SET-VALUED STOCHASTIC INTEGRALS AND STOCHASTIC INCLUSIONS IN A PLANE

WŁADYSŁAW SOSULSKI

Institute of Mathematics
University of Zielona Góra
Podgórna 50, 65-246 Zielona Góra, Poland

Abstract

We present the concepts of set-valued stochastic integrals in a plane and prove the existence of a solution to stochastic integral inclusions of the form

\[ z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]

Keywords: stochastic inclusions in the plane, set-valued random field, two-parameter stochastic process, weak compactness.

2000 Mathematics Subject Classification: 93E03, 93C30.

1. Introduction

In this paper, we shall consider a stochastic inclusion in a plane of the form:

\[ z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]  

(1)

The results of the paper generalize some problems dealing with stochastic integral equations in a plane of the form:

\[ x_{s,t} = x + \int_0^s \int_0^t a(u,v,x_{u,v}) \, dw_{u,v} + \int_0^s \int_0^t b(u,v,x_{u,v}) \, dv \]

(2)
that have been investigated by Tudor [8], Panomarenco [5] and others. The first integral in equation (2) is a stochastic integral in a plane considered with respect to the Wiener-Yeh process. It was defined by Cairoli [1], Panomarenco [5], Tsarenco [7]. Its generalization with respect to two parameters martingale is given by Cairoli and Walsh in [2].

In this paper, we introduce the stochastic integral in a plane for set-valued mappings taking values from space Comp ($\mathbb{R}^n$) of all nonempty closed subsets of $n$-dimensional Euclidean space $\mathbb{R}^n$ with respect to the Wiener-Yeh process and we establish some of its properties. The space Comp ($\mathbb{R}^n$) is considered with the Hausdorff metric $h$ defined in the usual way, i.e., $h(A, B) = \max\{\overline{h}(A, B), \overline{h}(B, A)\}$, for $A, B \in \text{Comp} (\mathbb{R}^n)$ where $\overline{h}(A, B) = \sup \{\text{dist} (a, B) : a \in A\}$ and $\overline{h}(B, A) = \sup \{\text{dist} (b, A) : b \in B\}$.

We assume, as given, a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{s,t})_{s,t \geq 0}, P)$ where a filtration $(\mathcal{F}_{s,t})_{s,t \geq 0}$ is assumed to satisfy:

(i) if $(s, t) \leq (s', t')$ then $\mathcal{F}_{s,t} \subset \mathcal{F}_{s', t'}$;

(ii) $\mathcal{F}_{0,0}$ contains all null sets of $\mathcal{F}$;

(iii) for each $(s, t)$, $\mathcal{F}_{s,t} = \bigcap_{(u,v) > (s,t)} \mathcal{F}_{u,v}$;

(iv) for each $(s, t)$, $\mathcal{F}^1_{s,t}$ and $\mathcal{F}^2_{s,t}$ are conditionally independent given $\mathcal{F}_{s,t}$.

2. Basic definitions and notations

Throughout the paper, we shall assume that filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{s,t})_{s,t \geq 0}, P)$ satisfies the following conditions:

(i) if $(s, t) \leq (s', t')$ then $\mathcal{F}_{s,t} \subset \mathcal{F}_{s', t'}$;

(ii) $\mathcal{F}_{0,0}$ contains all null sets of $\mathcal{F}$;

(iii) for each $(s, t)$, $\mathcal{F}_{s,t} = \bigcap_{(u,v) > (s,t)} \mathcal{F}_{u,v}$;

(iv) for each $(s, t)$, $\mathcal{F}^1_{s,t}$ and $\mathcal{F}^2_{s,t}$ are conditionally independent given $\mathcal{F}_{s,t}$. 

