SET-VALUED STOCHASTIC INTEGRALS AND
STOCHASTIC INCLUSIONS IN A PLANE

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Abstract

We present the concepts of set-valued stochastic integrals in a plane and prove the existence of a solution to stochastic integral inclusions of the form

$$z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}.$$  

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1. Introduction

In this paper, we shall consider a stochastic inclusion in a plane of the form:

$$z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}.$$  

The results of the paper generalize some problems dealing with stochastic integral equations in a plane of the form:

$$x_{s,t} = x + \int_0^s \int_0^t a(u,v,x_{u,v}) \, dw_{u,v} + \int_0^s \int_0^t b(u,v,x_{u,v}) \, du \, dv.$$
that have been investigated by Tudor [8], Panomarenco [5] and others. The first integral in equation (2) is a stochastic integral in a plane considered with respect to the Wiener-Yeh process. It was defined by Cairoli [1], Panomarenco [5], Tsarenco [7]. Its generalization with respect to two parameters martingale is given by Cairoli and Walsh in [2].

In this paper, we introduce the stochastic integral in a plane for set-valued mappings taking values from space $\text{Comp}(R^n)$ of all nonempty closed subsets of $n$-dimensional Euclidean space $R^n$ with respect to the Wiener-Yeh process and we establish some of its properties. The space $\text{Comp}(R^n)$ is considered with the Hausdorff metric $h$ defined in the usual way, i.e., $h(A, B) = \max\{\overline{h}(A, B), \overline{h}(B, A)\}$, for $A, B \in \text{Comp}(R^n)$ where $\overline{h}(A, B) = \sup \{\text{dist}(a, B) : a \in A\}$ and $\overline{h}(B, A) = \sup \{\text{dist}(b, A) : b \in B\}$.

We assume, as given, a complete filtered probability space $(\Omega, F, (F_{s,t})_{s,t \geq 0}, P)$ where a filtration $(F_{s,t})_{s,t \geq 0}$ is assumed to satisfy:

(i) if $(s, t) \leq (s', t')$ then $F_{s,t} \subset F_{s',t'}$;

(ii) $F_{0,0}$ contains all null sets of $F$;

(iii) for each $(s, t), F_{s,t} = \bigcap_{(u,v)>(s,t)} F_{u,v}$;

(iv) for each $(s, t), F^1_{s,t}$ and $F^2_{s,t}$ are conditionally independent given $F_{s,t}$.

2. Basic definitions and notations

Throughout the paper, we shall assume that filtered complete probability space $(\Omega, F, (F_{s,t})_{s,t \geq 0}, P)$ satisfies the following conditions:

(i) if $(s, t) \leq (s', t')$ then $F_{s,t} \subset F_{s',t'}$;

(ii) $F_{0,0}$ contains all null sets of $F$;

(iii) for each $(s, t), F_{s,t} = \bigcap_{(u,v)>(s,t)} F_{u,v}$;

(iv) for each $(s, t), F^1_{s,t}$ and $F^2_{s,t}$ are conditionally independent given $F_{s,t}$.
We write \((s, t) \leq (s', t')\) iff \(s \leq s'\) and \(t \leq t'\). As usual, we shall consider a set \(R^2_+ \times \Omega\) as a measurable space with the product \(\sigma\)-algebra \(B^2_+ \otimes \mathcal{F}\).

