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UNIT ROOT TEST IN THE PRESENCE OF A SINGLE ADDITIVE OUTLIER SMALL SAMPLE CASE*

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Abstract

The one sided unit root test of a first-order autoregressive model in the presence of an additive outlier is considered. In this paper, we present a formula to compute the size and the power of the test when an AO (additive outlier) occurs at a time k. A small sample case is considered only.

Keywords: autoregressive, outlier, power, size, unit root.

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1. Introduction

Consider a time series $\{x_t\}$ which follows the model

$$(1) \qquad (1-\rho B)x_t = \epsilon_t \qquad t = \ldots, -1, 0, 1, \ldots, n$$

where $\{\epsilon_t\}_{t=1,\ldots,n}$ is a sequence of independent normally distributed random variables with mean zero and variance σ^2 and B denotes the backshift operator such that $Bx_t = x_{t-1}$. We assume that $x_0 = 0$ and, without loss of generality, $\sigma = 1$.

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Suppose that all what we observe is the segment of observations

$$x_1, x_2, \ldots, x_n \tag{2}$$

and we want to test the hypothesis $H_0 : \rho = 1$ against $H_1 : \rho < 1$.

Testing for unit root in autoregressive time series has attracted great attention since the pioneer paper by Dickey and Fuller (1979). In their paper, the authors said that the hypothesis that $\rho = 1$ is of some interest in applications since it corresponds to the hypothesis that it is appropriate to transform the time series by differencing. For applications of unit root tests, see for example, Diebold (1988), Perron (1989) and Sims and Uhlig (1991).

There exists a vast litterature on the unit root tests. Dickey and Fuller (1979) proposed simple unit root tests. Phillips (1987) and Phillips and Perron (1988) suggested a criteria for correction of the bias in Dickey-Fuller statistics. Various aspects of the unit root tests are summarized in Fuller (1996, chap. 10). The tests are essentially based on the least square estimator of ρ defined by

$$\hat{\rho}_{LS} = \left[\sum_{t=2}^{n} x_t x_{t-1}\right] \left[\sum_{t=2}^{n} x_{t-1}^2\right]^{-1}. \tag{3}$$

Recently, Tiku and Wong (1998) derived a statistic based on a modified maximum likelihood estimator of ρ. They studied this statistic when $\rho = 1$ and the innovations have a symmetric distribution. Shin and Sarkar (1996) studied an AR(1) process with unit root for which observations available are irregular in nature. They proposed unit root tests which perform very well for small samples. To test the hypothesis H_0, we use the statistic

$$T = n(\hat{\rho}_{LS} - 1) \tag{4}$$

where $\hat{\rho}_{LS}$ is given by formula (3). Assume that, at a position $k \in]1, n[$, a single additive outlier of magnitude Δ occurs. Hence, instead of the segment (2), we observe the following observations $z_1, z_2, \ldots, z_n$ where

$$z_t = x_t \quad \forall t \neq k \quad \text{and} \quad z_k = x_k + \Delta \tag{5}$$

The process $\{z_t\}$ generated by the contaminant is called additive outlier model (AO) introduced by Fox (1972). Franses and Haldrup (1994) studied effects of outliers on unit root Dickey-Fuller tests. They showed that there is over-rejection of the unit root hypothesis when additive outliers occur. Also, Shin and *et al.* (1996) investigated the effects of outliers on unit root tests in AR(1) and more. They proved that the limiting distribution of T is affected by an additive outlier. Also, they proposed a method to detect outliers and to adjust the observations. Maddala and Rao (1997) argue that the standard Dikey-Fuller unit root tests face more trouble the more frequently AO occurs or the larger their impacts are. However, when n goes to infinity, the impacts of finite additive outliers will go to zero. Vogelsang (1999) proposed two robust procedures to detect outliers and adjust the observations.

In the case of small samples, very little is known to compute the power of the unit root test. In this paper, we derive a formula for the size and the power of the unit root test when a single AO contaminant occurs and when the statistic $n(\hat{\rho}_{LS} - 1)$ is used.

2. Gaussian model

In this section, we propose to calculate the value of the size $\alpha = P_{H_0}(n(\hat{\rho}_{LS} - 1) < c)$ for a given value of the constant c.