We put $R^2_+ = [0, \infty) \times [0, \infty)$ and $\beta^2_+$ denotes the Borel σ-algebra on $R^2_+$. We consider set-valued two parameter processes (set-valued random fields) $(\mathcal{F}_{s,t})_{s,t \geq 0}$ and $(\mathcal{G}_{s,t})_{s,t \geq 0}$ that are assumed to be nonanticipative and such that

$$\int_0^\infty \int_0^\infty ||\mathcal{F}_{s,t}||^2 \, ds \, dt < \infty \quad \text{and} \quad \int_0^\infty \int_0^\infty ||\mathcal{G}_{s,t}||^2 \, ds \, dt < \infty \quad \text{a.s.,}$$

where $||A|| := \sup\{|a| : a \in A\}$ for a nonempty set $A \subset \mathbb{R}^n$. Next, using the method which has been used by Kisielewicz in [4], we investigate the existence of solutions to stochastic inclusion (1).
We write \((s, t) \leq (s', t')\) iff \(s \leq s'\) and \(t \leq t'\). As usual, we shall consider a set \(R^2_+ \times \Omega\) as a measurable space with the product \(\sigma\)-algebra \(B^2_+ \otimes \mathcal{F}\).

An \(n\)-dimensional two parameter stochastic process \(z\) (random field) understood as a function \(z : R^2_+ \times \Omega \to R^n\) with \(\mathcal{F}\)-measurable sections \(z_{s,t}\), each \(s, t \geq 0\), is denoted by \((z_{s,t})_{s,t \geq 0}\). It is measurable if \(z\) is \(B^2_+ \otimes \mathcal{F}\)-measurable. The process \((z_{s,t})_{s,t \geq 0}\) is \(\mathcal{F}_{s,t}\)-adapted or adapted if \(z_{s,t}\) is \(\mathcal{F}_{s,t}\)-measurable for \(s, t \geq 0\). Every measurable and adapted process is called nonanticipative. In what follows, the Banach space \((L^2(R^2_+ \times \Omega, \beta^2_+ \otimes \mathcal{F}, dsdt \otimes P), \| \cdot \|_2)\), where \(dsdt\) denotes Lebesgue measure on \(R^2\), will be denoted by \(L^2_n\). Similarly, the Banach spaces \((L^2(\Omega, \mathcal{F}, P, R^n), \| \cdot \|)\) and \((L^2(\Omega, \mathcal{F}_{s,t}, P, R^n), \| \cdot \|)\) are denoted by \(L^2(\mathcal{F})\) and \(L^2(\mathcal{F}_{s,t})\), respectively. Let \(\mathcal{M}^2(\mathcal{F}_{s,t})\) denote the family of all (equivalence classes of) \(n\)-dimensional nonanticipative processes \((f_{s,t})_{s,t \geq 0}\) such that \(\int_0^\infty \int_0^\infty |f_{s,t}|^2 dsdt < \infty\), a.s.

We shall also consider a subspace \(L^2(\mathcal{F}_{s,t})\) of \(\mathcal{M}^2(\mathcal{F}_{s,t})\) defined by

\[
L^2(\mathcal{F}_{s,t}) = \left\{(f_{s,t})_{s,t \geq 0} \in \mathcal{M}^2(\mathcal{F}_{s,t}) : E \int_0^\infty \int_0^\infty |f_{s,t}|^2 dsdt < \infty \right\}.
\]

