

BAND COPULAS AS SPECTRAL MEASURES FOR TWO-DIMENSIONAL STABLE RANDOM VECTORS

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Abstract

In this paper, we study basic properties of symmetric stable random vectors for which the spectral measure is a copula, i.e., a distribution having uniformly distributed marginals.

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1. INTRODUCTION

We say that a symmetric random variable is stable if there exists a positive constant A and index of stability $\alpha \in (0, 2]$ such that

$$\mathbf{E}e^{itX} = \exp\{-A|t|^\alpha\}, \quad t \in \mathbb{R}.$$

A random vector $X = (X_1, \dots, X_n)$ is symmetric α -stable if for every $\xi = (\xi_1, \dots, \xi_n)$ the random variable $\langle \xi, X \rangle = \sum_{k=1}^n \xi_k X_k$ is symmetric α -stable. The following, well known theorem was proven by Feldheim in 1937 and presented in P. Levy [5] in 1937 (first edition).

Theorem 1.1. *A random vector $X = (X_1, \dots, X_n)$ is symmetric α -stable if and only if there exists a finite measure ν on the unit sphere $S_{n-1} \subset \mathbb{R}^n$ such that*

$$\mathbf{E}e^{i\langle \xi, X \rangle} = \exp \left\{ - \int \dots \int_{S_{n-1}} |\langle \xi, x \rangle|^\alpha \nu(dx) \right\}.$$

The measure ν on S_{n-1} is uniquely determined and it is called the canonical spectral measure for the symmetric α -stable random vector X .

Remark 1. Usually the measure ν given in the previous theorem is simply called the spectral measure for the symmetric α -stable vector X . However we will also consider other representations for the characteristic functions of X , so in this paper a *canonical spectral measure* will always mean the measure concentrated on the unit sphere. The existence of many representations of the characteristic functions for the given symmetric α -stable vector X follows from the following theorem:

Theorem 1.2. *For every symmetric finite measure ν on \mathbb{R}^n such that:*

$$\int \dots \int_{\mathbb{R}^n} \|x\|^\alpha \nu(dx) < \infty$$

the following function:

$$(1) \quad \varphi(\xi) \stackrel{\text{def}}{=} \exp \left\{ - \int \dots \int_{\mathbb{R}^n} |\langle \xi, x \rangle|^\alpha \nu(dx) \right\}$$

is a characteristic function of a symmetric α -stable vector $X = (X_1, \dots, X_n)$. The measure ν given by equality (1) we will call a spectral measure for the random vector X . This measure is not uniquely determined.

Proof. We shall prove that for every fixed $\xi \in \mathbb{R}^n$ the function $\varphi(t\xi)$, as function of $t \in \mathbb{R}$, is a characteristic function of an SaS random variable, i.e., there exists $A > 0$ such that $\varphi(\xi t) = e^{-A|t|^\alpha}$. Indeed:

$$\begin{aligned} \varphi(\xi t) &= \exp \left\{ - \int \dots \int_{\mathbb{R}^n} |\langle \xi t, x \rangle|^\alpha \nu(dx) \right\} \\ &= \exp \left\{ -|t|^\alpha \int \dots \int_{\mathbb{R}^n} |\langle \xi, x \rangle|^\alpha \nu(dx) \right\}. \end{aligned}$$

It is enough to take

$$A = A(\xi) = \int \dots \int_{\mathbb{R}^n} | \langle \xi, x \rangle |^\alpha \nu(dx).$$

■

Remark 2. If the characteristic function of a symmetric α -stable random vector X is given by the formula (1) with the spectral measure ν , then the canonical spectral measure ν_0 for this vector we obtain substituting $u = x/\|x\|$, and $r = \|x\|$ and integrating with respect to r . Notice that if ν has an atom at zero, then this atom has no influence on the formula (1), thus we can always assume that $\nu(\{0\}) = 0$.

