SET-VALUED STOCHASTIC INTEGRALS AND STOCHASTIC INCLUSIONS IN A PLANE

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Abstract

We present the concepts of set-valued stochastic integrals in a plane and prove the existence of a solution to stochastic integral inclusions of the form

\[ z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]

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1. Introduction

In this paper, we shall consider a stochastic inclusion in a plane of the form:

\[ z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]

The results of the paper generalize some problems dealing with stochastic integral equations in a plane of the form:

\[ x_{s,t} = x + \int_0^s \int_0^t a(u,v,x_{u,v}) \, dw_{u,v} + \int_0^s \int_0^t b(u,v,x_{u,v}) \, du \, dv \]
that have been investigated by Tudor [8], Panomarenco [5] and others. The first integral in equation (2) is a stochastic integral in a plane considered with respect to the Wiener-Yeh process. It was defined by Cairoli [1], Panomarenco [5], Tsarenco [7]. Its generalization with respect to two parameters martingale is given by Cairoli and Walsh in [2].

In this paper, we introduce the stochastic integral in a plane for set-valued mappings taking values from space Comp (\(\mathbb{R}^n\)) of all nonempty closed subsets of \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) with respect to the Wiener-Yeh process and we establish some of its properties. The space Comp (\(\mathbb{R}^n\)) is considered with the Hausdorff metric \(h\) defined in the usual way, i.e.,

\[ h(A, B) = \max\{\overline{h}(A, B), \overline{\overline{h}}(B, A)\}, \]

where \(\overline{\overline{h}}(B, A) = \sup\{\text{dist} (b, A) : b \in B\}\) and \(\overline{h}(A, B) = \sup\{\text{dist} (a, B) : a \in A\}\).

We assume, as given, a complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_{s,t})_{s,t \geq 0}, P)\) where a filtration \((\mathcal{F}_{s,t})_{s,t \geq 0}\) is assumed to satisfy:

(i) if \((s, t) \leq (s', t')\) then \(\mathcal{F}_{s,t} \leq \mathcal{F}_{s', t'}\);
(ii) \(\mathcal{F}_{0,0}\) contains all null sets of \(\mathcal{F}\);
(iii) for each \((s, t)\), \(\mathcal{F}_{s,t} = \bigcap_{(u,v) > (s,t)} \mathcal{F}_{u,v}\);
(iv) for each \((s, t)\), \(\mathcal{F}_{s,t}^1\) and \(\mathcal{F}_{s,t}^2\) are conditionally independent given \(\mathcal{F}_{s,t}\).

2. Basic definitions and notations

Throughout the paper, we shall assume that filtered complete probability space \((\Omega, \mathcal{F}, (\mathcal{F}_{s,t})_{s,t \geq 0}, P)\) satisfies the following conditions:

(i) if \((s, t) \leq (s', t')\) then \(\mathcal{F}_{s,t} \leq \mathcal{F}_{s', t'}\);
(ii) \(\mathcal{F}_{0,0}\) contains all null sets of \(\mathcal{F}\);
(iii) for each \((s, t)\), \(\mathcal{F}_{s,t} = \bigcap_{(u,v) > (s,t)} \mathcal{F}_{u,v}\);
(iv) for each \((s, t)\), \(\mathcal{F}_{s,t}^1\) and \(\mathcal{F}_{s,t}^2\) are conditionally independent given \(\mathcal{F}_{s,t}\).
We write \((s, t) \leq (s', t')\) iff \(s \leq s'\) and \(t \leq t'\). As usual, we shall consider a set \(R^2_+ \times \Omega\) as a measurable space with the product \(\sigma\)-algebra \(B^2_+ \otimes \mathcal{F}\).

An \(n\)-dimensional two parameter stochastic process \(z\) (random field) understood as a function \(z : R^2_+ \times \Omega \to R^n\) with \(\mathcal{F}\)-measurable sections \(z_{s,t}\), each \(s, t \geq 0\), is denoted by \((z_{s,t})_{s,t \geq 0}\). It is measurable if \(z\) is \(B^2_+ \otimes \Omega\)-measurable.

As usual, we shall consider a subspace \(L^2_2(\mathcal{F}_{s,t})\) of \(M^2(\mathcal{F}_{s,t})\) defined by \(L^2_2(\mathcal{F}_{s,t}) = \left\{ (f_{s,t})_{s,t \geq 0} \in M^2(\mathcal{F}_{s,t}) : E \int_0^\infty \int_0^\infty |f_{s,t}|^2 \, ds \, dt < \infty \right\}\).

