SET-VALUED STOCHASTIC INTEGRALS AND
STOCHASTIC INCLUSIONS IN A PLANE

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Abstract

We present the concepts of set-valued stochastic integrals in a plane and prove the existence of a solution to stochastic integral inclusions of the form

\[ z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]

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1. Introduction

In this paper, we shall consider a stochastic inclusion in a plane of the form:

\[ z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]  

The results of the paper generalize some problems dealing with stochastic integral equations in a plane of the form:

\[ x_{s,t} = x + \int_0^s \int_0^t a(u,v,x_{u,v}) \, dw_{u,v} + \int_0^s \int_0^t b(u,v,x_{u,v}) \, dv \]
that have been investigated by Tudor [8], Panomarenco [5] and others. The first integral in equation (2) is a stochastic integral in a plane considered with respect to the Wiener-Yeh process. It was defined by Cairoli [1], Panomarenco [5], Tsarenco [7]. Its generalization with respect to two parameters martingale is given by Cairoli and Walsh in [2].

In this paper, we introduce the stochastic integral in a plane for set-valued mappings taking values from space Comp ($\mathbb{R}^n$) of all nonempty closed subsets of $n$-dimensional Euclidean space $\mathbb{R}^n$ with respect to the Wiener-Yeh process and we establish some of its properties. The space Comp ($\mathbb{R}^n$) is considered with the Hausdorff metric $h$ defined in the usual way, i.e.,

$$h(A, B) = \max\{h(A, B), h(B, A)\},$$

for $A, B \in \text{Comp (} \mathbb{R}^n \text{)}$ where $h(A, B) = \sup \{\text{dist} (a, B) : a \in A\}$ and $h(B, A) = \sup \{\text{dist} (b, A) : b \in B\}$. We assume, as given, a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{s,t})_{s,t \geq 0}, P)$ where a filtration $(\mathcal{F}_{s,t})_{s,t \geq 0}$ is assumed to satisfy:

- (i) if $(s, t) \leq (s', t')$ then $\mathcal{F}_{s,t} < \mathcal{F}_{s',t'}$;
- (ii) $\mathcal{F}_{0,0}$ contains all null sets of $\mathcal{F}$;
- (iii) for each $(s, t), \mathcal{F}_{s,t} = \bigcap_{(u,v)>(s,t)} \mathcal{F}_{u,v}$;
- (iv) for each $(s, t), \mathcal{F}_{s,t}^1$ and $\mathcal{F}_{s,t}^2$ are conditionally independent given $\mathcal{F}_{s,t}$.

2. Basic definitions and notations

Throughout the paper, we shall assume that filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{s,t})_{s,t \geq 0}, P)$ satisfies the following conditions:

(i) if $(s, t) \leq (s', t')$ then $\mathcal{F}_{s,t} < \mathcal{F}_{s',t'}$;
(ii) $\mathcal{F}_{0,0}$ contains all null sets of $\mathcal{F}$;
(iii) for each $(s, t), \mathcal{F}_{s,t} = \bigcap_{(u,v)>(s,t)} \mathcal{F}_{u,v}$;
(iv) for each $(s, t), \mathcal{F}_{s,t}^1$ and $\mathcal{F}_{s,t}^2$ are conditionally independent given $\mathcal{F}_{s,t}$.
We write \((s, t) \leq (s', t')\) iff \(s \leq s'\) and \(t \leq t'\). As usual, we shall consider a set \(R^2_+ \times \Omega\) as a measurable space with the product \(\sigma\)-algebra \(B^2_2 \otimes \mathcal{F}\).