An \(n\)-dimensional two parameter stochastic process \(z\) (random field) understood as a function \(z : R^2_+ \times \Omega \rightarrow \mathbb{R}^n\) with \(\mathcal{F}\)-measurable sections \(z_{s,t}\), each \(s, t \geq 0\), is denoted by \((z_{s,t})_{s,t \geq 0}\). It is measurable if \(z\) is \(B^2_+ \otimes \mathcal{F}\)-measurable. The process \((z_{s,t})_{s,t \geq 0}\) is \(\mathcal{F}_{s,t}\)-adapted or adapted if \(z_{s,t}\) is \(\mathcal{F}_{s,t}\)-measurable for \(s, t \geq 0\). Every measurable and adapted process is called nonanticipative. In what follows, the Banach space \((L^2(R^2_+ \times \Omega, \beta^2_+ \otimes \mathcal{F},\, dsdt \otimes P), \| \cdot \|_2)\), where \(dsdt\) denotes Lebesgue measure on \(R^2\), will be denoted by \(L^2_0\). Similarly, the Banach spaces \((L^2(\Omega, \mathcal{F}, P, R^n), \| \cdot \|)\) and \((L^2(\Omega, \mathcal{F}_{s,t}, P, R^n), \| \cdot \|)\) are denoted by \(L^2(\mathcal{F})\) and \(L^2(\mathcal{F}_{s,t})\), respectively. Let \(\mathcal{M}^2(\mathcal{F}_{s,t})\) denote the family of all (equivalence classes of) \(n\)-dimensional nonanticipative processes \((f_{s,t})_{s,t \geq 0}\) such that \(\int_0^\infty \int_0^\infty |f_{s,t}|^2\, dsdt < \infty\), a.s.

We shall also consider a subspace \(L^2(\mathcal{F}_{s,t})\) of \(\mathcal{M}^2(\mathcal{F}_{s,t})\) defined by

\[
L^2(\mathcal{F}_{s,t}) = \left\{ (f_{s,t})_{s,t \geq 0} \in \mathcal{M}^2(\mathcal{F}_{s,t}) : E \int_0^\infty \int_0^\infty |f_{s,t}|^2\, dsdt < \infty \right\}.
\]

It is a closed subset of the Banach space \(L^2_0\). Finally, by \(M_n(\mathcal{F}_{s,t})\) we denote a space of all (equivalence classes of) \(n\)-dimensional \(\mathcal{F}_{s,t}\)-measurable mappings. Throughout, by \((w_{s,t})_{s,t \geq 0}\) we mean two parameter \(\mathcal{F}_{s,t}\)-Brownian motion, i.e., a continuous Gaussian process such that \(E(w_{s,t}) = 0\) and \(E(w_{s,t} \cdot w_{s',t'}) = \min(s, s') \cdot \min(t, t')\) for every \(s, s', t, t'\). Given \(g \in \mathcal{M}^2(\mathcal{F}_{s,t})\), by \((\int_0^t \int_0^s g_{u,v} \, dw_{u,v})_{s,t \geq 0}\) we denote its stochastic integral with respect to an \(\mathcal{F}_{s,t}\)-Brownian motion \((w_{s,t})_{s,t \geq 0}\). Let us denote by \(D\) the family of all \(n\)-dimensional \(\mathcal{F}_{s,t}\)-adapted continuous processes \((z_{s,t})_{s,t \geq 0}\) such that \(E \sup_{s,t \geq 0} |z_{s,t}|^2 < \infty\). The space \(D\) is considered as a normed space with the norm \(\| \cdot \|_1\) defined by \(|x|_1 = \sup_{s,t \geq 0} |x_{s,t}| \|L^2\) for \(x = (x_{s,t})_{s,t \in D}\), where \(\| \cdot \|_L^2\) is a norm of \(L^2(\Omega, \mathcal{F}, P, R)\). It can be verified that \((D, \| \cdot \|_1)\) is a Banach space. In what follows, we shall deal with upper and lower semicontinuous set-valued mappings. Recall that a set-valued mapping \(\mathcal{R}\) with nonempty values in a topological space \((Y, \mathcal{T}_Y)\) is said to be upper (lower) semicontinuous \([\text{s.c.}]/\text{l.s.c.}\) on a topological space \((X, \mathcal{T}_X)\) if \(\mathcal{R}^-(C) := \{ x \in X : \mathcal{R}(x) \cap C \neq \emptyset \} \) (resp. \(\mathcal{R}^-(C) := \{ x \in X : \mathcal{R}(x) \subset C \} \) is a closed subset of \(X\) for every closed set \(C \subset Y\). In particular, for \(\mathcal{R}\) defined on a metric space \((X, d)\) with values in \(\text{Comp } (R^n)\), it is equivalent (see \([4]\)) to \(\lim_{n \to \infty} \overline{h}(\mathcal{R}(x_n), \mathcal{R}(x)) = 0\) (resp. \(\lim_{n \to \infty} \overline{h}(\mathcal{R}(x), \mathcal{R}(x_n)) = 0\)) for every \(x \in X\).
and every sequence \((x_n)\) of \(X\) converging to \(x\). In what follows, we shall need the following well-known (see [4]) fixed point and continuous selection theorems.