It is a closed subset of the Banach space \(L^2_n\). Finally, by \(M_n(\mathcal{F}_{s,t})\) we denote a space of all (equivalence classes of) \(n\)-dimensional \(\mathcal{F}_{s,t}\)-measurable mappings. Throughout, by \((w_{s,t})_{s,t \geq 0}\) we mean two parameter \(\mathcal{F}_{s,t}\)-Brownian motion, i.e., a continuous Gaussian process such that \(E(w_{s,t}) = 0\) and \(E(w_{s,t} \cdot w_{s',t'}) = \min(s, s') \cdot \min(t, t')\) for every \(s, s', t, t'\). Given \(g \in \mathcal{M}^2(\mathcal{F}_{s,t})\), by \((\int_0^s \int_0^t g_{u,v} dw_{u,v})_{s,t \geq 0}\) we denote its stochastic integral with respect to an \(\mathcal{F}_{s,t}\)-Brownian motion \((w_{s,t})_{s,t \geq 0}\). Let us denote by \(D\) the family of all \(n\)-dimensional \(\mathcal{F}_{s,t}\)-adapted continuous processes \((z_{s,t})_{s,t \geq 0}\) such that \(E \sup_{s,t \geq 0} |z_{s,t}|^2 < \infty\). The space \(D\) is considered as a normed space with the norm \(\| \cdot \|\) defined by \(\|x\| = \sup_{s,t \geq 0} |x_{s,t}| \|L^2\|\) for \(x = (x_{s,t})_{s,t \in D}\), where \(\| \cdot \|_{L^2}\) is a norm of \(L^2(\Omega, \mathcal{F}, P, R)\). It can be verified that \((D, \| \cdot \|)\) is a Banach space. In what follows, we shall deal with upper and lower semicontinuous set-valued mappings. Recall that a set-valued mapping \(\mathcal{R}\) with nonempty values in a topological space \((Y, T_Y)\) is said to be upper (lower) semicontinuous [u.s.c. (l.s.c.)] on a topological space \((X, T_X)\) if \(\mathcal{R}^{-}(C) := \{ x \in X : \mathcal{R}(x) \cap C \neq \emptyset \} \) (resp. \(\mathcal{R}^{-}(C) := \{ x \in X : \mathcal{R}(x) \subset C \} \) ) is a closed subset of \(X\) for every closed set \(C \subset Y\). In particular, for \(\mathcal{R}\) defined on a metric space \((X, d)\) with values in \(\text{Comp}(R^n)\), it is equivalent (see [4]) to \(\lim_{n \to \infty} h(\mathcal{R}(x_n), \mathcal{R}(x)) = 0\) for every \(x \in X\).
and every sequence \((x_n)\) of \(X\) converging to \(x\). In what follows, we shall need the following well-known (see [4]) fixed point and continuous selection theorems.

**Theorem 1** (Schauder, Tikhonov). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(f\) a continuous mapping of \(K\) into itself. Then \(f\) has fixed point in \(K\).

**Theorem 2** (Kakutani, Fan). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(CCl(X)\) a family of all nonempty closed convex subsets of \(K\). If \(K : K \rightarrow CCl(K)\) is u.s.c. on \(K\), then there exists \(x \in K\) such that \(x \in K(x)\).

**Theorem 3** (Michael). Let \((X, \mathcal{F}_X)\) be a paracompact space and let \(K\) be a set-valued mapping from \(X\) to a Banach space \((Y, \| \cdot \|)\) whose values are closed and convex. Suppose, further \(K\) is l.s.c. on \(X\). Then there is a continuous function \(f : X \rightarrow Y\) such that \(f(x) \in K(x)\), for each \(x \in X\).

### 3. Set-valued stochastic integrals in the plane

Given measure space \((X, \beta, m)\), a set-valued function \(\mathcal{R} : X \rightarrow \text{Comp}(\mathbb{R}^n)\) is said to be \(\beta\)-measurable if \(\mathcal{R}^{-}(C) \in \beta\) for every closed \(C \subset \mathbb{R}^n\). For such a multifunction we define integrals subtrajectory (see [6]) as a set

\[
S^p(\mathcal{R}) = \{ g \in L^p(X, \beta, m, \mathbb{R}^n) : g(x) \in \mathcal{R}(x) \text{ a.e.} \}.
\]

It is clear that for nonemptiness of \(S^p(\mathcal{R})\) we have to assume more than \(\beta\)-measurability of \(\mathcal{R}\). In what follows, we shall assume that \(\beta\)-measurable set-valued function \(\mathcal{R} : X \rightarrow \text{Comp}(\mathbb{R}^n)\) is \(p\)-integrably bounded, \(p \geq 1\), i.e., that a real-valued mapping \(X \ni x \rightarrow \|\mathcal{R}(x)\| \in \mathbb{R}_+\) belongs to \(L^p(X, \beta, m, R_+)\) where \(\|A\| := \sup\{|a| : a \in A\}\) for \(A \in \text{Comp}(\mathbb{R}^n)\). It can be verified that a \(\beta\)-measurable set-valued mapping \(\mathcal{R} : X \rightarrow \text{Comp}(\mathbb{R}^n)\) is \(p\)-integrably bounded \(p \geq 1\), if and only if \(S^p(\mathcal{R})\) is nonempty and bounded in \(L^p(X, \beta, m, \mathbb{R}^n)\) (see [4]). Finally, it is easy to see that \(S^p(\mathcal{R})\) is decomposable, i.e., such that \(1|A|f_1 + 1_{X\setminus A}f_2 \in S^p(\mathcal{R})\) for \(A \in \beta\) and \(f_1, f_2 \in S^p(\mathcal{R})\). We have the following general result dealing with properties of subtrajectory integrals (see [4], [5]).