An \(n\)-dimensional two parameter stochastic process \(z\) (random field) understood as a function \(z : R^2_+ \times \Omega \rightarrow R^n\) with \(\mathcal{F}\)-measurable sections \(z_{s,t}\), each \(s, t \geq 0\), is denoted by \((z_{s,t})_{s,t \geq 0}\). It is measurable if \(z\) is \(B^2_2 \otimes \Omega\)-measurable. The process \((z_{s,t})_{s,t \geq 0}\) is \(\mathcal{F}_{s,t}\)-adapted or adapted if \(z_{s,t}\) is \(\mathcal{F}_{s,t}\)-measurable for \(s, t \geq 0\). Every measurable and adapted process is called nonanticipative. In what follows, the Banach space \((L^2(R^2_+ \times \Omega, \beta^2_2 \otimes \mathcal{F}, dsdt \otimes \mathcal{P}), \| \cdot \|_2)\), where \(dsdt\) denotes Lebesgue measure on \(R^2\), will be denoted by \(L^2_n\). Similarly, the Banach spaces \((L^2(\Omega, \mathcal{F}, P, R^n), \| \cdot \|)\) and \((L^2(\Omega, \mathcal{F}_{s,t}, P, R^n), \| \cdot \|)\) are denoted by \(L^2(\mathcal{F})\) and \(L^2(\mathcal{F}_{s,t})\), respectively. Let \(\mathcal{M}^2(\mathcal{F}_{s,t})\) denote the family of all (equivalence classes of) \(n\)-dimensional nonanticipative processes \((f_{s,t})_{s,t \geq 0}\) such that \(\int_0^\infty \int_0^\infty \| f_{s,t} \|^2 dsdt < \infty\), a.s.

We shall also consider a subspace \(L^2(\mathcal{F}_{s,t})\) of \(\mathcal{M}^2(\mathcal{F}_{s,t})\) defined by

\[
L^2(\mathcal{F}_{s,t}) = \{(f_{s,t})_{s,t \geq 0} \in \mathcal{M}^2(\mathcal{F}_{s,t}) : E \int_0^\infty \int_0^\infty \| f_{s,t} \|^2 dsdt < \infty\}.
\]

It is a closed subset of the Banach space \(L^2_n\). Finally, by \(M_n(\mathcal{F}_{s,t})\) we denote a space of all (equivalence classes of) \(n\)-dimensional \(\mathcal{F}_{s,t}\)-measurable mappings. Throughout, by \((w_{s,t})_{s,t \geq 0}\) we mean two parameter \(\mathcal{F}_{s,t}\)-Brownian motion, i.e., a continuous Gaussian process such that \(E(w_{s,t}) = 0\) and \(E(w_{s,t}w_{s',t'}) = \min(s, s') \cdot \min(t, t')\) for every \(s, s', t, t'\). Given \(g \in \mathcal{M}^2(\mathcal{F}_{s,t})\), by \((\int_0^s \int_0^t g_{u,v} dw_{u,v})_{s,t \geq 0}\) we denote its stochastic integral with respect to an \(\mathcal{F}_{s,t}\)-Brownian motion \((w_{s,t})_{s,t \geq 0}\). Let us denote by \(D\) the family of all \(n\)-dimensional \(\mathcal{F}_{s,t}\)-adapted continuous processes \((z_{s,t})_{s,t \geq 0}\) such that \(E \sup_{s,t \geq 0} \| z_{s,t} \|^2 < \infty\). The space \(D\) is considered as a normed space with the norm \(\| \cdot \|_1\) defined by \(\| x \|_1 = \sup_{s,t \geq 0} \| x_{s,t} \|_{L^2}\) for \(x = (x_{s,t})_{s,t \in D}\), where \(\| \cdot \|_{L^2}\) is a norm of \(L^2(\Omega, \mathcal{F}, P, R)\). It can be verified that \((D, \| \cdot \|_1)\) is a Banach space. In what follows, we shall deal with upper and lower semicontinuous set-valued mappings. Recall that a set-valued mapping \(\mathcal{R}\) with nonempty values in a topological space \((Y, \mathcal{T}_Y)\) is said to be upper (lower) semicontinuous \([\text{n.s.c.}, \text{l.s.c.}]\) on a topological space \((X, \mathcal{T}_X)\) if \(\mathcal{R}^-(C) := \{x \in X : \mathcal{R}(x) \cap C \neq \emptyset\}\) (resp. \(\mathcal{R}^-(C) := \{x \in X : \mathcal{R}(x) \subset C\}\)) is a closed subset of \(X\) for every closed set \(C \subset Y\). In particular, for \(\mathcal{R}\) defined on a metric space \((X, d)\) with values in \(\text{Comp} \(R^n)\), it is equivalent (see [4]) to \(\lim_{n \to \infty} \overline{\mathcal{R}(x_n)}(\mathcal{R}(x)) = 0\) (resp. \(\lim_{n \to \infty} \overline{\mathcal{R}(x_n)}) = 0\) for every \(x \in X\).
and every sequence \((x_n)\) of \(X\) converging to \(x\). In what follows, we shall need the following well-known (see [4]) fixed point and continuous selection theorems.