Let us introduce the following notations: $\mathbf{X} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})^{\mathbf{T}}$ - vector of the observations, $\mathbf{E} = (\varepsilon_{\mathbf{1}}, \varepsilon_{\mathbf{2}}, \ldots, \varepsilon_{\mathbf{n}})^{\mathbf{T}}$ - vector of innovations and the nxn-matrix generated by the coefficient ρ denoted $A = (a_{ij})_{i,j=1,\ldots,n}$ and defined by

$$a_{ij} = \begin{cases} \rho^{i-j} & \text{if} \quad i \geq j \quad i,j = 1,\ldots,n \\ 0 & \text{else where.} \end{cases}$$

Then, we have the relation $X = AE$. Also, the least square estimator of ρ can be written as a ratio of two quadratic forms:

$$\hat{\rho}_{LS} = \left[X^T R_1 X\right] \left[X^T R_2 X\right]^{-1}$$

with

$$R_1 = \begin{pmatrix} 0 & 1/2 & 0 & \dots & 0 \\ 1/2 & 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 1/2 \\ 0 & 0 & \dots & 1/2 & 0 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Since $X = A.E$, we obtain $\hat{\rho}_{LS} = \left[E^T LE\right]\left[E^T ME\right]^{-1}$ with $L = A^T R_1 A$ and $M = A^T R_2 A$. If we use the notation $c' = 1 + c/n$, we have,

$$P_{H_0}(n(\hat{\rho}_{LS} - 1) < c) = P_{H_0}(\hat{\rho}_{LS} < c') = P_{H_0}\left(E^T.(L - c'M).E < 0\right)$$
$$= P_{H_0}\left(E^T.A^T(R_1 - c'R_2).A.E < 0\right) = P_{H_0}\left(E^T.B.E < 0\right)$$

where $B = A^T(R_1 - c'R_2).A$.

To compute $P_{H_0}(n(\hat{\rho}_{LS} - 1) < c)$, we use the formula given by Imhof (1961) which gives

$$P_{H_0}(n(\hat{\rho}_{LS} - 1) < c) = 0.5 - \frac{1}{\pi}\int_0^\infty \frac{\sin f(u)}{ug(u)}du \tag{6}$$

where

$$f(u) = \frac{1}{2}\sum_{i=1}^{n} Tan^{-1}(\lambda_i u) \quad \text{and} \quad g(u) = \prod_1^n \left(1 + \lambda_i^2 u^2\right)^{1/4}$$

$\lambda_1, \lambda_2, \dots, \lambda_n$ being the eigenvalues of the matrix B. The value of the integral is obtained using a numerical method. In this paper, the Gauss method is used.

3. ADDITIVE OUTLIER MODEL

Assume that an additive outlier of magnitude Δ occurs at a time k $(1 < k < n)$. Then, the observed sample is $z_1, z_2, \ldots, z_n$ obtained with formula (5). Now, we compute the value of $P_{H_0}(n(\hat{\rho}^*_{LS} - 1) < c)$ where

$$\begin{aligned} \hat{\rho}^*_{LS} &= \left[\sum_{t=2}^{n} z_t z_{t-1}\right] \left[\sum_{t=2}^{n} z^2_{t-1}\right]^{-1} \\ &= \left[\sum_{t=2}^{n} x_t x_{t-1} + \Delta(x_{k-1} + x_{k+1})\right] \left[\sum_{t=2}^{n} x^2_{t-1} + 2\Delta x_k + \Delta^2\right]^{-1}. \end{aligned} \tag{7}$$

Let us introduce the following notations:

$Z = (z_1, z_2, \ldots, z_n)^T$ and $Y = (y_1, y_2, \ldots, y_n)^T$ where

$$y_t = \varepsilon_t \quad \forall t \neq k, k+1 \quad \text{and} \quad y_k = \varepsilon_k + \Delta, \qquad y_{k+1} = \varepsilon_{k+1} - \Delta.$$