**Proposition 4.** Let \(\mathcal{R} : X \rightarrow \text{Comp}(\mathbb{R}^n)\) be \(\beta\)-measurable and \(p\)-integrably bounded, \(p \geq 1\). The set \(S^p(\mathcal{R})\) is a nonempty bounded and closed subset
of $L^p(X, \beta, m, R^n)$. Moreover, if $\mathcal{R}$ takes on convex values, then $S^p(\mathcal{R})$ is convex and weakly compact in $L^p(X, \beta, m, R^n)$.

Let $G = (G_{s,t})_{s,t \geq 0}$ be a set-valued two parameter stochastic process with values in $\text{Comp}(R^n)$, i.e., a family of $\mathcal{F}$-measurable set-valued mappings $G_{s,t} : \Omega \to \text{Comp}(R^n), s, t \geq 0$. We call $G$ measurable if it is $\beta^2_+ \otimes \mathcal{F}$-measurable. Similarly, $G$ is said to be $\mathcal{F}_{s,t}$-adapted or adapted if $G_{s,t}$ is $\mathcal{F}_{s,t}$-measurable for each $s, t \geq 0$. A measurable and adapted set-valued two parameter stochastic process is called nonanticipative. Denote by $\mathcal{M}^2_{s-t}(\mathcal{F}_{s,t})$ a family of all nonanticipative set-valued processes $G = (G_{s,t})_{s,t \geq 0}$ such that $\int_0^\infty \int_0^\infty \| G_{s,t} \|^2 \, ds \, dt < \infty$, a.s. Immediately from the Kuratowski and Ryll-Nardzewski measurable selection theorem (see [4]) it follows that for every $F, G \in \mathcal{M}^2_{s-t}(\mathcal{F}_{s,t})$ their subtrajectory integrals

$$S^2(F) = f \in \{ M^2(\mathcal{F}_{s,t} : f_{s,t} \in (\omega) \in \mathcal{F}_{s,t}(\omega), ds \, dt \times P - \text{a.e.} \}$$

and

$$S^2(G) = \{ g \in M^2(\mathcal{F}_{s,t} : g_{s,t}(\omega) \in G_{s,t}(\omega), ds \, dt \times P - \text{a.e.} \},$$

are nonempty. Indeed, let $\sum = \{ Z \in B^2_+ \otimes \mathcal{F} : Z_{s,t} \in \mathcal{F}_{s,t}, \text{each } s, t \geq 0 \}$, where $Z_{s,t}$ denotes a section of $Z$ determined by $s, t \geq 0$. It is a $\sigma$-algebra on $R^2_+ \times \Omega$ and function $f : R^2_+ \times \Omega \to R^n$ (a multifunction $F : R^2_+ \times \Omega \to \text{Comp}(R^n)$) is nonanticipative if and only if it is $\sum$-measurable. Therefore, by the Kuratowski and Ryll-Nardzewski measurable selection theorem, every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in \mathcal{M}^2_{s-t}(\mathcal{F}_{s,t})$ such selectors belong to $\mathcal{M}^2(\mathcal{F}_{s,t})$. Finally, denote

$$\mathcal{L}^2_{s-t}(\mathcal{F}_{s,t}) = \left\{ G \in \mathcal{M}_{s-t}(\mathcal{F}_{s,t} : E \int_0^\infty \int_0^\infty \| G_{s,t} \|^2 \, ds \, dt < \infty \right\}.$$