**Theorem 1** (Schauder, Tikhonov). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(f\) a continuous mapping of \(K\) into itself. Then \(f\) has fixed point in \(K\).

**Theorem 2** (Kakutani, Fan). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(CCl(X)\) a family of all nonempty closed convex subsets of \(K\). If \(K : K \to CCl(K)\) is u.s.c. on \(K\), then there exists \(x \in K\) such that \(x \in K(x)\).

**Theorem 3** (Michael). Let \((X, \mathcal{F}_X)\) be a paracompact space and let \(K\) be a set-valued mapping from \(X\) to a Banach space \((Y, \| \cdot \|)\) whose values are closed and convex. Suppose, further \(K\) is l.s.c. on \(X\). Then there is a continuous function \(f : X \to Y\) such that \(f(x) \in K(x)\), for each \(x \in X\).

### 3. Set-valued stochastic integrals in the plane

Given measure space \((X, \beta, m)\), a set-valued function \(R : X \to \text{Comp}(\mathbb{R}^n)\) is said to be \(\beta\)-measurable if \(R^-(C) \in \beta\) for every closed \(C \subset \mathbb{R}^n\). For such a multifunction we define integrals subtrajectory (see [6]) as a set

\[
S^p(R) = \{ g \in L^p(X, \beta, m, \mathbb{R}^n) : g(x) \in R(x) \text{ a.e.} \}.
\]

It is clear that for nonemptiness of \(S^p(R)\) we have to assume more than \(\beta\)-measurability of \(R\). In what follows, we shall assume that \(\beta\)-measurable set-valued function \(R : X \to \text{Comp}(\mathbb{R}^n)\) is \(p\)-integrably bounded, \(p \geq 1\), i.e., that a real-valued mapping \(X \ni x \to \|R(x)\| \in \mathbb{R}_+\) belongs to \(L^p(X, \beta, m, R_+)\) where \(\|A\| := \sup\{|a| : a \in A\}\) for \(A \in \text{Comp}(\mathbb{R}^n)\). It can be verified that a \(\beta\)-measurable set-valued mapping \(R : X \to \text{Comp}(\mathbb{R}^n)\) is \(p\)-integrably bounded \(p \geq 1\), if and only if \(S^p(R)\) is nonempty and bounded in \(L^p(X, \beta, m, R^n)\) (see [4]). Finally, it is easy to see that \(S^p(R)\) is decomposable, i.e., such that \(1|_{Af_1} + 1|_{X^c \setminus A}f_2 \in S^p(R)\) for \(A \in \beta\) and \(f_1, f_2 \in S^p(R)\).

We have the following general result dealing with properties of subtrajectory integrals (see [4], [5]).

**Proposition 4.** Let \(R : X \to \text{Comp}(\mathbb{R}^n)\) be \(\beta\)-measurable and \(p\)-integrably bounded, \(p \geq 1\). The set \(S^p(R)\) is a nonempty bounded and closed subset
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of $L^p(X, \beta, m, R^n)$. Moreover, if $R$ takes on convex values, then $S^p(R)$ is convex and weakly compact in $L^p(X, \beta, m, R^n)$.

Let $G = (G_{s,t})_{s,t \geq 0}$ be a set-valued two parameter stochastic process with values in $\text{Comp}(R^n)$, i.e., a family of $\mathcal{F}$-measurable set-valued mappings $G_{s,t} : \Omega \rightarrow \text{Comp}(R^n), s, t \geq 0$. We call $G$ measurable if it is $\beta^2_+ \otimes \mathcal{F}$-measurable. Similarly, $G$ is said to be $\mathcal{F}_{s,t}$-adapted or adapted if $G_{s,t}$ is $\mathcal{F}_{s,t}$-measurable for each $s, t \geq 0$. A measurable and adapted set-valued two parameter stochastic process is called nonanticipative. Denote by $\mathcal{M}_{s,t}^2(\mathcal{F}_{s,t})$ a family of all nonanticipative set-valued processes $G = (G_{s,t})_{s,t \geq 0}$ such that $\int_0^\infty \int_0^\infty \|G_{s,t}\|^2 dsdt < \infty$, a.s. Immediately from the Kuratowski and Ryll-Nardzewski measurable selection theorem (see [4]) it follows that for every $F, G \in \mathcal{M}_{s,t}^2(\mathcal{F}_{s,t})$ their subtrajectory integrals