It is a closed subset of the Banach space \(L^2_2\). Finally, by \(M_n(\mathcal{F}_{s,t})\) we denote a space of all (equivalence classes of) \(n\)-dimensional \(\mathcal{F}_{s,t}\)-measurable mappings. Throughout, by \((w_{s,t})_{s,t \geq 0}\) we mean two parameter \(\mathcal{F}_{s,t}\)-Brownian motion, i.e., a continuous Gaussian process such that \(E(w_{s,t}) = 0\) and \(E(w_{s,t}w_{s',t'}) = \min(s, s') \cdot \min(t, t')\) for every \(s, s', t, t'\). Given \(g \in M^2(\mathcal{F}_{s,t})\), by \(\int_0^t \int_0^s g_{u,v} \, dw_{u,v} \) we denote its stochastic integral with respect to an \(\mathcal{F}_{s,t}\)-Brownian motion \((w_{s,t})_{s,t \geq 0}\). Let us denote by \(D\) the family of all \(n\)-dimensional \(\mathcal{F}_{s,t}\)-adapted continuous processes \((z_{s,t})_{s,t \geq 0}\) such that \(E \sup_{s,t \geq 0} |z_{s,t}|^2 < \infty\). The space \(D\) is considered as a normed space with the norm \(|\cdot|_1\) defined by \(|x|_1 = \sup_{s,t \geq 0} |x_{s,t}| L^2\) for \(x = (x_{s,t})_{s,t \in D}\), where \(\|\cdot\|_{L^2}\) is a norm of \(L^2(\Omega, \mathcal{F}, P, R)\). It can be verified that \((D, \|\cdot\|_1)\) is a Banach space. In what follows, we shall deal with upper and lower semicontinuous set-valued mappings. Recall that a set-valued mapping \(\mathcal{R}\) with nonempty values in a topological space \((Y, T_Y)\) is said to be upper (lower) semicontinuous [u.s.c.(l.s.c.)] on a topological space \((X, T_X)\) if \(\mathcal{R}^{-}(C) := \{x \in X : \mathcal{R}(x) \cap C \neq \emptyset\}\) (resp. \(\mathcal{R}^{-}(C) := \{x \in X : \mathcal{R}(x) \subset C\}\)) is a closed subset of \(X\) for every closed set \(C \subset Y\). In particular, for \(\mathcal{R}\) defined on a metric space \((X, d)\) with values in \(\text{Comp} (R^n)\), it is equivalent (see [4]) to \(\lim_{n \to \infty} \overline{\mathcal{R}}(\mathcal{R}(x_n), \mathcal{R}(x)) = 0\) (\(\lim_{n \to \infty} \overline{\mathcal{R}}(\mathcal{R}(x), \mathcal{R}(x_n)) = 0\)) for every \(x \in X\).
and every sequence \((x_n)\) of \(X\) converging to \(x\). In what follows, we shall need the following well-known (see [4]) fixed point and continuous selection theorems.

**Theorem 1** (Schauder, Tikhonov). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(f\) a continuous mapping of \(K\) into itself. Then \(f\) has fixed point in \(K\).

**Theorem 2** (Kakutani, Fan). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(CCl(X)\) a family of all nonempty closed convex subsets of \(K\). If \(K : K \to CCl(K)\) is u.s.c. on \(K\), then there exists \(x \in K\) such that \(x \in K(x)\).

**Theorem 3** (Michael). Let \((X, \mathcal{F}_X)\) be a paracompact space and let \(K\) be a set-valued mapping from \(X\) to a Banach space \((Y, \| \cdot \|)\) whose values are closed and convex. Suppose, further \(K\) is l.s.c. on \(X\). Then there is a continuous function \(f : X \to Y\) such that \(f(x) \in K(x)\), for each \(x \in X\).

