km-ks+1)\Gamma(m(n-k)+\gamma ks)}}, & z=ns, \\ \dfrac{(z+1) \sum\limits_{k=1}^{n} \binom{n}{k} \alpha^k \beta^{n-k} [(\gamma z+m+2)(\gamma z+m+1)(\gamma z+m)]^{-1}}{\sum\limits_{k=1}^{n} \binom{n}{k} \alpha^k \beta^{n-k} [(\gamma z+m+1)(\gamma z+m)]^{-1}}, & z>ns. \end{cases}$$

(b) If the prior distribution of θ is the Beta distribution with the parameters $a > 0,\ b > 0$, then we have

$$\delta(z)=\begin{cases}\dfrac{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}\frac{k\gamma(\gamma ks+km)B(a+ks+1,b+m(n-k)+ks(\gamma-1))}{\Gamma(ks+1)\Gamma(\gamma ks+km-ks+1)}}{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}\frac{k\gamma(\gamma ks+km)B(a+ks,b+m(n-k)+ks(\gamma-1))}{\Gamma(ks+1)\Gamma(\gamma ks+km-ks+1)}}, & z=ns,\\ \dfrac{\sum\limits_{k=1}^{n}\binom{n}{k}\alpha^k\beta^{n-k}B(a+1,\gamma z-z+b+km)B(1-z,km+\gamma z)}{\sum\limits_{k=1}^{n}\binom{n}{k}\alpha^k\beta^{n-k}B(a,\gamma z-z+b+km)B(1-z,km+\gamma z)}, & z>ns.\end{cases}$$

Example 2.7. If a sample $X_1, X_2, ..., X_n$ is from the inflated generalized Poisson distribution, then from Gupta, Gupta and Tripathi (1995) we have

$$a_k(z) = \frac{k(k+\gamma z)^{z-1}}{z!}.$$

If the prior distribution of θ is the Gamma distribution with the parameters $p > 0$ and $b > 0$, then we obtain

$$\delta(z)=\begin{cases}\dfrac{p\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}\frac{k^{z-1}}{(ks-1)!}\frac{\Gamma(p+ks+1)}{(b+n-k+\gamma ks)^{p+ks+1}}}{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}\frac{k^{z-1}}{(ks-1)!}\frac{\Gamma(p+ks)}{(b+n-k+\gamma ks)^{p+ks}}}, & z=ns,\\ \dfrac{(z+p)\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}k(k+\gamma z)^{z-1}(k+b+\gamma z)^{p+z+1}}{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}k(k+\gamma z)^{z-1}(k+b+\gamma z)^{p+z}}, & z>ns.\end{cases}$$

3. Generalized pareto distribution as a prior distribution

In this Section, we assume that the prior distribution of θ is the generalized Pareto distribution, given by the following probability function

$$\phi=\begin{cases}(1+b\theta)^{-1-\frac{1}{b}}, & \theta\geq 0, & b>0,\\ e^{-\theta}, & \theta\geq 0, & b=0\\ (1+b\theta)^{-1-\frac{1}{b}}, & \theta\in(0,-\frac{1}{b}), & b<0,\end{cases} \tag{3.1}$$

(cf. Pikands (1975) and Ahsanullah (1994)).

In this case we obtain what follows.

Theorem 3.1. *Let X be an observation from modified power series distribution inflated at the point s. If the prior distribution of θ is the distribution* (3.1), *then the posterior moments of θ are given by the following formula*

(i) *for* $b > 0$

$$E[\theta^r \mid \boldsymbol{x}] = \begin{cases} \dfrac{C\beta + \alpha a(s) \int_0^\infty \theta^r (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^s [f(\theta)]^{-1} d\theta}{\beta + \alpha a(s) \int_0^\infty (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^s [f(\theta)]^{-1} d\theta}, & x = s, \\[2ex] \dfrac{\int_0^\infty \theta^r (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^x [f(\theta)]^{-1} d\theta}{\int_0^\infty (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^x [f(\theta)]^{-1} d\theta}, & x \neq s; \end{cases}$$

(ii) *for* $b = 0$

$$E[\theta^r \mid \boldsymbol{x}] = \begin{cases} \dfrac{\beta\Gamma(r+1) + \alpha a(s) \int_0^\infty \theta^r e^{-\theta} [g(\theta)]^s [f(\theta)]^{-1} d\theta}{\beta + \alpha a(s) \int_0^\infty e^{-\theta} [g(\theta)]^s [f(\theta)]^{-1} d\theta}, & x = s, \\[2ex] \dfrac{\int_0^\infty \theta^r e^{-\theta} [g(\theta)]^x [f(\theta)]^{-1} d\theta}{\int_0^\infty e^{-\theta} [g(\theta)]^x [f(\theta)]^{-1} d\theta}, & x \neq s; \end{cases}$$