Given set-valued two parameter processes

$$F = (F_{s,t})_{s,t \geq 0} \in \mathcal{M}^2_{s-t}(\mathcal{F}_{s,t}) \text{ and } G = (G_{s,t})_{s,t \geq 0} \in \mathcal{M}^2_{s-t}(\mathcal{F}_{s,t})$$

by their stochastic integrals we mean families $(\int_0^s f_{u,v} \, du \, dv)_{s,t \geq 0}$ and $(\int_0^s g_{u,v} \, dw \, du)_{s,t \geq 0}$ of subsets of $\mathcal{M}_{u}(\mathcal{F}_{s,t})$, defined by

$$\int_0^s \int_0^t f_{u,v} \, du \, dv := \left\{ \int_0^s \int_0^t f_{u,v} \, du \, dv : f \in S^2(F) \right\}$$
and
\[ \int_0^s \int_0^t G_{u,v} \, dw_{u,v} := \left\{ \int_0^s \int_0^t g_{u,v} \, dw_{u,w} : g \in S^2(G) \right\}. \]

Immediately, from the above definitions the following simple results follow.

**Theorem 5.** Let \( F, G \in \mathcal{M}^2_{s-v}(\mathcal{F}_{s,t}) \). Then
1. \( \int_0^s \int_0^t F_{u,v} \, dv \) and \( \int_0^s \int_0^t G_{u,v} \, dv \) are nonempty subsets of \( \mathcal{M}(\mathcal{F}_{s,t}) \), each \( s, t \geq 0 \). They are convex if \( F \) and \( G \) take on convex values.
2. If \( G \in L^2_{s-v}(\mathcal{F}_{s,t}) \), then \( \int_0^s \int_0^t G_{u,v} \, dv \) is a nonempty subset of \( L^2(\mathcal{F}_{s,t}) \), each \( s \geq 0, t \geq 0 \).
3. If \( F, G \in L^2_{s-v}(\mathcal{F}_{s,t}) \) take on convex values then \( \int_0^s \int_0^t F_{u,v} \, dv \) and \( \int_0^s \int_0^t G_{u,v} \, dv \) are nonempty convex and weakly compact subsets of \( L^2(\mathcal{F}_{s,t}) \), each \( s \geq 0, t \geq 0 \).

**Proof.** It is clear that (i) follows immediately from the definitions of set-valued stochastic integrals. To verify (ii) let us observe that the operator \( I_{s,t} \), defined by the formula \( I_{s,t}g = \int_0^s \int_0^t g_{u,v} \, dv \), is a linear continuous mapping from \( L^2(\mathcal{F}_{s,t}) \) into \( L^2(\mathcal{F}_{s,t}) \). By the properties of stochastic integrals in the plane (see [6]) one has \( E|I_{s,t}g|^2 = \|g\|^2_2 \), each \( s, t \geq 0 \) and \( g \in L^2(\mathcal{F}_{s,t}) \), where \( \|: \|_2 \) denotes the norm of \( L^2(\mathcal{F}_{s,t}) \). Therefore, \( I_{s,t} \) maps closed subsets of \( L^2(\mathcal{F}_{s,t}) \) into closed subsets of \( L^2(\mathcal{F}_{s,t}) \). Thus (ii) is satisfied. To verify (iii), it suffices only to observe that for every \( s, t \geq 0 \), all sets \( S^2(F) \) and \( S^2(G) \) are convex and weakly compact in \( L^2(\mathcal{F}_{s,t}) \). Hence, \( J_{s,t}(S^2(F)) \) and \( J_{s,t}(S^2(G)) \) (where \( J_{s,t} \), is defined by \( J_{s,t}f = \int_0^s \int_0^t f \, dv \)) are convex and weakly compact subsets of \( L^2(\mathcal{F}_{s,t}) \) because \( J_{s,t} \) and \( I_{s,t} \) are linear and continuous for fixed \( s, t \geq 0 \). Then (iii) also holds. □

4. Stochastic inclusions in the plane

Let \( F = \{(F_{s,t}(z))_{s,t} \geq 0 : z \in \mathbb{R}^n\} \) and \( G = \{(G_{s,t}(z))_{s,t} \geq 0 : z \in \mathbb{R}^n\} \). Assume \( F \) and \( G \) are such that \( (F_{s,t}(z))_{s,t} \in \mathcal{M}^2_{s-v}(\mathcal{F}_{s,t}) \) and \( (G_{s,t}(z))_{s,t} \in \mathcal{M}^2_{s-v}(\mathcal{F}_{s,t}) \) for each \( z \in \mathbb{R}^n \). By a stochastic inclusion in the plane we mean the relation