Remark 3. Assume that $n = 2$ and assume that the canonical measure ν in formula (1) is absolutely continuous with the density function $f(x, y)$. Then we can write:

$$\begin{aligned} \int \dots \int_{\mathbb{R}^n} | \langle \xi, x \rangle |^\alpha \nu(dx) &= \\ &= \int_0^{2\pi} |\xi_1 \cos t + \xi_2 \sin t|^\alpha \int_0^\infty r^{\alpha+1} f(r \cos t, r \sin t) dr dt. \end{aligned}$$

This means that the canonical spectral measure ν_0 for this random vector has the density given by:

$$g(u) = \int_0^\infty f(ru) r^{\alpha+1} dr, \quad u \in S_1 \subset \mathbb{R}^2.$$

2. COPULAE S

In general, by the term copula we understand a two dimensional (or n -dimensional) distribution with given marginals. The inversion method restricts the problem of constructing such distributions into constructing distributions on $[0, 1]^2$ (or $[-1, 1]^2$) having marginals uniform on the interval $[0, 1]$ (or $[-1, 1]$ respectively). Many types of copulas are well known in the literature. Recently there appeared a book written by Nelsen [6] which is entirely devoted to the theory of copulae and a two dimensional distribution on $[0, 1]^2$. In this paper, we will use copulae from a very wide class constructed independently by T.S. Ferguson in [2] and J. Bojarski in [1]. The construction is follows:

Construction:

Let Z be a random variable with a density function $f(z)$, concentrated on an interval $[-2, 2]$ such that $f(z) = f(-z)$. We define a two-dimensional density function $g(x, y)$ concentrated on $[-1, 1]^2$ by the formula

$$g(x, y) = \begin{cases} f(x - y) + f(x + y - 2) & \text{for } x + y \geq 0, \\ f(x - y) + f(x + y + 2) & \text{for } x + y \leq 0. \end{cases}$$

The density $g(x, y)$ has marginals uniform on the interval $[-1, 1]$, thus it defines a two-dimensional copulae.

3. COPULAE AS A SPECTRAL MEASURE FOR AN $S\alpha S$ RANDOM VECTOR

Let $x^{<p>} = |x|^p \text{sign}(x)$. This notation is very useful in describing properties and moments of random variables with a infinite variance. In our considerations, we will use the following formulas:

$$(2) \quad \begin{aligned} \int (ax + b)^{<\alpha>} dx &= \frac{1}{a(\alpha + 1)} |ax + b|^{\alpha+1} + C, \\ \int |ax + b|^\alpha dx &= \frac{1}{a(\alpha + 1)} (ax + b)^{<\alpha+1>} + C. \end{aligned}$$

Theorem 3.1. *Assume that Z is a random variable with the density function $f(z)$ concentrated on $[-2, 2]$. If the spectral measure ν of an $S\alpha S$ random vector (X, Y) has the density $g(x, y)$ given by formula (2), then the characteristic function of (X, Y) at the point (a, b) is given by $\exp\{-c(a, b)^\alpha\}$ where*

$$\begin{aligned} c(a, b)^\alpha &= \\ &= \frac{2(1 + \alpha)^{-1}}{(a^2 - b^2)} \mathbf{E} \left[b(b - a(1 - |Z|))^{<\alpha+1>} + a(b(1 - |Z|) - a)^{<\alpha+1>} \right]. \end{aligned}$$

The James correlation coefficient for the random vector (X, Y) is given by:

$$\begin{aligned} \rho_\alpha(X, Y) &\stackrel{\text{def}}{=} \int \dots \int x^{\langle \alpha-1 \rangle} y \nu(dx, dy) \\ &= \frac{2}{\alpha(\alpha+1)} \mathbf{E} \left[(\alpha+1)(1-|Z|) - (1-|Z|)^{\langle \alpha+1 \rangle} \right]. \end{aligned}$$

Proof. The proof is only a matter of laborious calculations, and the integral formulas given at the beginning of this section simplify these calculations slightly. The formula for ρ_α holds for every $\alpha \in (0, 2]$ as long as the right hand side makes sense. ■

Examples. In the following three examples we want to illustrate the dependence between the distribution $f(x)$, the distribution of spectral measure $g(x, y)$, the shape of level curves of the characteristic function of the corresponding $S\alpha S$ vector for different α 's. For each example we give also

$$h(\alpha) = \frac{\rho_\alpha(X, Y)}{\rho_\alpha(X, X)}$$

describing the dependence between α and the James correlation function. In the definition of the function $h(\alpha)$ we shall explain something more. Since $g(x, y)$ is a copula density function, then it has identical marginals, and from the Hölder inequality we obtain