3. Set-valued stochastic integrals in the plane

Given measure space \((X, \beta, m)\), a set-valued function \(R : X \to \text{Comp}(R^n)\) is said to be \(\beta\)-measurable if \(R^{-}(C) \in \beta\) for every closed \(C \subset R^n\). For such a multifunction we define integrals subtrajectory (see [6]) as a set

\[
S^p(R) = \{ g \in L^p(X, \beta, m, R^n) : g(x) \in R(x) \text{ a.e.} \}.
\]

It is clear that for nonemptiness of \(S^p(R)\) we have to assume more than \(\beta\)-measurability of \(R\). In what follows, we shall assume that \(\beta\)-measurable set-valued function \(R : X \to \text{Comp}(R^n)\) is \(p\)-integrably bounded, \(p \geq 1\), i.e., that a real-valued mapping \(X \ni x \to \|R(x)\| \in R_+\) belongs to \(L^p(X, \beta, m, R_+)\) where \(\|A\| := \sup\{|a| : a \in A\}\) for \(A \in \text{Comp}(R^n)\). It can be verified that a \(\beta\)-measurable set-valued mapping \(R : X \to \text{Comp}(R^n)\) is \(p\)-integrably bounded \(p \geq 1\), if and only if \(S^p(R)\) is nonempty and bounded in \(L^p(X, \beta, m, R^n)\) (see [4]). Finally, it is easy to see that \(S^p(R)\) is decomposable, i.e., such that \(1|Af_1 + 1|A \setminus Af_2 \in S^p(R)\) for \(A \in \beta\) and \(f_1, f_2 \in S^p(R)\).

We have the following general result dealing with properties of subtrajectory integrals (see [4], [5]).

**Proposition 4.** Let \(R : X \to \text{Comp}(R^n)\) be \(\beta\)-measurable and \(p\)-integrably bounded, \(p \geq 1\). The set \(S^p(R)\) is a nonempty bounded and closed subset...
of $L^p(X, \beta, m, R^n)$. Moreover, if $R$ takes on convex values, then $S^p(R)$ is convex and weakly compact in $L^p(X, \beta, m, R^n)$.

Let $G = (G_{s,t})_{s,t \geq 0}$ be a set-valued two parameter stochastic process with values in $\text{Comp}(R^n)$, i.e., a family of $\mathcal{F}$-measurable set-valued mappings $G_{s,t} : \Omega \to \text{Comp}(R^n), s, t \geq 0$. We call $G$ measurable if it is $\beta^p_+ \otimes \mathcal{F}$-measurable. Similarly, $G$ is said to be $F_{s,t}$-adapted or adapted if $G_{s,t}$ is $F_{s,t}$-measurable for each $s, t \geq 0$. A measurable and adapted set-valued two parameter stochastic process is called nonanticipative. Denote by $\mathcal{M}^2_{s,t}(F_{s,t})$ a family of all nonanticipative set-valued processes $G = (G_{s,t})_{s,t \geq 0}$ such that $\int_0^\infty \int_0^\infty \|G_{s,t}\|^2 dsdt < \infty$, a.s. Immediately from the Kuratowski and Ryll-Nardzewski measurable selection theorem (see [4]) it follows that for every $F, G \in \mathcal{M}^2_{s,t}(F_{s,t})$ their subtrajectory integrals

$$S^2(F) = f \in \mathcal{M}^2(F_{s,t}) : f_{s,t} \in (\omega) \in F_{s,t}(\omega), dsdt \times P - \text{a.e.} \}$$

and

$$S^2(G) = \{ g \in \mathcal{M}^2(F_{s,t}) : g_{s,t}(\omega) \in G_{s,t}(\omega), dsdt \times P - \text{a.e.} \},$$

are nonempty. Indeed, let $\sum = \{ Z \in B^2_+ \otimes \mathcal{F} : Z_{s,t} \in F_{s,t}, \text{each } s, t \geq 0 \}$, where $Z_{s,t}$ denotes a section of $Z$ determined by $s, t \geq 0$. It is a $\sigma$-algebra on $R^2_+ \times \Omega$ and function $f : R^2_+ \times \Omega \to R^n$ (a multifunction $F : R^2_+ \times \Omega \to \text{Comp}(R^n)$) is nonanticipative if and only if it is $\sum$-measurable. Therefore, by the Kuratowski and Ryll-Nardzewski measurable selection theorem, every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in \mathcal{M}^2_{s,t}(F_{s,t})$ such selectors belong to $\mathcal{M}^2(F_{s,t})$. Finally, denote

$$L^2_{s,t}(F_{s,t}) = \left\{ G \in \mathcal{M}^2_{s,t}(F_{s,t}) : E \int_0^\infty \int_0^\infty \|G_{s,t}\|^2 dsdt < \infty \right\}.$$