**Theorem 1** (Schauder, Tikhonov). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(f\) a continuous mapping of \(K\) into itself. Then \(f\) has fixed point in \(K\).

**Theorem 2** (Kakutani, Fan). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(CCl(X)\) a family of all nonempty closed convex subsets of \(K\). If \(K : K \to CCl(K)\) is u.s.c. on \(K\), then there exists \(x \in K\) such that \(x \in K(x)\).

**Theorem 3** (Michael). Let \((X, \mathcal{F}_X)\) be a paracompact space and let \(K\) be a set-valued mapping from \(X\) to a Banach space \((Y, \| \cdot \|)\) whose values are closed and convex. Suppose, further \(K\) is l.s.c. on \(X\). Then there is a continuous function \(f : X \to Y\) such that \(f(x) \in K(x)\), for each \(x \in X\).

### 3. Set-valued stochastic integrals in the plane

Given measure space \((X, \beta, m)\), a set-valued function \(R : X \to \text{Comp } (\mathbb{R}^n)\) is said to be \(\beta\)-measurable if \(R^-(C) \in \beta\) for every closed \(C \subset \mathbb{R}^n\). For such a multifunction we define integrals subtrajectory (see [6]) as a set

\[
S_p(R) = \{ g \in L^p(X, \beta, m, \mathbb{R}^n) : g(x) \in R(x) \text{ a.e.} \}.
\]

It is clear that for nonemptiness of \(S_p(R)\) we have to assume more than \(\beta\)-measurability of \(R\). In what follows, we shall assume that \(\beta\)-measurable set-valued function \(R : X \to \text{Comp } (\mathbb{R}^n)\) is \(p\)-integrably bounded, \(p \geq 1\), i.e., that a real-valued mapping \(X \ni x \to \| R(x) \| \in \mathbb{R}_+\) belongs to \(L^p(X, \beta, m, \mathbb{R}_+)\) where \(\| A \| := \sup\{|a| : a \in A\}\) for \(A \in \text{Comp } (\mathbb{R}^n)\). It can be verified that a \(\beta\)-measurable set-valued mapping \(R : X \to \text{Comp } (\mathbb{R}^n)\) is \(p\)-integrably bounded \(p \geq 1\), if and only if \(S_p(R)\) is nonempty and bounded in \(L^p(X, \beta, m, \mathbb{R}^n)\) (see [4]). Finally, it is easy to see that \(S_p(R)\) is decomposable, i.e., such that \(1|_{A}f_1 + 1|_{X\setminus A}f_2 \in S_p(R)\) for \(A \in \beta\) and \(f_1, f_2 \in S_p(R)\). We have the following general result dealing with properties of subtrajectory integrals (see [4], [5]).

**Proposition 4.** Let \(R : X \to \text{Comp } (\mathbb{R}^n)\) be \(\beta\)-measurable and \(p\)-integrably bounded, \(p \geq 1\). The set \(S_p(R)\) is a nonempty bounded and closed subset
of $L^p(X, \beta, m, R^n)$. Moreover, if $\mathcal{R}$ takes on convex values, then $S^p(\mathcal{R})$ is convex and weakly compact in $L^p(X, \beta, m, R^n)$.