\[ z_{s,t} \in z_{0,0} + \int_0^s \int_0^t F_{u,v}(x_{u,v}) \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]
Every two parameter stochastic process $(z_{s,t})_{s,t \geq 0} \in D$ such that $F_{s,t}(z_{s,t}) \in M^2_{2-v}(\mathcal{F}_{s,t})$ and $G_{s,t}(z_{s,t}) \in M^2_{2-v}(\mathcal{F}_{s,t})$ satisfying a.s. the relation (3) is said to be a global solution to (3). Suppose $F$ and $G$ satisfy the following conditions $(C_1)$

(i) $F = \{(F_{s,t}(z))_{s,t \geq 0} : z \in R^n \}$ and $G = \{(G_{s,t}(z))_{s,t \geq 0} : z \in R^n \}$ are such that mappings $R^2_+ \times \Omega \times R^n \ni (s,t,w,z) \mapsto F_{s,t}(z) \in Cl(R^n)$ and $R^2_+ \times \Omega \times R^n \ni (s,t,w,z) \mapsto G_{s,t}(z) \in Cl(R^n)$ are $\beta^n_+ \otimes \Omega \otimes \beta^n$ measurable.

(ii) $(F_{s,t}(z)_{s,t \geq 0})$ and $(G_{s,t}(z))_{s,t \geq 0}$ are uniformly square-integrably bounded, i.e.,

$$\sup_{z \in \Omega} \|F_{s,t}(z)\| \in L^2_1 \quad \text{and} \quad \sup_{z \in \Omega} \|G_{s,t}(z)\| \in L^2_1.$$ 

Now define a linear continuous mapping $\Phi$ on $L^2_n \times L^2_n$ taking $\Phi(f,g) = (I_n f + J_n g)_{s,t \geq 0}$ for $(f,g) \in L^2_n \times L^2_n$. It is clear that $\Phi$ maps $L^2_n \times L^2_n$ into $D$. Let $\varphi = \varphi_{s,t} \in D$ be given. For $F$ and $G$ satisfying $(C_1)$ define a set-valued mapping $K$ on $D$ by setting

$$K(z) = \varphi + \Phi(S^2 F_{s,t}(z))_{s,t \geq 0} \times S^2(G_{s,t}(z))_{s,t \geq 0}$$

for $z = (z_{s,t})_{s,t \geq 0} \in D$.

Let $F$ and $G$ satisfy conditions $(C_1)$ and the following condition $(C_2)$ : set-valued functions

$$D \ni z \mapsto (F_{s,t}(z))_{s,t \geq 0}(\omega) \subset R^n \quad \text{and} \quad D \ni z \mapsto (G_{s,t}(z))_{s,t \geq 0}(\omega) \subset R^n$$

are w-w.s.u.s.c. on $D$, i.e., for every $z \in D$ and every sequence $(z_n)$ of $(D, \| \cdot \|)$ converging weakly to $z$, one has

$$\overline{h} \iint_A (F_{s,t}(z_{s,t}^{(n)})) \, dsdt \, dP, \quad \iint_A (G_{s,t}(z_{s,t})) \, dsdt \, dP \to 0$$

and

$$\overline{h} \iint_A (G_{s,t}(z_{s,t}^{(n)})) \, dsdt \, dP, \quad \iint_A (G_{s,t}(z_{s,t})) \, dsdt \, dP \to 0,$$

for every $A \in \beta_+^2 \otimes \Omega$.