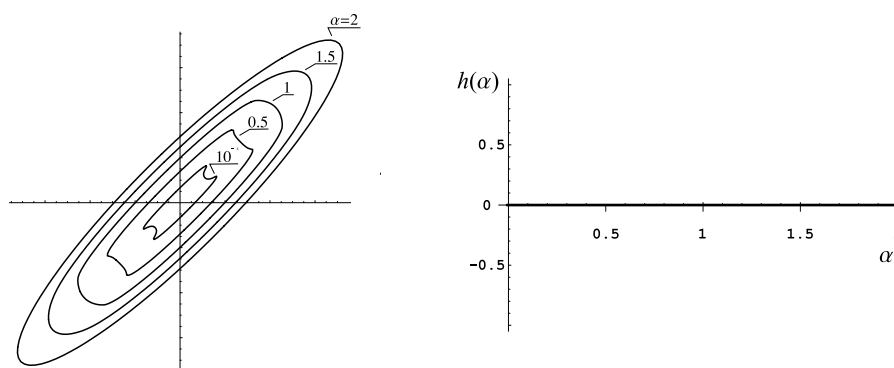
$$\begin{aligned} &\left| \int \dots \int x^{\langle \alpha-1 \rangle} y \nu(dx, dy) \right| \\ &\leq \left(\int \dots \int |x|^\alpha \nu(dx, dy) \right)^{\frac{\alpha-1}{\alpha}} \left(\int \dots \int |y|^\alpha \nu(dx, dy) \right)^{\frac{1}{\alpha}} \\ &= \int \dots \int |x|^\alpha \nu(dx, dy). \end{aligned}$$

This means that

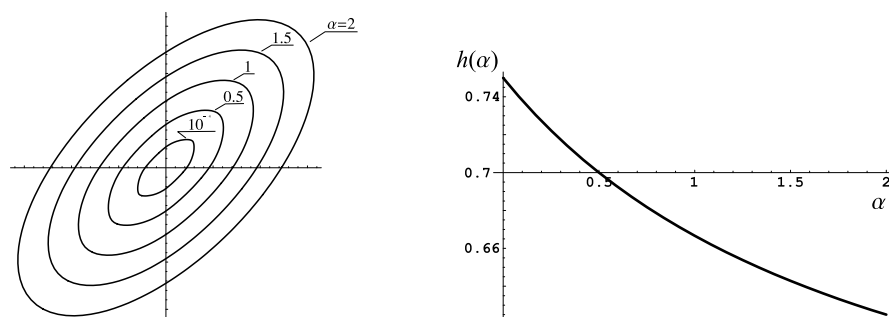
$$|h(\alpha)| \leq 1,$$

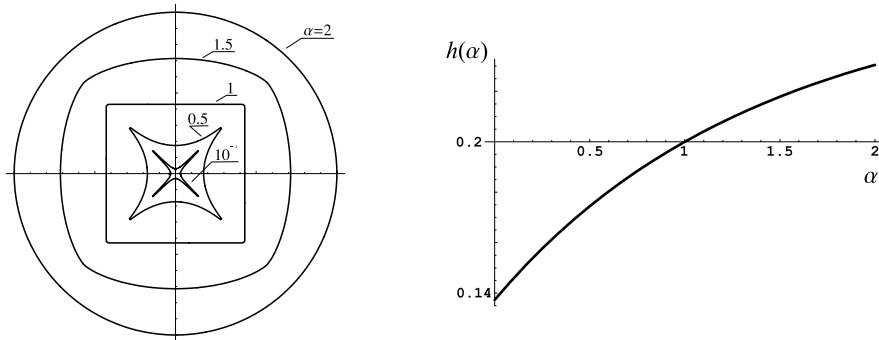
thus the function $h(\alpha)$ can play the same role for the $S\alpha S$ random vector as the correlation coefficient for the second order random vector. In our examples the function $h(\alpha)$ makes sense on the whole interval $(0, 2]$.

Example 1. Notice that the shape of level curves for the characteristic function suggests positive dependence coefficients, while in fact we have here $h(\alpha) = 0$ for every $\alpha \in (0, 2]$.



Example 2.



Example 3.

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