Let $G = (G_{s,t})_{s,t \geq 0}$ be a set-valued two parameter stochastic process with values in $\text{Comp}(R^n)$, i.e., a family of $\mathcal{F}$-measurable set-valued mappings $G_{s,t} : \Omega \rightarrow \text{Comp}(R^n), s, t \geq 0$. We call $G$ measurable if it is $\beta_2^+ \otimes \mathcal{F}$-measurable. Similarly, $G$ is said to be $\mathcal{F}_{s,t}$-adapted or adapted if $G_{s,t}$ is $\mathcal{F}_{s,t}$-measurable for each $s, t \geq 0$. A measurable and adapted set-valued two parameter stochastic process is called nonanticipative. Denote by $\mathcal{M}^2_{s,t}(\mathcal{F}_{s,t})$ a family of all nonanticipative set-valued processes $G = (G_{s,t})_{s,t \geq 0}$ such that $\int_0^\infty \int_0^\infty ||G_{s,t}||^2 ds dt < \infty$, a.s. Immediately from the Kuratowski and Ryll-Nardzewski measurable selection theorem (see [4]) it follows that for every $F, G \in \mathcal{M}^2_{s,t}(\mathcal{F}_{s,t})$ their subtrajectory integrals

$$S^2(F) = f \in \{M^2(\mathcal{F}_{s,t} : f_{s,t} \in (\omega) \in \mathcal{F}_{s,t}(\omega), ds dt \times P - a.e.\}$$

and

$$S^2(G) = \{g \in M^2(\mathcal{F}_{s,t}) : g_{s,t}(\omega) \in \mathcal{G}_{s,t}(\omega), ds dt \times P - a.e.\},$$

are nonempty. Indeed, let $\sum = \{Z \in B^2_+ \otimes \mathcal{F} : Z_{s,t} \in \mathcal{F}_{s,t}, \text{each } s, t \geq 0\}$, where $Z_{s,t}$ denotes a section of $Z$ determined by $s, t \geq 0$. It is a $\sigma$-algebra on $R^2_+ \times \Omega$ and function $f : R^2_+ \times \Omega \rightarrow R^n$ (a multifunction $F : R^2_+ \times \Omega \rightarrow \text{Comp}(R^n)$) is nonanticipative if and only if it is $\sum$-measurable. Therefore, by the Kuratowski and Ryll-Nardzewski measurable selection theorem, every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in \mathcal{M}^2_{s-t}(\mathcal{F}_{s,t})$ such selectors belong to $\mathcal{M}^2(\mathcal{F}_{s,t})$. Finally, denote

$$\mathcal{L}^2_{s-t}(\mathcal{F}_{s,t}) = \left\{G \in \mathcal{M}^2_{s-t}(\mathcal{F}_{s,t}) : E \int_0^\infty \int_0^\infty ||G_{s,t}||^2 ds dt < \infty\right\}.$$  

Given set-valued two parameter processes

$$F = (F_{s,t})_{s, t \geq 0} \in \mathcal{M}^2_{s-t}(\mathcal{F}_{s,t}) \quad \text{and} \quad G = (G_{s,t})_{s, t \geq 0} \in \mathcal{M}^2_{s-t}(\mathcal{F}_{s,t})$$

by their stochastic integrals we mean families $(\int_0^s \int_0^t F_{u,v} dudv)_{s, t \geq 0}$ and $(\int_0^s \int_0^t G_{u,v} dudv)_{s, t \geq 0}$ of subsets of $M_u(\mathcal{F}_{s,t})$, defined by

$$\int_0^s \int_0^t F_{u,v} dudv := \left\{f : f_{s,t} \in S^2(F) : \int_0^s \int_0^t f_{u,v} dudv \right\}$$
and
\[
\int_0^s \int_0^t G_{u,v} \, dw_{u,v} := \left\{ \int_0^s \int_0^t g_{u,v} \, dw_{u,w} : g \in S^2(G) \right\}.
\]

Immediately, from the above definitions the following simple results follow.