Given set-valued two parameter processes

$$F = (F_{s,t})_{s,t \geq 0} \in \mathcal{M}^2_{s,t}(F_{s,t}) \quad \text{and} \quad G = (G_{s,t})_{s,t \geq 0} \in \mathcal{M}^2_{s,t}(F_{s,t})$$

by their stochastic integrals we mean families $(\int_0^t f_{u,v} \, du \, dv)_{s,t \geq 0}$ and $(\int_0^t f_{u,v} \, dw_{u,v})_{s,t \geq 0}$ of subsets of $\mathcal{M}_{n}(F_{s,t})$, defined by

$$\int_0^s \int_0^t F_{u,v} \, du \, dv = \left\{ \int_0^s \int_0^t f_{u,v} \, du \, dv : f \in S^2(F) \right\}$$
and
\[ \int_0^s \int_0^t G_{u,v} \, dw_u, v := \left\{ \int_0^s \int_0^t g_{u,v} \, dw_u, w : g \in S^2(G) \right\}. \]

Immediately, from the above definitions the following simple results follow.

**Theorem 5.** Let \( F, G \in M^2_{s \to v}(\mathcal{F}_{s,t}) \). Then

(i) \( \int_0^t \int_0^s F_{u,v} \, du \, dv \) and \( \int_0^t \int_0^s G_{u,v} \, du \, dv \) are nonempty subsets of \( \mathcal{M}_{s \to v}(\mathcal{F}_{s,t}) \), each \( s, t \geq 0 \). They are convex if \( F \) and \( G \) take on convex values.

(ii) If \( G \in L^2_{s \to v}(\mathcal{F}_{s,t}) \), then \( \int_0^t \int_0^s G_{u,v} \, du \, dv \) is a nonempty subset of \( L^2(\mathcal{F}_{s,t}) \), each \( s \geq 0, t \geq 0 \).

(iii) If \( F, G \in L^2_{s \to v}(\mathcal{F}_{s,t}) \) take on convex values then \( \int_0^t \int_0^s F_{u,v} \, du \, dv \) and \( \int_0^t \int_0^s G_{u,v} \, du \, dv \) are nonempty convex and weakly compact subsets of \( L^2(\mathcal{F}_{s,t}) \), each \( s \geq 0, t \geq 0 \).

**Proof.** It is clear that (i) follows immediately from the definitions of set-valued stochastic integrals. To verify (ii) let us observe that the operator \( I_{s,t} \) defined by the formula \( I_{s,t} g = \int_0^t \int_0^s g_{u,v} \, du \, dv \) is a linear continuous mapping from \( L^2(\mathcal{F}_{s,t}) \) into \( L^2(\mathcal{F}_{s,t}) \). By the properties of stochastic integrals in the plane (see [6]) one has \( E|I_{s,t} g|^2 = \| g \|^2_2 \), each \( s, t \geq 0 \) and \( g \in L^2(\mathcal{F}_{s,t}) \), where \( \| \cdot \|_2 \) denotes the norm of \( L^2(\mathcal{F}_{s,t}) \). Therefore, \( I_{s,t} \) maps closed subsets of \( L^2(\mathcal{F}_{s,t}) \) into closed subsets of \( L^2(\mathcal{F}_{s,t}) \). Thus (ii) is satisfied. To verify (iii), it suffices only to observe that for every \( s, t \geq 0 \), all sets \( S^2(F) \) and \( S^2(G) \) are convex and weakly compact in \( L^2(\mathcal{F}_{s,t}) \). Hence, \( J_{s,t}(S^2(F)) \) and \( I_{s,t}(S^2(G)) \) (where \( J_{s,t} \) is defined by \( J_{s,t} f = \int_0^t \int_0^s f_{u,v} \, du \, dv \)) are convex and weakly compact subsets of \( L^2(\mathcal{F}_{s,t}) \) because \( J_{s,t} \) and \( I_{s,t} \) are linear and continuous for fixed \( s, t \geq 0 \). Then (iii) also holds. \( \blacksquare \)