(iii) *for* $b < 0$

$$E[\theta^r \mid \boldsymbol{x}] = \begin{cases} \dfrac{C\beta + \alpha a(s) \int_0^{-\frac{1}{b}} \theta^r (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^s [f(\theta)]^{-1} d\theta}{\beta + \alpha a(s) \int_0^{-\frac{1}{b}} (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^s [f(\theta)]^{-1} d\theta}, & x = s, \\[2ex] \dfrac{\int_0^{-\frac{1}{b}} \theta^r (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^x [f(\theta)]^{-1} d\theta}{\int_0^{-\frac{1}{b}} (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^x [f(\theta)]^{-1} d\theta}, & x \neq s, \end{cases}$$

where $C = \dfrac{(-1)^r r!}{(rb-1)((r-1)b-1)...(2b-1)(b-1)}$.

Now we consider some special cases.

Example 3.2. Let X be a random variable with generalized Poisson distribution inflated at the point s and let the prior distribution of θ be the generalized Pareto distribution with $b = 0$. Then the Bayes estimator of θ is given by the formula

$$\delta(x) = \begin{cases} \dfrac{\beta(\gamma s+2)^{s+2} + \alpha(1+\gamma s)^{s-1}(s+1)}{\beta(\gamma s+2)^{s+2} + \alpha(1+\gamma s)^{s-1}(\gamma s+2)}, & x = s, \\ \dfrac{x+1}{\gamma x+2}, & x \neq s. \end{cases}$$

Example 3.3. Let X be an observation from lost games distribution inflated at the point γ with the folowing probability function

$$p(x;\theta) = \begin{cases} \beta + \alpha\frac{\gamma}{2x-\gamma}\binom{2x-\gamma}{x}\frac{(\theta(1-\theta))^x}{\theta^\gamma}, & x = s, \\ \alpha\frac{\gamma}{2x-\gamma}\binom{2x-\gamma}{x}\frac{(\theta(1-\theta))^x}{\theta^\gamma}, & x \neq s, \end{cases}$$

where $0 < \theta < \frac{1}{2}, \gamma \geq 1$.

One can see that for $\alpha = 1$ we get the lost games distribution considered by Janardan (1984) and Kemp and Kemp (1968).

Let the prior distribution of θ be the generalized Pareto distribution with $b = -2$. Then the Bayes estimator of θ is given by the folowing formula

$$\delta(x) = \begin{cases} \dfrac{(2s-\gamma)2^{s-\gamma+2}\Gamma\left(s-\gamma+\frac{5}{2}\right)\beta + F_2}{3(2s-\gamma)2^{s-\gamma+2}\Gamma\left(s-\gamma+\frac{5}{2}\right)\beta + 2\left(s-\gamma+\frac{3}{2}\right)F_1}, & x = s, \\ \dfrac{(x-\gamma+1)F\left(-x, x-\gamma+2, x-\gamma+\frac{5}{2}, \frac{1}{2}\right)}{(2x-2\gamma+3)F\left(-x, x-\gamma+1, x-\gamma+\frac{3}{2}, \frac{1}{2}\right)}, & x \neq s, \end{cases}$$

where

$$F_i = F\left(\gamma(n-k)-z, z-n\gamma+i, z-n\gamma+i+\frac{1}{2}, \frac{1}{2}\right), \ i=1,2,$$

and $F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$ is a hypergeometric function with parameters a, b, c.

Example 3.4. Let X be an observation from distribution of the number of customers served in a busy period of the queue M/M/1 inflated at the point γ with the folowing probability function

$$p(x;\theta) = \begin{cases} \beta + \alpha\frac{1}{2\gamma-1}\binom{2\gamma-1}{\gamma}\left[\frac{\theta}{(1+\theta)^2}\right]^x\left(\frac{\theta}{1+\theta}\right)^\gamma, & x = s, \\ \alpha\frac{1}{2x-\gamma}\binom{2x-\gamma}{x}\left[\frac{\theta}{(1+\theta)^2}\right]^x\left(\frac{\theta}{1+\theta}\right)^\gamma, & x \neq s, \end{cases}$$

where $0 < \theta < \frac{1}{2}, \gamma \geq 1$.

One can see that for $\alpha = 1$ we get the distribution considered by Kemp and Kemp (1968).