**Theorem 5.** Let \( F, G \in M^2_{s-t}(F_{s,t}) \). Then
(i) \( \int_0^s \int_0^t F_{u,v} \, dudv \) and \( \int_0^s \int_0^t G_{u,v} \, dw_{u,v} \) are nonempty subsets of \( M_n(F_{s,t}) \), each \( s, t \geq 0 \). They are convex if \( F \) and \( G \) take on convex values.
(ii) If \( G \in L^2_{s-t}(F_{s,t}) \), then \( \int_0^s \int_0^t G_{u,v} \, dw_{u,v} \) is a nonempty subset of \( L^2(F_{s,t}) \), each \( s \geq 0, t \geq 0 \),
(iii) If \( F, G \in L^2_{s-t}(F_{s,t}) \) take on convex values then \( \int_0^s \int_0^t F_{u,v} \, dudv \) and \( \int_0^s \int_0^t G_{u,v} \, dw_{u,v} \) are nonempty convex and weakly compact subsets of \( L^2(F_{s,t}) \), each \( s \geq 0, t \geq 0 \).

**Proof.** It is clear that (i) follows immediately from the definitions of set-valued stochastic integrals. To verify (ii) let us observe that the operator \( I_{s,t} \) defined by the formula \( I_{s,t} g = \int_0^s \int_0^t g_{u,v} \, dw_{u,v} \) is a linear continuous mapping from \( L^2(F_{s,t}) \) into \( L^2(F_{s,t}) \). By the properties of stochastic integrals in the plane (see [6]) one has \( E|I_{s,t}|^2 = \| g \|_2^2 \), each \( s, t \geq 0 \) and \( g \in L^2(F_{s,t}) \), where \( \| \cdot \|_2 \) denotes the norm of \( L^2(F_{s,t}) \). Therefore, \( I_{s,t} \) maps closed subsets of \( L^2(F_{s,t}) \) into closed subsets of \( L^2(F_{s,t}) \). Thus (ii) is satisfied. To verify (iii), it suffices only to observe that for every \( s, t \geq 0 \), all sets \( S^2(F) \) and \( S^2(G) \) are convex and weakly compact in \( L^2(F_{s,t}) \). Hence, \( J_{s,t}(S^2(F)) \) and \( I_{s,t}(S^2(G)) \) (where \( J_{s,t} \) is defined by \( J_{s,t} f = \int_0^s \int_0^t f_{u,v} \, dudv \)) are convex and weakly compact subsets of \( L^2(F_{s,t}) \) because \( J_{s,t} \) and \( I_{s,t} \) are linear and continuous for fixed \( s, t \geq 0 \). Then (iii) also holds.

\[\square\]

4. **Stochastic inclusions in the plane**

Let \( F = \{(F_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n\} \) and \( G = \{(G_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n\} \). Assume \( F \) and \( G \) are such that \( (F_{s,t}(z))_{s,t \geq 0} \in M^2_{s-t}(F_{s,t}) \) and \( (G_{s,t}(z))_{s,t \geq 0} \in M^2_{s-t}(F_{s,t}) \) for each \( z \in \mathbb{R}^n \). By a stochastic inclusion in the plane we mean the relation

\[ z_{s,t} \in z_{0,0} + \int_0^s \int_0^t F_{u,v}(x_{u,v}) \, dudv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]
Every two parameter stochastic process \( (z_{s,t})_{s,t \geq 0} \in D \) such that \( F_{s,t}(z_{s,t}) \in \mathcal{M}_{2-\nu}(\mathcal{F}_{s,t}) \) and \( G_{s,t}(z_{s,t}) \in \mathcal{M}_{2-\nu}(\mathcal{F}_{s,t}) \) satisfying a.s. the relation (3) is said to be a global solution to (3). Suppose \( F \) and \( G \) satisfy the following conditions (\( C_1 \))

(i) \( F = \{(F_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n\} \) and \( G = \{(G_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n\} \) are such that mappings \( R_+^2 \times \Omega \times \mathbb{R}^n \ni (s,t,w,z) \rightarrow F_{s,t}(z) \in Cl(\mathbb{R}^n) \) and \( R_+^2 \times \Omega \times \mathbb{R}^n \ni (s,t,w,z) \rightarrow G_{s,t}(z) \in Cl(\mathbb{R}^n) \) are \( \beta_+^2 \otimes \Omega \otimes \beta^n \) measurable.