$$S^2(F) = \{ f \in \mathcal{M}^2(\mathcal{F}_{s,t} : f_{s,t} \in (\omega) \in \mathcal{F}_{s,t}(\omega), dsdt \times P - \text{a.e.}\}$$

and

$$S^2(G) = \{ g \in \mathcal{M}^2(\mathcal{F}_{s,t} : g_{s,t}(\omega) \in G_{s,t}(\omega), dsdt \times P - \text{a.e.}\},$$

are nonempty. Indeed, let $\sum = \{ Z \in B^2_+ \otimes \mathcal{F} : Z_{s,t} \in \mathcal{F}_{s,t}, \text{each } s, t \geq 0\}$, where $Z_{s,t}$ denotes a section of $Z$ determined by $s, t \geq 0$. It is a $\sigma$-algebra on $R^2_+ \times \Omega$ and function $f : R^2_+ \times \Omega \rightarrow R^n$ (a multifunction $F : R^2_+ \times \Omega \rightarrow \text{Comp } (R^n)$) is nonanticipative if and only if it is $\sum$-measurable. Therefore, by the Kuratowski and Ryll-Nardzewski measurable selection theorem, every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in \mathcal{M}_{s,t}^2(\mathcal{F}_{s,t})$ such selectors belong to $\mathcal{M}^2(\mathcal{F}_{s,t})$. Finally, denote

$$\mathcal{L}_{s,t}^2(\mathcal{F}_{s,t}) = \left\{ G \in \mathcal{M}_{s,t}^2(\mathcal{F}_{s,t} : E \int_0^\infty \int_0^\infty \|G_{s,t}\|^2 dsdt < \infty \right\}.$$

Given set-valued two parameter processes

$$F = (F_{s,t})_{s,t \geq 0} \in \mathcal{M}_{s,t}^2(\mathcal{F}_{s,t}) \quad \text{and} \quad G = (G_{s,t})_{s,t \geq 0} \in \mathcal{M}_{s,t}^2(\mathcal{F}_{s,t})$$

by their stochastic integrals we mean families $(\int_0^s \int_0^t F_{u,v} dudv)_{s,t \geq 0}$ and $(\int_0^s \int_0^t G_{u,v} dudv)_{s,t \geq 0}$ of subsets of $\mathcal{M}(\mathcal{F}_{s,t})$, defined by

$$\int_0^s \int_0^t F_{u,v} dudv := \left\{ \int_0^s \int_0^t f_{u,v} dudv : f \in S^2(F) \right\}$$

and
and
\[ \int_0^s \int_0^t G_{u,v} \, dw_{u,v} := \left\{ \int_0^s \int_0^t g_{u,v} \, dw_{u,w} : g \in S^2(G) \right\}. \]

Immediately, from the above definitions the following simple results follow.

**Theorem 5.** Let \( F, G \in \mathcal{M}_2(F_{s,t}) \). Then

(i) \( \int_0^s \int_0^t F_{u,v} \, du \, dv \) and \( \int_0^s \int_0^t G_{u,v} \, dw_{u,v} \) are nonempty subsets of \( M_n(F_{s,t}) \), each \( s, t \geq 0 \). They are convex if \( F \) and \( G \) take on convex values.

(ii) If \( G \in L^2(F_{s,t}) \), then \( \int_0^s \int_0^t G_{u,v} \, dw_{u,v} \) is a nonempty subset of \( L^2(F_{s,t}) \), each \( s \geq 0, t \geq 0 \).

(iii) If \( F, G \in L^2(F_{s,t}) \) take on convex values then \( \int_0^s \int_0^t F_{u,v} \, du \, dv \) and \( \int_0^s \int_0^t G_{u,v} \, dw_{u,v} \) are nonempty convex and weakly compact subsets of \( L^2(F_{s,t}) \), each \( s \geq 0, t \geq 0 \).