Let the prior distribution of θ be the generalized Pareto distribution with $b = -2$. Then the Bayes estimator of θ is given by the folowing formula

$$\delta(x) = \begin{cases} \dfrac{(2s-\gamma)2^{\gamma-2s+2}\Gamma\left(\gamma-2s+\dfrac{5}{2}\right)\beta + F_2}{3(2s-\gamma)2^{\gamma-2s+2}\Gamma\left(\gamma-2s+\dfrac{5}{2}\right)\beta + (4s+2\gamma+3)F_1}, & x = s, \\ \dfrac{(\gamma-2x+1)F\left(2x+\gamma, \gamma-2x+2, \gamma-2x+\dfrac{5}{2}, -\dfrac{1}{2}\right)}{(2\gamma-4x+3)F\left(2x+\gamma, \gamma-2x+1, \gamma-2x+\dfrac{3}{2}, -\dfrac{1}{2}\right)}, & x \neq s. \end{cases}$$

Now we consider the problem of estimating θ based on a given sample $X_1, X_2, ..., X_n$ from (2.1). Using Theorem 2.4 we obtain.

Theorem 3.5. *Let $X_1, X_2, ..., X_n$ be a sample from* (2.1). *If the prior distribution of θ is the generalized Pareto distribution, then the posterior moments of θ are given by the formula*

(i) *for* $b > 0$

$$E[\theta^r|\boldsymbol{x}]=\begin{cases} \dfrac{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(ks)\int\limits_0^{\infty}\theta^r(1+b\theta)^{-1-\frac{1}{b}}[g(\theta)]^{ks}[f(\theta)]^{-k}d\theta}{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(ks)\int\limits_0^{\infty}(1+b\theta)^{-1-\frac{1}{b}}[g(\theta)]^{ks}[f(\theta)]^{-k}d\theta}, & z=ns,\\ \dfrac{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(z-(n-k)s)\int\limits_0^{\infty}\theta^r(1+b\theta)^{-1-\frac{1}{b}}[g(\theta)]^{z-(n-k)s}[f(\theta)]^{-k}d\theta}{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(z-(n-k)s)\int\limits_0^{\infty}(1+b\theta)^{-1-\frac{1}{b}}[g(\theta)]^{z-(n-k)s}[f(\theta)]^{-k}d\theta}, & z>ns;\end{cases}$$

(ii) *for* $b = 0$

$$E[\theta^r\,|\boldsymbol{x}]=\begin{cases} \dfrac{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(ks)\int\limits_0^{\infty}\theta^r e^{-\theta}[g(\theta)]^{ks}[f(\theta)]^{-k}d\theta}{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(ks)\int\limits_0^{\infty}e^{-\theta}[g(\theta)]^{ks}[f(\theta)]^{-k}d\theta}, & z=ns,\\ \dfrac{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(z-(n-k)s)\int\limits_0^{\infty}\theta^r e^{-\theta}[g(\theta)]^{z-(n-k)s}[f(\theta)]^{-k}d\theta}{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(z-(n-k)s)\int\limits_0^{\infty}e^{-\theta}[g(\theta)]^{z-(n-k)s}[f(\theta)]^{-k}d\theta}, & z>ns;\end{cases}$$

(iii) *for* $b < 0$

$$E[\theta^r|\boldsymbol{x}]=\begin{cases} \dfrac{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(ks)\int\limits_0^{-\frac{1}{b}}\theta^r(1+b\theta)^{-1-\frac{1}{b}}[g(\theta)]^{ks}[f(\theta)]^{-k}d\theta}{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(ks)\int\limits_0^{-\frac{1}{b}}(1+b\theta)^{-1-\frac{1}{b}}[g(\theta)]^{ks}[f(\theta)]^{-k}d\theta}, & z=ns,\\ \dfrac{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(z-(n-k)s)\int\limits_0^{-\frac{1}{b}}\theta^r(1+b\theta)^{-1-\frac{1}{b}}[g(\theta)]^{z-(n-k)s}[f(\theta)]^{-k}d\theta}{\sum\limits_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}a_k(z-(n-k)s)\int\limits_0^{-\frac{1}{b}}(1+b\theta)^{-1-\frac{1}{b}}[g(\theta)]^{z-(n-k)s}[f(\theta)]^{-k}d\theta}, & z>ns.\end{cases}$$

Now we give some examples.