(ii) \((F_{s,t}(z))_{s,t \geq 0}\) and \((G_{s,t}(z))_{s,t \geq 0}\) are uniformly square-integrably bounded, i.e.,
\[
\sup_{z \in \mathbb{R}^n} \|F_{s,t}(z)\| \in L_1^2 \text{ and } \sup_{z \in \mathbb{R}^n} \|G_{s,t}(z)\| \in L_1^2.
\]
Now define a linear continuous mapping \( \Phi \) on \( L_n^2 \times L_n^2 \) taking \( \Phi(f,g) = (I_{s,t}f + J_{s,t}g)_{s,t \geq 0} \) for \((f,g) \in L_n^2 \times L_n^2\). It is clear that \( \Phi \) maps \( L_n^2 \times L_n^2 \) into \( D \). Let \( \varphi = \varphi_{s,t} \in D \) be given. For \( F \) and \( G \) satisfying \((C_1)\) define a set-valued mapping \( K \) on \( D \) by setting
\[
K(z) = \varphi + \Phi(S^2F_{s,t}(z))_{s,t \geq 0} \times S^2(G_{s,t}(z))_{s,t \geq 0}
\]
for \( z = (z_{s,t})_{s,t \geq 0} \in D \).

Let \( F \) and \( G \) satisfy conditions \((C_1)\) and the following condition \((C_2)\) : set-valued functions
\[
D \ni z \rightarrow (F_{s,t}(z))_{s,t \geq 0}(\omega) \subset \mathbb{R}^n \text{ and } D \ni z \rightarrow (G_{s,t}(z))_{s,t \geq 0}(\omega) \subset \mathbb{R}^n
\]
are w-w.s.u.s.c. on \( D \), i.e., for every \( z \in D \) and every sequence \((z_n)\) of \((D, \| \cdot \|_1)\) converging weakly to \( z \), one has
\[
\left( F_{s,t}\left(z^{(n)}_{s,t}\right) \right) dsdtP, \left( G_{s,t}\left(z_{s,t}\right) \right) dsdtP \rightarrow 0
\]
and
\[
\left( F_{s,t}\left(z^{(n)}_{s,t}\right) \right) dsdtP, \left( G_{s,t}\left(z_{s,t}\right) \right) dsdtP \rightarrow 0,
\]
for every \( A \in \beta_+^2 \otimes \Omega \).

**Lemma 6.** Assume \( F \) and \( G \) take on convex values and satisfy \((C_1)\) and \((C_2)\). Then a set-valued mapping \( K \) is u.s.c. as a multifunction defined on a locally convex topological Hausdorff space \((D, \sigma(D,D^*))\) with nonempty values in \((D, \sigma(D,D^*))\).
**Proof.** Let \( C \) be a nonempty weakly closed subset of \( D \) and select a sequence \((z^n)\) of \( K^-(C) \) weakly converging to \( z \in D \). There is a sequence \((y^n)\) of \( C \) such that \( y^n \in K(z_n) \) for \( n = 1, 2, \ldots \). By the uniform square-integrable boundedness of \( F \) and \( G \), there is a convex weakly compact subset \( B \subset \mathcal{L}_n^2 \times \mathcal{L}_n^2 \) such that \( K(z) \subset K \) for \( z \in D \). By (iii) of Theorem 5 which, by the weak compactness of \( \Phi(B) \) implies the existence of a subsequence \((y^k)\) of \((y^n)\) weakly converging to \( y \in \Phi(B) \). We have \( y^k \in K(z^k) \) for \( k = 1, 2, \ldots \). Let \((f^k, g^k) \in S^2(F_{s,t}(z^k_{s,t})) \times S^2(G_{s,t}(z^k_{s,t})) \) be such that \( \Phi(f^k, g^k) = y^k \), for each \( k = 1, 2, \ldots \). We have of course \((f^k, g^k) \in B \). Therefore, there is a subsequence, say again \((f^k, g^k) \) of \((f^k, g^k) \) weakly converging in \( \mathcal{L}_n^2 \times \mathcal{L}_n^2 \) to \((f, g) \in B \). Now, for every \( A \in \beta_+^2 \otimes \Omega \) one obtains