**Proof.** It is clear that (i) follows immediately from the definitions of set-valued stochastic integrals. To verify (ii) let us observe that the operator \( I_{s,t} \) defined by the formula \( I_{s,t}g = \int_0^s \int_0^t g_{u,v} \, dw_{u,v} \) is a linear continuous mapping from \( L^2(F_{s,t}) \) into \( L^2(F_{s,t}) \). By the properties of stochastic integrals in the plane (see [6]) one has \( E|I_{s,t}|^2 = \|g\|_2^2 \), each \( s, t \geq 0 \) and \( g \in L^2(F_{s,t}) \), where \( \| \cdot \|_2 \) denotes the norm of \( L^2(F_{s,t}) \). Therefore, \( I_{s,t} \) maps closed subsets of \( L^2(F_{s,t}) \) into closed subsets of \( L^2(F_{s,t}) \). Thus (ii) is satisfied. To verify (iii), it suffices only to observe that for every \( s, t \geq 0 \), all sets \( S^2(F) \) and \( S^2(G) \) are convex and weakly compact in \( L^2(F_{s,t}) \). Hence, \( J_{s,t}(S^2(F)) \) and \( I_{s,t}(S^2(G)) \) (where \( J_{s,t} \) is defined by \( J_{s,t}f = \int_0^s \int_0^t f_{u,v} \, du \, dv \)) are convex and weakly compact subsets of \( L^2(F_{s,t}) \) because \( J_{s,t} \) and \( I_{s,t} \) are linear and continuous for fixed \( s, t \geq 0 \). Then (iii) also holds.

\[ \int_0^s \int_0^t F_{u,v} \, du \, dv + \int_0^s \int_0^t G_{u,v} \, dw_{u,v}. \]

4. **Stochastic inclusions in the plane**

Let \( F = \{ (F_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n \} \) and \( G = \{ (G_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n \} \). Assume \( F \) and \( G \) are such that \( (F_{s,t}(z))_{s,t \geq 0} \in \mathcal{M}_2(F_{s,t}) \) and \( (G_{s,t}(z))_{s,t \geq 0} \in \mathcal{M}_2(F_{s,t}) \) for each \( z \in \mathbb{R}^n \). By a stochastic inclusion in the plane we mean the relation

\[ z_{s,t} \in z_{0,0} + \int_0^s \int_0^t F_{u,v}(x_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]
Every two parameter stochastic process \((z_{s,t})_{s,t \geq 0} \in D\) such that \(F_{s,t}(z_{s,t}) \in \mathcal{M}_{2-\nu}^{2}(\mathcal{F}_{\cdot,\cdot})\) and \(G_{s,t}(z_{s,t}) \in \mathcal{M}_{2-\nu}^{2}(\mathcal{F}_{\cdot,\cdot})\) satisfying a.s. the relation (3) is said to be a global solution to (3). Suppose \(F\) and \(G\) satisfy the following conditions \((C_1)\)

\[
\begin{align*}
(i) \quad & F = \{(F_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n\} \text{ and } G = \{(G_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n\} \text{ are such that mappings } R^2_+ \times \Omega \times \mathbb{R}^n \ni (s,t,w,z) \to F_{s,t}(z) \in \text{Cl}(\mathbb{R}^n) \text{ and } R^2_+ \times \Omega \times \mathbb{R}^n \ni (s,t,w,z) \to G_{s,t}(z) \in \text{Cl}(\mathbb{R}^n) \text{ are } \beta^2_+ \otimes \Omega \otimes \beta^n \text{ measurable.} \\
(ii) \quad & (F_{s,t}(z))_{s,t \geq 0} \text{ and } (G_{s,t}(z))_{s,t \geq 0} \text{ are uniformly square-integrably bounded, i.e.,} \\
& \sup_{z \in \mathbb{R}^n} \|F_{s,t}(z)\| \in L^2_1 \text{ and } \sup_{z \in \mathbb{R}^n} \|G_{s,t}(z)\| \in L^2_1. \\
\end{align*}
\]