Example 3.6. If a sample $X_1, X_2, ..., X_n$ is from the inflated generalized Poisson distribution, then using (ii) we obtain

$$\delta(z) = \begin{cases} \dfrac{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} (k+1)^{-2}}{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} (k+1)^{-1}}, & z = ns \\ \dfrac{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} k(\gamma z + k + 1)^{-z-2} (\gamma z + k)^{z-1}}{\sum\limits_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} k(\gamma z + k + 1)^{-z-1} (\gamma z + k)^{z-1}}, & z > ns. \end{cases}$$

Example 3.7. If a sample $X_1, X_2, ..., X_n$ is from the lost games distribution inflated at the point γ, then using (iii) for $b = -2$ we obtain

$$\delta(z) = \begin{cases} \dfrac{\sum\limits_{k=0}^{n} \binom{n}{k} a_k(k\gamma) F(-k\gamma, 2, \frac{5}{2}, \frac{1}{2})}{3 \sum\limits_{k=0}^{n} \binom{n}{k} a_k(k\gamma) F(-k\gamma, 1, \frac{3}{2}, \frac{1}{2})}, & z = n\gamma, \\ \dfrac{(z + n\gamma + 1) \sum\limits_{k=0}^{n} \binom{n}{k} a_k(z - (n-k)\gamma) F_2}{(2z - 2n\gamma + 3) \sum\limits_{k=0}^{n} \binom{n}{k} a_k(z - (n-k)\gamma) F_1}, & z > n\gamma. \end{cases}$$

Example 3.8. If a sample $X_1, X_2, ..., X_n$ is from (3.2), then using (iii) for $b = -2$ we obtain

$$\delta(z) = \begin{cases} \dfrac{\sum\limits_{k=0}^{n} \binom{n}{k} a_k(k\gamma) 2^{-2k\gamma} G_2}{3 \sum\limits_{k=0}^{n} \binom{n}{k} a_k(k\gamma) 2^{-2k\gamma} G_1}, & z = n\gamma, \\ \dfrac{\sum\limits_{k=0}^{n} \binom{n}{k} a_k(z - (n-k)\gamma) H_2}{\sum\limits_{k=0}^{n} \binom{n}{k} a_k(z - (n-k)\gamma) H_1}, & z > n\gamma, \end{cases}$$

where

$$G_i=\frac{\Gamma(2k\gamma+i)}{\Gamma(2k\gamma+i+\frac{1}{2})}F\left(3k\gamma, 2k\gamma+i, 2k\gamma+i+\frac{1}{2}, -\frac{1}{2}\right), \quad i=1,2,$$

and

$$H_i=\frac{\Gamma(z-n\gamma+2k\gamma+i)}{\Gamma(z-n\gamma+2k\gamma+i+\frac{1}{2})}$$

$$\times F\left(2(n\gamma-z)-3k\gamma, z-n\gamma+2k\gamma+i, z-n\gamma+2k\gamma+i+\frac{1}{2}, -\frac{1}{2}\right), \ i=1,2.$$

4. Posterior distributions and moments of IMPSD based on likelihood function

In Sections 2 and 3, we give Bayesian estimators using, among others, the distribution of the sum $Z_n = X_1+X_2+...+X_n$. In these Sections, we assume that the random variables $X_1, X_2, ..., X_n$ have the distribution inflated at the first point of their support. In this Section, we consider random variables with probability function inflated at any of support points. In such situations the distribution of the sum Z_n is unknown. To obtain the Bayes estimator of the parameter θ we use the following likelihood function

$$L(\theta,\alpha \mid \boldsymbol{x}) = \tag{4.1}$$
$$= \sum_{j=0}^{n_s}\binom{n_s}{j}\alpha^j\beta^{N-j}a(s)^{-j}\prod_{i=1}^{N-n_s}a(x_i)^{n_i}[g(\theta)]^{y-sj}[f(\theta)]^{j-N},$$

where $\boldsymbol{x} = (x_1, x_2, ..., x_N)$, $y = \sum_{i=1}^{N-n_s} x_i$ and n_i is the number of observations in the i'th class such that $\sum_{i\geq 0} n_i = N$.

Theorem 4.1. *Let X be a random variable with modified power series distribution inflated at the point s.*

(i) *If the prior distribution of* θ *is the uniform distribution on* $(0,1)$, *then the posterior moments of* θ *are given by the formula*

$$E(\theta^r \mid \boldsymbol{x}) = \frac{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^1 \theta^r [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} \int_0^1 [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}.$$

(ii) *If the prior distribution of* θ *is the Beta distribution with parameters* $a > 0$ *and* $b > 0$, *then the posterior moments of* θ *are given by the formula*

$$E(\theta^r \mid \boldsymbol{x}) = \frac{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^1 \theta^{a+r-1} (1-\theta)^{b-1} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^1 \theta^{a-1} (1-\theta)^{b-1} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}.$$