\[
\text{dist} \left( \int_A \int_A f_{s,t} \, dsdtdP, \int_A \int_A F_{s,t}(z) \, dsdtdP \right) \leq \int_A \int_A \left( f_{s,t} - f^k_{s,t} \right) \, dsdtdP \quad + \quad \text{dist} \left( \int_A \int_A f^k_{s,t} \, dsdtdP, \int_A \int_A F_{s,t}(z^k) \, dsdtdP \right) \quad + \quad \beta \left( \int_A \int_A F_{s,t}(z^k) \, dsdtdP, \int_A \int_A F_{s,t}(z) \, dsdtdP \right).
\]

Therefore, \( f \in S^2(F_{s,t}(z_{s,t}))_{s,t \geq 0} \). Quite similarly, we also get \( g \in S^2(G_{s,t}(z_{s,t}))_{s,t \geq 0} \). Thus, \( \varphi + \Phi(f, g) \in K(z) \), which implies \( y \in K(z) \). On the other hand, we also have \( y \in C \), because \( C \) is weakly closed. Therefore, \( z \in K^-(C) \). Now the result follows immediately from Eberlein and Smulian’s Theorem.

**Theorem 7.** If \( F \) and \( G \) take on convex values and satisfy \((C_1)\) and \((C_2)\), then there is \( z \in D \) such that

\[
z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, dudv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}.
\]

**Proof.** Let

\[
\mathcal{B} = \{ (f, g) \in \mathcal{L}_n^2 \times \mathcal{L}_n^2 : |f_{s,t}(\omega)| \leq \|F_{s,t}(\omega)\|, |g_{s,t}(\omega)| \leq \|G_{s,t}(\omega)\| \}
\]

and put \( K = \varphi + \Phi(\mathcal{B}) \). It is clear that \( K \) is a nonempty convex weakly compact subset of \( D \) such that \( K(z) \subset K \) for \( z \in D \). By (iii) of Theorem 5
$K(z)$ is a convex and weakly compact subset of $D$, for each $z \in D$. By Lemma 6 $K$ is u.s.c. on a locally convex topological Hausdorff space $(D, \sigma(D, D^*)$. Therefore, by the Kakutani and Fan fixed point theorem there exists $z \in K$ such that $z \in K(z)$, i.e.,

$$z_{s,t} \in \varphi_{s,t} + \int_0^s \int_t^l F_{u,v}(z_{u,v}) \, dudv + \int_0^s \int_t^l G_{u,v}(z_{u,v}) \, dw_{u,v}.$$  

Assume now that $F$ and $G$ satisfy conditions $(C_1)$ and the following condition $(C_3) F$ and $G$ are such that set-valued functions:

$$D \ni z \mapsto \{ F_{s,t}(z) \}_{s,t \geq 0} \subset \mathbb{R}^n \quad \text{and} \quad D \ni z \mapsto \{ G_{s,t}(z) \}_{s,t \geq 0} \subset \mathbb{R}^n$$

are s.-w.s.l.s.c. on $D$, i.e., for every $z \in D$ and every sequence $(z_n)$ of $(D, | \cdot |)$ converging weakly to $z$ one has

$$\mathcal{H} \left( (F_{s,t}(z))_{s,t \geq 0}(w), (F_{s,t}(z_n))_{s,t \geq 0}(\omega) \right) \to 0$$

and

$$\mathcal{H} \left( (G_{s,t}(z))_{s,t \geq 0}(w), (G_{s,t}(z_n))_{s,t \geq 0}(\omega) \right) \to 0 \quad \text{a.e.}$$

**Lemma 8.** Assume $F$ and $G$ take on convex values and satisfy $(C_1)$ and $(C_3)$. Then a set-valued mapping $K$ is l.s.c. as a multifunction defined on a locally convex topological Hausdorff space $(D, \sigma(D, D^*)$).