**Lemma 6.** Assume $F$ and $G$ take on convex values and satisfy $(C_1)$ and $(C_2)$. Then a set-valued mapping $K$ is u.s.c. as a multifunction defined on a locally convex topological Hausdorff space $(D, \sigma(D, D^*))$ with nonempty values in $(D, \sigma(D, D^*))$. 
Proof. Let $C$ be a nonempty weakly closed subset of $D$ and select a sequence $(z^n)$ of $K^-(C)$ weakly converging to $z \in D$. There is a sequence $(y^n)$ of $C$ such that $y^n \in K(z_n)$ for $n = 1, 2, \ldots$. By the uniform square-integrable boundedness of $F$ and $G$, there is a convex weakly compact subset $B \subset L^2_n \times L^2_n$ such that $K\{z_n\} \subset \Phi(B)$. Therefore, $y^n \in \Phi(B)$ for $n = 1, 2, \ldots$ which, by the weak compactness of $\Phi(B)$ implies the existence of a subsequence $(y^k)$ of $(y^n)$ weakly converging to $y \in \Phi(B)$. We have $y^k \in K(z^k)$ for $k = 1, 2, \ldots$. Let $(f^k, g^k) \in S^2(F_{s,t}(z^k_{s,t})) \times S^2(G_{s,t}(z^k_{s,t}))$ be such that $\Phi(f^k, g^k) = y^k$, for each $k = 1, 2, \ldots$. We have of course $(f^k, g^k) \in B$. Therefore, there is a subsequence, say again $(f^k, g^k)$ of $(f^k, g^k)$ weakly converging in $L^2_n \times L^2_n$ to $(f, g) \in B$. Now, for every $A \in \beta^+ \otimes \Omega$ one obtains

$$\text{dist} \left( \int_A \int f_{s,t} \, dsdtP, \int_A \int F_{s,t}(z) \, dsdtP \right) \leq \left| \int_A \int (f_{s,t} - f^k_{s,t}) \, dsdtP \right|$$

$$+ \text{dist} \left( \int_A \int f^k_{s,t} \, dsdtP, \int_A \int F_{s,t}(z^k) \, dsdtP \right)$$

$$+ h \left( \int_A \int F_{s,t}(z^k) \, dsdtP, \int_A \int F_{s,t}(z) \, dsdtP \right).$$

Therefore, $f \in S^2(F_{s,t}(z_{s,t}))_{s,t \geq 0}$. Quite similarly, we also get $g \in S^2(G_{s,t}(z_{s,t}))_{s,t \geq 0}$. Thus, \( \varphi + \Phi(f, g) \in K(z) \), which implies $y \in K(z)$. On the other hand, we also have $y \in C$, because $C$ is weakly closed. Therefore, $z \in K^-(C)$. Now the result follows immediately from Eberlein and Smulian’s Theorem.

Theorem 7. If $F$ and $G$ take on convex values and satisfy $(C_1)$ and $(C_2)$, then there is $z \in D$ such that

$$z_{s,t} \in \varphi_{s,t} + \int_0^t \int_0^s F_{u,v}(z_{u,v}) \, dudv + \int_0^t \int_0^s G_{u,v}(z_{u,v}) \, dw_{u,v}.$$  

Proof. Let

$$\{B = (f, g) \in L^2_n \times L^2_n : |f_{s,t}(\omega)| \leq \|F_{s,t}(\omega)\|, |g_{s,t}(\omega)| \leq \|G_{s,t}(\omega)\| \}$$

and put $K = \varphi + \Phi(B)$. It is clear that $K$ is a nonempty convex weakly compact subset of $D$ such that $K(z) \subset K$ for $z \in D$. By (iii) of Theorem 5
\( \mathcal{K}(z) \) is a convex and weakly compact subset of \( D \), for each \( z \in D \). By Lemma 6 \( \mathcal{K} \) is u.s.c. on a locally convex topological Hausdorff space \( (D, \sigma(D, D^*)) \). Therefore, by the Kakutani and Fan fixed point theorem there exists \( z \in K \) such that \( z \in \mathcal{K}(z) \), i.e.,

\[
z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, du \, dv.
\]

Assume now that \( F \) and \( G \) satisfy conditions \((C_1)\) and the following condition \((C_3)\) \( F \) and \( G \) are such that set-valued functions:

\[
D \ni z \rightarrow (F_{s,t}(z)_{s,t \geq 0}(\omega) \subset R^m) \quad \text{and} \quad D \ni z \rightarrow (G_{s,t}(z)_{s,t \geq 0}(\omega) \subset R^n)
\]

are s.-w.s.l.s.c. on \( D \), i.e., for every \( z \in D \) and every sequence \((z_n)\) of \((D, |_D)\) converging weakly to \( z \) one has

\[
\mathcal{R}\left[(F_{s,t}(z))_{s,t \geq 0}(w), (F_{s,t}(z_n))_{s,t \geq 0}(\omega)\right] \rightarrow 0
\]

and

\[
\mathcal{R}\left[(G_{s,t}(z))_{s,t \geq 0}(w), (G_{s,t}(z_n))_{s,t \geq 0}(\omega)\right] \rightarrow 0 \quad \text{a.e.}
\]

**Lemma 8.** Assume \( F \) and \( G \) take on convex values and satisfy \((C_1)\) and \((C_3)\). Then a set-valued mapping \( \mathcal{K} \) is l.s.c. as a multifunction defined on a locally convex topological Hausdorff space \((D, \sigma(D, D^*))\).