**4. Stochastic inclusions in the plane**

Let \( F = \{(F_{s,t}(z))_{s,t \geq 0} : z \in R^n\} \) and \( G = \{(G_{s,t}(z))_{s,t \geq 0} : z \in R^n\} \). Assume \( F \) and \( G \) are such that \( (F_{s,t}(z))_{s,t \geq 0} \in M^2_{s \to v}(\mathcal{F}_{s,t}) \) and \( (G_{s,t}(z))_{s,t \geq 0} \in M^2_{s \to v}(\mathcal{F}_{s,t}) \) for each \( z \in R^n \). By a stochastic inclusion in the plane we mean the relation

\[ z_{s,t} \in z_{0,0} + \int_0^s \int_0^t F_{u,v}(x_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_u, v. \]
Every two parameter stochastic process \((z_{s,t})_{s,t \geq 0} \in D\) such that \(F_{s,t}(z_{s,t}) \in \mathcal{M}_{s-t}^2(\mathcal{F}_{s,t})\) and \(G_{s,t}(z_{s,t}) \in \mathcal{M}_{s-t}^2(\mathcal{F}_{s,t})\) satisfying a.s. the relation (3) is said to be a global solution to (3). Suppose \(F\) and \(G\) satisfy the following conditions \((C_1)\)

(i) \(F = \{(F_{s,t}(z))_{s,t \geq 0} : z \in R^n\}\) and \(G = \{(G_{s,t}(z))_{s,t \geq 0} : z \in R^n\}\) are such that mappings \(R^2_+ \times \Omega \times R^n \ni (s,t,w,z) \mapsto F_{s,t}(z) \in Cl(R^n)\) and \(R^2_+ \times \Omega \times R^n \ni (s,t,w,z) \mapsto G_{s,t}(z) \in Cl(R^n)\) are \(\beta^2_+ \otimes \Omega \otimes \beta^n\) measurable.

(ii) \((F_{s,t}(z))_{s,t \geq 0}\) and \((G_{s,t}(z))_{s,t \geq 0}\) are uniformly square-integrably bounded, i.e.,
\[
\sup_{z \in R^n} \|F_{s,t}(z)\| \in L^2_1 \text{ and } \sup_{z \in R^n} \|G_{s,t}(z)\| \in L^2_1.
\]

Now define a linear continuous mapping \(\Phi\) on \(L^2_1 \times L^2_2\) taking \(\Phi(f,g) = (I_{s,t}f + J_{s,t}g)_{s,t \geq 0}\) for \((f,g) \in L^2_1 \times L^2_2\). It is clear that \(\Phi\) maps \(L^2_1 \times L^2_2\) into \(D\). Let \(\varphi = \varphi_{s,t} \in D\) be given. For \(F\) and \(G\) satisfying \((C_1)\) define a set-valued mapping \(\mathcal{K}\) on \(D\) by setting
\[
\mathcal{K}(z) = \varphi + \Phi(S^2F_{s,t}(z))_{s,t \geq 0} \times S^2(G_{s,t}(z))_{s,t \geq 0}
\]
for \(z = (z_{s,t})_{s,t \geq 0} \in D\).

Let \(F\) and \(G\) satisfy conditions \((C_1)\) and the following condition \((C_2)\): set-valued functions
\[
D \ni z \mapsto (F_{s,t}(z))_{s,t \geq 0}(\omega) \in R^n\text{ and } D \ni z \mapsto (G_{s,t}(z))_{s,t \geq 0}(\omega) \in R^n
\]
are w-w.s.u.s.c. on \(D\), i.e., for every \(z \in D\) and every sequence \((z_n)\) of \((D, \| \cdot \|)\) converging weakly to \(z\), one has
\[
\overline{h} \iint_A (F_{s,t}(z_{s,t}^{(n)})) \, ds dt d\mathcal{P}, \quad \iint_A (G_{s,t}(z_{s,t}^{(n)})) \, ds dt d\mathcal{P} \rightarrow 0
\]
and
\[
\overline{h} \iint_A (G_{s,t}(z_{s,t}^{(n)})) \, ds dt d\mathcal{P}, \quad \iint_A (G_{s,t}(z_{s,t})) \, ds dt d\mathcal{P} \rightarrow 0,
\]
for every \(A \in \beta^2_+ \otimes \Omega\).