(iii) *If the prior distribution of* θ *is the Gamma distribution with parameters* $p > 0$ *and* $b > 0$, *then the posterior moments of* θ *are given by the formula*

$$hE(\theta^r \mid \boldsymbol{x}) = \frac{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^\infty \theta^{p+r-1} e^{-\theta b} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^\infty \theta^{p-1} e^{-\theta b} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}.$$

Proof. Let the prior distribution of θ be the uniform distribution on $(0,1)$. Then applying Bayes theorem to (4.1), we obtain the posterior distribution of θ which is given by

$$h(\theta \mid \boldsymbol{x})=\frac{\sum_{j=0}^{n_s}\binom{n_s}{j}\alpha^j\beta^{N-j}a(s)^{-j}[g(\theta)]^{y-sj}[f(\theta)]^{j-N}}{\sum_{j=0}^{n_s}\binom{n_s}{j}\alpha^j\beta^{N-j}a(s)^{-j}\int_0^1[g(\theta)]^{y-sj}[f(\theta)]^{j-N}d\theta}. \tag{4.2}$$

Hence the posterior moments of θ out of (4.2) by multipying $h(\theta \mid \boldsymbol{x})$ by θ^r and integrating by θ.

If we assume that the prior distribution of θ is the Beta distribution with parameters $a > 0$ and $b > 0$ and if we apply Bayes theorem to (4.1), we have the posterior distribution of θ in the form

$$h(\theta \mid \boldsymbol{x})=\frac{\sum_{j=0}^{n_s}\binom{n_s}{j}\alpha^j\beta^{N-j}a(s)^{-j}\theta^{a-1}(1-\theta)^{b-1}[g(\theta)]^{y-sj}[f(\theta)]^{j-N}}{\sum_{j=0}^{n_s}\binom{n_s}{j}\alpha^j\beta^{N-j}a(s)^{-j}\int_0^1\theta^{a-1}(1-\theta)^{b-1}[g(\theta)]^{y-sj}[f(\theta)]^{j-N}d\theta}.$$

Hence we get the posterior moments of θ.

Assuming that the prior distribution of θ is the Gamma distribution with parameters $p > 0$ and $b > 0$ and applying Bayes theorem to (4.1), we obtain the posterior distribution of θ which is given by

$$h(\theta \mid \boldsymbol{x})=\frac{\sum_{j=0}^{n_s}\binom{n_s}{j}\alpha^j\beta^{N-j}a(s)^{-j}[g(\theta)]^{y-sj}[f(\theta)]^{j-N}\theta^p e^{-\theta b}}{\sum_{j=0}^{n_s}\binom{n_s}{j}\alpha^j\beta^{N-j}a(s)^{-j}\int_0^\infty\theta^{p-1}e^{-\theta b}[g(\theta)]^{y-sj}[f(\theta)]^{j-N}d\theta}.$$

Hence we obtain the posterior moments of θ.

Theorem 4.2. *Let X be a random variable with modified power series distribution inflated at the point s. If the prior distribution of θ is the generalized Pareto distribution, then the posterior moments of θ are as follows*

(i) *for $b > 0$*

$$E(\theta^r \mid \boldsymbol{x}) = \frac{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^{\infty} \theta^r (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^{\infty} (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta},$$

(ii) *for* $b = 0$

$$E(\theta^r \mid \boldsymbol{x}) = \frac{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^{\infty} \theta^r e^{-\theta} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^{\infty} e^{-\theta} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta},$$

(iii) *for* $b < 0$

$$E(\theta^r \mid \boldsymbol{x}) = \frac{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^{-\frac{1}{b}} \theta^r (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^{-\frac{1}{b}} (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta},$$

respectively.

Proof. Applying Bayes theorem and assuming that the prior distribution of θ is the generalized Pareto distribution, we obtain the following posterior distribution

(i) for $b > 0$

$$h(\theta \mid \boldsymbol{x}) = \frac{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^{y-sj} [f(\theta)]^{j-N}}{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^{\infty} (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta},$$

(ii) for $b = 0$

$$h(\theta \mid \boldsymbol{x}) = \frac{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} e^{-\theta} [g(\theta)]^{y-sj} [f(\theta)]^{j-N}}{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^{\infty} e^{-\theta} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta},$$

(iii) for $b < 0$

$$h(\theta \mid \boldsymbol{x}) = \frac{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^{y-sj} [f(\theta)]^{j-N}}{\sum_{j=0}^{n_s} \binom{n_s}{j} \alpha^j \beta^{N-j} a(s)^{-j} \int_0^{-\frac{1}{b}} (1+b\theta)^{-1-\frac{1}{b}} [g(\theta)]^{y-sj} [f(\theta)]^{j-N} d\theta}.$$

Hence we get the posterior moments.

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