Then, we have $Z = AY$ and

$$P_{H_0}(n(\hat{\rho}^*_{LS} - 1) < c) = P_{H_0}(\hat{\rho}^*_{LS} < c') = P_{H_0}(Y^T A^T (R_1 - c' R_2) A Y < 0)$$

where c' is defined above. Using Imhof's formula, we obtain the value of $P_{H_0}(n(\hat{\rho}^*_{LS} - 1) < c)$ in the presence of an additive outlier contamination

$$P_{H_0}(n(\hat{\rho}^*_{LS} - 1) < c) = 0.5 - \frac{1}{\pi} \int_0^\infty \frac{\sin f^*(u, \Delta)}{u g^*(u, \Delta)} du \tag{8}$$

where

$$f^*(u,\Delta) = f(u) + \frac{\Delta^2 u}{2}\sum_{i=1}^{n}\frac{\lambda_i(Q_{k,i}-Q_{k+1,i})^2}{1+\lambda_i^2u^2}$$

and

$$g^*(u,\Delta) = g(u)exp\left\{\frac{\Delta^2u^2}{2}\sum_{i=1}^{n}\frac{\lambda_i^2(Q_{k,i}-Q_{k+1,i})^2}{1+\lambda_i^2u^2}\right\}$$

$\lambda_1, \lambda_2, \ldots, \lambda_n$ being the same eigenvalues of the matrix B (see Section 2). $Q_{k,i}$ and $Q_{k+1,i}$ are the (k,i) and $(k+1,i)$ element of the orthogonal matrix Q containing the normalized eigenvectors of B respectively. Using numerical integration of the integral given in (8), we can obtain numerical approximations of the size of the unit root test in the contaminated model.

The power of the test for a given $\rho = \rho_0$ is easily obtained in the same way by assuming that $y_{k+1} = \epsilon_{k+1} - \rho_0\Delta$. Then, we derive the formula given above by replacing $(Q_{k,i} - Q_{k+1,i})^2$ with $(Q_{k,i} - \rho_0 Q_{k+1,i})^2$ in the expression of $f^*(u,\Delta)$ and $g^*(u,\Delta)$.

Since the formula (8) depends on Δ^2 only, we can say that, positive and negative values of Δ have the same effect on the size and the power.

Table 1 presents, as application, some values of the power of the 5 % level test when $\rho = 0.5, 0.7, 0.8, 0.9, 0.95, 1.0$ and $\Delta = 0, 1, 2, 3, 4, 5$. The critical values of the 5 % level test obtained using simulation procedure are, for $n = 5, 10, 25$, -5.612, -6.5575 and - 7.3800 respectively.

Notice that when $\rho = 1.0$, we obtain the size of the unit root test and the value corresponding to $n = 25$ is the same than of Fuller (1996, p. 641).

We note that the size and the power of the test are really affected by the presence of an AO. When Δ tends to infinity, $\hat{\rho}^*_{LS}$ tends to zero (see formula 7) and then, the size and the power tend to one.

When an additive outlier is detected and the magnitude is estimated (see, e.g. Shin *et al.*, 1996 or Abraham and Box, 1979), the method presented in this paper allows us to assess exactly the impact of this contaminant on the size and the power of the unit root test.

Table 1. Variation of the power of the test with Δ

n	ρ	Δ					
		0	1	2	3	4	5
5	0.5	0.1466	0.1856	0.2531	0.2881	0.3041	0.2996
	0.7	0.0948	0.1315	0.1986	0.2463	0.2702	0.2788
	0.8	0.0764	0.1103	0.1711	0.2190	0.2458	0.2593
	0.9	0.0618	0.0924	0.1481	0.1905	0.2172	0.2334
	0.95	0.0557	0.0846	0.1366	0.1762	0.2020	0.2186
	1.0	0.0502	0.0775	0.1257	0.1623	0.1867	0.2031
10	0.5	0.3606	0.4093	0.5235	0.6455	0.7447	0.8187
	0.7	0.1799	0.2239	0.3352	0.4709	0.5946	0.6932
	0.8	0.1202	0.1554	0.2510	0.3747	0.4967	0.6011
	0.9	0.0781	0.1051	0.1803	0.2824	0.3894	0.4886
	0.95	0.0625	0.0857	0.1507	0.2403	0.3362	0.4279
	1.0	0.0500	0.0696	0.1250	0.2020	0.2856	0.3674
25	0.5	0.9339	0.9441	0.9659	0.9846	0.9948	0.9986
	0.7	0.5917	0.6299	0.7245	0.8314	0.9141	0.9628
	0.8	0.3392	0.3761	0.4769	0.6126	0.7460	0.8509
	0.9	0.1490	0.1717	0.2387	0.3425	0.4665	0.5908
	0.95	0.0848	0.1002	0.1468	0.2223	0.3183	0.4227
	1.0	0.0500	0.0582	0.0883	0.1371	0.2014	0.2744

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