**Lemma 6.** Assume \(F\) and \(G\) take on convex values and satisfy \((C_1)\) and \((C_2)\). Then a set-valued mapping \(\mathcal{K}\) is u.s.c. as a multifunction defined on a locally convex topological Hausdorff space \((D, \sigma(D,D^*))\) with nonempty values in \((D, \sigma(D,D^*))\).
Proof. Let $C$ be a nonempty weakly closed subset of $D$ and select a sequence $(z^n)$ of $K^-(C)$ weakly converging to $z \in D$. There is a sequence $(y^n)$ of $C$ such that $y^n \in K(z_n)$ for $n = 1, 2, \ldots$. By the uniform square-integrable boundedness of $F$ and $G$, there is a convex weakly compact subset $B \subset L_n^2 \times L_n^2$ such that $K(z_n) \subset B$ for $z \in D$. Therefore, $y^n \in \Phi(B)$ for $n = 1, 2, \ldots$ which, by the weak compactness of $\Phi(B)$ implies the existence of a subsequence $(y^k)$ of $(y^n)$ weakly converging to $y \in \Phi(B)$. We have $y^k \in K(z^K)$ for $k = 1, 2, \ldots$. Let $(f^k, g^k) \in S^2(F_s,t(z^K)) \times S^2(G_{s,t}(z^K))$ be such that $\Phi(f^k, g^k) = y^k$, for each $k = 1, 2, \ldots$. We have of course $(f^k, g^k) \in B$. Therefore, there is a subsequence, say again $(f^k, g^k)$ of $(f^k, g^k)$ weakly converging in $L_n^2 \times L_n^2$ to $(f, g) \in B$. Now, for every $A \in \beta_n^2 \otimes \Omega$ one obtains

$$
\begin{align*}
\text{dist} \left( \int \int f_{s,t} \ ds dt dP, \int \int F_{s,t}(z) \ ds dt dP \right) & \leq \left| \int \int \left( f_{s,t} - f_{s,t}^k \right) \ ds dt dP \right| \\
+ \text{dist} \left( \int \int f^k_{s,t} \ ds dt dP, \int \int F_{s,t}(z^k) \ ds dt dP \right) \\
+ \overline{h} \left( \int \int F_{s,t}(z^k) \ ds dt dP, \int \int F_{s,t}(z) \ ds dt dP \right) .
\end{align*}
$$

Therefore, $f \in S^2(F_{s,t}(z),s,t \geq 0)$. Quite similarly, we also get $g \in S^2(G_{s,t}(z),s,t \geq 0)$. Thus, $\varphi + \Phi(f, g) \in K(z)$, which implies $y \in K(z)$. On the other hand, we also have $y \in C$, because $C$ is weakly closed. Therefore, $z \in K^-(C)$. Now the result follows immediately from Eberlein and Smulian’s Theorem.

Theorem 7. If $F$ and $G$ take on convex values and satisfy $(C_1)$ and $(C_2)$, then there is $z \in D$ such that

$$z_{s,t} \in \varphi_{s,t} + \int_0^t \int_0^t F_{u,v}(z_{a,v}) \ du dv + \int_0^t \int_0^t G_{u,v}(z_{a,v}) \ dw_{u,v}.$$ 

Proof. Let

$$\{B = (f, g) \in L_n^2 \times L_n^2 : |f_{s,t}(\omega)| \leq \|F_{s,t}(\omega)\|, |g_{s,t}(\omega)| \leq \|G_{s,t}(\omega)\| \}$$

and put $K = \varphi + \Phi(B)$. It is clear that $K$ is a nonempty convex weakly compact subset of $D$ such that $K(z) \subset K$ for $z \in D$. By (iii) of Theorem 5
\( K(z) \) is a convex and weakly compact subset of \( D \), for each \( z \in D \). By Lemma 6 \( K \) is u.s.c. on a locally convex topological Hausdorff space \((D, \sigma(D, D^*))\). Therefore, by the Kakutani and Fan fixed point theorem there exists \( z \in K \) such that \( z \in K(z) \), i.e.,

\[
z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}.
\]