**Proof.** Let \( C \) be a nonempty weakly closed subset of \( D \) and \((z^{(n)})\) a sequence of \( \mathcal{R}_-(C) \) weakly converging to \( z \in D \). Select arbitrary \( u \in \mathcal{K}(z) \) and suppose \((f, g) \in S^2(F_{s,t}(z)_{s,t \geq 0}) \times S^2(G_{s,t}(z)_{s,t \geq 0})\) is such that \( u = \Phi(f, g) \). Let \((f^{(n)}, g^{(n)}) \in S^2(F_{s,t}(z^{(n)})_{s,t \geq 0}) \times S^2(G_{s,t}(z^{(n)})_{s,t \geq 0})\) be such that

\[
\left| f_{s,t}(\omega) - f_{s,t}^{(n)}(\omega) \right| = \text{dist} \left( f_{s,t}(\omega), \left( F_{s,t} \left( z^{(n)} \right) \right)(\omega) \right)
\]

and

\[
\left| g_{s,t}(\omega) - g_{s,t}^{(n)}(\omega) \right| = \text{dist} \left( g_{s,t}(\omega), \left( G_{s,t} \left( z^{(n)} \right) \right)(\omega) \right)
\]

on \( R^2_+ \times \Omega \), for each \( n = 1, 2, \ldots \). By virtue of \((C_3)\) one gets \( |f_{s,t}(\omega) - f_{s,t}^{(n)}(\omega)| \rightarrow 0 \) and \( |g_{s,t}(\omega) - g_{s,t}^{(n)}(\omega)| \rightarrow 0 \) a.e., as \( n \rightarrow \infty \). Hence by \((C_1)\) we can see that a sequence \((u^{(n)})\), defined by \( u^{(n)} = \Phi(f^{(n)}, g^{(n)}) \) weakly converges to \( u \). But \( u^{(n)} \in \mathcal{K}(z^{(n)}) \subset C \) for \( n = 1, 2, \ldots \) and \( C \) is weakly closed. Then \( u \in C \), which implies \( \mathcal{K}(z) \subset C \). Thus \( z \in \mathcal{K}_-(C) \).
Theorem 9. If $F$ and $G$ take on convex values and satisfy $(C_1)$ and $(C_3)$ then stochastic integral inclusion (1) admits a solution.

Proof. Let

$$B = \{(f,g) \in L^2_n \times L^2_n : |f_{s,t}(\omega)| \leq \|F_{s,t}(\omega)\|, |g_{s,t}(\omega)| \leq \|G_{s,t}(\omega)\|\}$$

and put $K = \varphi + \Phi(B)$. It is clear that $K$ is a nonempty convex weakly compact subset of $D$ such that $K(z) \subset K$ for $z \in D$. By virtue of Lemma 8, $K$ is l.s.c. as a set-valued mapping from a paracompact space $K$ considered with its relative topology induced by a weak topology $\sigma(D, D^*)$ on $D$ into a Banach space $(D, \|\cdot\|)$. By (iii) of Theorem 5, $K(z)$ is a closed and convex subset of $D$, for each $z \in K$. Therefore, by Michael’s theorem, there is a continuous selection $k : K \to D$ for $K$. But $K(K) \subset K$. Then $K$ maps $K$ into itself and is continuous with respect to the relative topology on $K$, defined above. Therefore, by the Schauder and Tikhonov fixed point theorem, there is $z \in K$ such that $z = k(z) \in K(z)$, i.e.,

$$z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v} \text{ a.s.}$$

References


Received 10 October 2001
Revised 18 November 2001