Now define a linear continuous mapping \(\Phi\) on \(L^2_1 \times L^2_1\) taking \(\Phi(f,g) = (I_{s,t}f + J_{s,t}g)_{s,t \geq 0}\) for \((f,g) \in L^2_1 \times L^2_1\). It is clear that \(\Phi\) maps \(L^2_1 \times L^2_1\) into \(D\). Let \(\varphi = \varphi_{s,t} \in D\) be given. For \(F\) and \(G\) satisfying \((C_1)\) define a set-valued mapping \(\mathcal{K}\) on \(D\) by setting

\[
\mathcal{K}(z) = \varphi + \Phi(S^2F_{s,t}(z))_{s,t \geq 0} \times S^2(G_{s,t}(z))_{s,t \geq 0}
\]

for \(z = (z_{s,t})_{s,t \geq 0} \in D\).

Let \(F\) and \(G\) satisfy conditions \((C_1)\) and the following condition \((C_2)\) : set-valued functions

\[
D \ni z \to (F_{s,t}(z))_{s,t \geq 0}(\omega) \subset \mathbb{R}^n \text{ and } D \ni z \to (G_{s,t}(z))_{s,t \geq 0}(\omega) \subset \mathbb{R}^n
\]

are w-w.s.u.s.c. on \(D\), i.e., for every \(z \in D\) and every sequence \((z_n)\) of \((D, \| \cdot \|_1)\) converging weakly to \(z\), one has

\[
\bar{h} \iint_{A} (F_{s,t}(z_{s,t}^{(n)})) \, dsdtP, \quad \iint_{A} (G_{s,t}(z_{s,t}^{(n)})) \, dsdtP \to 0
\]

and

\[
\bar{h} \iint_{A} (G_{s,t}(z_{s,t}^{(n)})) \, dsdtP, \quad \iint_{A} (G_{s,t}(z_{s,t}^{(n)})) \, dsdtP \to 0,
\]

for every \(A \in \beta^2_+ \otimes \Omega\).

**Lemma 6.** Assume \(F\) and \(G\) take on convex values and satisfy \((C_1)\) and \((C_2)\). Then a set-valued mapping \(\mathcal{K}\) is u.s.c. as a multifunction defined on a locally convex topological Hausdorff space \((D, \sigma(D,D^*))\) with nonempty values in \((D, \sigma(D,D^*))\).
Proof. Let \( C \) be a nonempty weakly closed subset of \( D \) and select a sequence \( (z^n) \) of \( K^-(C) \) weakly converging to \( z \in D \). There is a sequence \( (y^n) \) of \( C \) such that \( y^n \in K(z^n) \) for \( n = 1, 2, \ldots \). By the uniform square-integrable boundedness of \( F \) and \( G \), there is a convex weakly compact subset \( B \subset \mathcal{L}_2^n \times \mathcal{L}_n^2 \) such that \( K(z^n) \subset K \) for \( z \in D \). By (iii) of Theorem 5 and put \( K \subset \mathcal{L}_2^n \times \mathcal{L}_n^2 \). Therefore, \( y^n \in \Phi(B) \) for \( n = 1, 2, \ldots \) which, by the weak compactness of \( \Phi(B) \) implies the existence of a subsequence \( (y^k) \) of \( (y^n) \) weakly converging to \( y \in \Phi(B) \). We have \( y^k \in K(z^k) \) for \( k = 1, 2, \ldots \). Let \( (f^k, g^k) \in S^2(F_{s,t}(z^k_{s,t})) \times S^2(G_{s,t}(z^k_{s,t})) \) be such that \( \Phi(f^k, g^k) = y^k \), for each \( k = 1, 2, \ldots \). We have of course \( (f^k, g^k) \in B \). Therefore, there is a subsequence, say again \( (f^k, g^k) \) of \( (f^k, g^k) \) weakly converging in \( \mathcal{L}_2^n \times \mathcal{L}_n^2 \) to \( (f, g) \in B \). Now, for every \( A \in \beta_2^2 \times \Omega \) one obtains

\[
\operatorname{dist}\left(\int\int_A f_{s,t} \, dsdtdP, \int\int_A F_{s,t}(z) \, dsdtdP\right) \leq \left| \int\int_A \left( f_{s,t} - f^k_{s,t} \right) \, dsdtdP \right| \\
+ \operatorname{dist}\left(\int\int_A f^k_{s,t} \, dsdtdP, \int\int_A F_{s,t}(z^k) \, dsdtdP\right) \\
+ \left| \int\int_A \left( F_{s,t}(z^k) - F_{s,t}(z) \right) \, dsdtdP \right|.
\]