Assume now that \( F \) and \( G \) satisfy conditions \((C_1)\) and the following condition \((C_3)\) \( F \) and \( G \) are such that set-valued functions:

\[
D \ni z \rightarrow (F_{s,t}(z)_{s,t \geq 0}(\omega) \subset \mathbb{R}^n) \quad \text{and} \quad D \ni z \rightarrow (G_{s,t}(z)_{s,t \geq 0}(\omega) \subset \mathbb{R}^n)
\]

are s.-w.l.s.c. on \( D \), i.e., for every \( z \in D \) and every sequence \((z_n)\) of \((D, \| \cdot \|)\) converging weakly to \( z \) one has

\[
\mathfrak{H}[ (F_{s,t}(z))_{s,t \geq 0}(w), (F_{s,t}(z_n))_{s,t \geq 0}(\omega) ] \to 0
\]

and

\[
\mathfrak{H}[ (G_{s,t}(z))_{s,t \geq 0}(w), (G_{s,t}(z_n))_{s,t \geq 0}(\omega) ] \to 0 \quad \text{a.e.}
\]

**Lemma 8.** Assume \( F \) and \( G \) take on convex values and satisfy \((C_1)\) and \((C_3)\). Then a set-valued mapping \( K \) is l.s.c. as a multifunction defined on a locally convex topological Hausdorff space \((D, \sigma(D, D^*))\).

**Proof.** Let \( C \) be a nonempty weakly closed subset of \( D \) and \((z^{(n)})\) a sequence of \( \mathcal{R}_-(C) \) weakly converging to \( z \in D \). Select arbitrary \( u \in K(z) \) and suppose \((f, g) \in S^2(F_{s,t}(z)_{s,t \geq 0}) \times S^2(G_{s,t}(z)_{s,t \geq 0})\) is such that \( u = \Phi(f, g) \).

Let \((f^{(n)}, g^{(n)}) \in S^2(F_{s,t}(z^{(n)})_{s,t \geq 0}) \times S^2(G_{s,t}(z^{(n)})_{s,t \geq 0})\) be such that

\[
| f_{s,t}(\omega) - f^{(n)}_{s,t}(\omega) | = \text{dist} \left( f_{s,t}(\omega), \left( F_{s,t} \left( z^{(n)} \right) \right) (\omega) \right)
\]

and

\[
| g_{s,t}(\omega) - g^{(n)}_{s,t}(\omega) | = \text{dist} \left( g_{s,t}(\omega), \left( G_{s,t} \left( z^{(n)} \right) \right) (\omega) \right)
\]

on \( R^2_+ \times \Omega \), for each \( n = 1, 2, \ldots \). By virtue of \((C_3)\) one gets \( | f_{s,t}(\omega) - f^{(n)}_{s,t}(\omega) | \to 0 \) and \( | g_{s,t}(\omega) - g^{(n)}_{s,t}(\omega) | \to 0 \) a.e., as \( n \to \infty \). Hence by \((C_1)\) we can see that a sequence \((u^{(n)})\), defined by \( u^{(n)} = \Phi(f^{(n)}, g^{(n)}) \) weakly converges to \( u \). But \( u^{(n)} \in K(z^{(n)}) \subset C \) for \( n = 1, 2, \ldots \) and \( C \) is weakly closed. Then \( u \in C \), which implies \( K(z) \subset C \). Thus \( z \in K_-(C) \).
Theorem 9. If $F$ and $G$ take on convex values and satisfy $(C_1)$ and $(C_3)$ then stochastic integral inclusion (1) admits a solution.

Proof. Let

$$\mathcal{B} = \{(f, g) \in L^2_n \times L^2_n : |f_s,t(\omega)| \leq \|F_s,t(\omega)\|, |g_s,t(\omega)| \leq \|G_s,t(\omega)\|\}$$

and put $K = \phi + \Phi(\mathcal{B})$. It is clear that $K$ is a nonempty convex weakly compact subset of $D$ such that $\mathcal{K}(z) \subset K$ for $z \in D$. By virtue of Lemma 8, $K$ is l.s.c. as a set-valued mapping from a paracompact space $K$ considered with its relative topology induced by a weak topology $\sigma(D, D^*)$ on $D$ into a Banach space $(D, \| \cdot \|)$. By (iii) of Theorem 5, $\mathcal{K}(z)$ is a closed and convex subset of $D$, for each $z \in K$. Therefore, by Michael’s theorem, there is a continuous selection $k : K \to D$ for $K$. But $\mathcal{K}(K) \subset K$. Then $\mathcal{K}$ maps $K$ into itself and is continuous with respect to the relative topology on $K$, defined above. Therefore, by the Schauder and Tikhonov fixed point theorem, there is $z \in K$ such that $z = k(z) \in \mathcal{K}(z)$, i.e.,

$$z_{s,t} \in \phi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v} \text{ a.s.}$$

References


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