**Proof.** Let $C$ be a nonempty weakly closed subset of $D$ and $(z^{(n)})$ a sequence of $\mathcal{R}_-(C)$ weakly converging to $z \in D$. Select arbitrary $u \in K(z)$ and suppose $(f, g) \in S^2(\{ F_{s,t}(z) \}_{s,t \geq 0}) \times S^2(\{ G_{s,t}(z) \}_{s,t \geq 0})$ is such that $u = \Phi(f, g)$. Let $(f^{(n)}, g^{(n)}) \in S^2(\{ F_{s,t}(z^{(n)}) \}_{s,t \geq 0}) \times S^2(\{ G_{s,t}(z^{(n)}) \}_{s,t \geq 0})$ be such that

$$|f_{s,t}(\omega) - f^{(n)}_{s,t}(\omega)| = \text{dist} \left( f_{s,t}(\omega), \left( F_{s,t}(z^{(n)}) \right)(\omega) \right)$$

and

$$|g_{s,t}(\omega) - g^{(n)}_{s,t}(\omega)| = \text{dist} \left( g_{s,t}(\omega), \left( G_{s,t}(z^{(n)}) \right)(\omega) \right)$$

on $\mathbb{R}_+^2 \times \Omega$, for each $n = 1, 2, \ldots$. By virtue of $(C_3)$ one gets $|f_{s,t}(\omega) - f^{(n)}_{s,t}(\omega)| \to 0$ and $|g_{s,t}(\omega) - g^{(n)}_{s,t}(\omega)| \to 0$ a.e., as $n \to \infty$. Hence by $(C_1)$ we can see that a sequence $(u^{(n)})$, defined by $u^{(n)} = \Phi(f^{(n)}, g^{(n)})$ weakly converges to $u$. But $u^{(n)} \in K(z^{(n)}) \subset C$ for $n = 1, 2, \ldots$ and $C$ is weakly closed. Then $u \in C$, which implies $K(z) \subset C$. Thus $z \in K(C)$. 

\[ \square \]
Theorem 9. If \( F \) and \( G \) take on convex values and satisfy \((C_1)\) and \((C_3)\) then stochastic integral inclusion (1) admits a solution.

Proof. Let
\[
\mathcal{B} = \{(f, g) \in L^2_n \times L^2_n : |f_{s,t}(\omega)| \leq \|F_{s,t}(\omega)\|, |g_{s,t}(\omega)| \leq \|G_{s,t}(\omega)\|\}
\]
and put \( K = \varphi + \Phi(\mathcal{B}) \). It is clear that \( K \) is a nonempty convex weakly compact subset of \( D \) such that \( K(z) \subset K \) for \( z \in D \). By virtue of Lemma 8, \( \mathcal{K} \) is l.s.c. as a set-valued mapping from a paracompact space \( K \) considered with its relative topology induced by a weak topology \( \sigma(D, D^\ast) \) on \( D \) into a Banach space \( (D, \| \cdot \|) \). By (iii) of Theorem 5, \( \mathcal{K}(z) \) is a closed and convex subset of \( D \), for each \( z \in K \). Therefore, by Michael’s theorem, there is a continuous selection \( k : K \to D \) for \( \mathcal{K} \). But \( K(K) \subset K \). Then \( \mathcal{K} \) maps \( K \) into itself and is continuous with respect to the relative topology on \( K \), defined above. Therefore, by the Schauder and Tikhonov fixed point theorem, there is \( z \in K \) such that \( z = k(z) \in K(z) \), i.e.,
\[
z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, dudv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v} \text{ a.s.}
\]

References


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