Therefore, \( f \in S^2(F_{s,t}(z_{s,t}))_{s,t \geq 0} \). Quite similarly, we also get \( g \in S^2(G_{s,t}(z_{s,t}))_{s,t \geq 0} \). Thus, \( \varphi + \Phi(f, g) \in K(z) \), which implies \( y \in K(z) \). On the other hand, we also have \( y \in C \), because \( C \) is weakly closed. Therefore, \( z \in K^-(C) \). Now the result follows immediately from Eberlein and Smulian’s Theorem.

Theorem 7. If \( F \) and \( G \) take on convex values and satisfy \((C_1)\) and \((C_2)\), then there is \( z \in D \) such that

\[
z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du\, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, du\, dv.
\]

Proof. Let

\[
\mathcal{B} = \{(f, g) \in \mathcal{L}_2^n \times \mathcal{L}_n^2 : |f_{s,t}(\omega)| \leq \|F_{s,t}(\omega)\|, |g_{s,t}(\omega)| \leq \|G_{s,t}(\omega)\|\}
\]

and put \( K = \varphi + \Phi(\mathcal{B}) \). It is clear that \( K \) is a nonempty convex weakly compact subset of \( D \) such that \( K(z) \subset K \) for \( z \in D \). By (iii) of Theorem 5
\[ K(z) \text{ is a convex and weakly compact subset of } D, \text{ for each } z \in D. \text{ By Lemma } 6 \text{ } K \text{ is u.s.c. on a locally convex topological Hausdorff space } (D, \sigma(D, D^*)) \text{. Therefore, by the Kakutani and Fan fixed point theorem there exists } z \in K \text{ such that } z \in K(z), \text{ i.e.,} \]

\[
z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^l F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^l G_{u,v}(z_{u,v}) \, dw_{u,v}. \]

Assume now that \( F \) and \( G \) satisfy conditions (C1) and the following condition (C3) \( F \) and \( G \) are such that set-valued functions:

\[ D \ni z \to (F_{s,t}(z)_{s,t \geq 0}(\omega) \subset R^n) \text{ and } D \ni z \to (G_{s,t}(z)_{s,t \geq 0}(\omega) \subset R^n) \]

are s.-w.s.l.s.c. on \( D \), i.e., for every \( z \in D \) and every sequence \((z_n)\) of \((D, | \cdot |)\) converging weakly to \( z \) one has

\[
\bar{R} \left[ (F_{s,t}(z))_{s,t \geq 0}(w), (F_{s,t}(z_n))_{s,t \geq 0}(\omega) \right] \to 0 \]

and

\[
\bar{R} \left[ (G_{s,t}(z))_{s,t \geq 0}(w), (G_{s,t}(z_n))_{s,t \geq 0}(\omega) \right] \to 0 \text{ a.e.} \]

**Lemma 8.** Assume \( F \) and \( G \) take on convex values and satisfy (C1) and (C3). Then a set-valued mapping \( K \) is l.s.c. as a multifunction defined on a locally convex topological Hausdorff space \((D, \sigma(D, D^*))\).

**Proof.** Let \( C \) be a nonempty weakly closed subset of \( D \) and \((z^{(n)})\) a sequence of \( \mathcal{R}_{-}(C) \) weakly converging to \( z \in D \). Select arbitrary \( u \in K(z) \) and suppose \((f, g) \in S^2(F_{s,t}(z)_{s,t \geq 0}) \times S^2(G_{s,t}(z)_{s,t \geq 0})\) is such that \( u = \Phi(f, g) \). Let \((f^{(n)}, g^{(n)}) \in S^2(F_{s,t}(z^{(n)})_{s,t \geq 0}) \times S^2(G_{s,t}(z^{(n)})_{s,t \geq 0})\) be such that

\[
\left| f_{s,t}(\omega) - f_{s,t}^{(n)}(\omega) \right| = \text{dist} \left( f_{s,t}(\omega), \left( F_{s,t}(z^{(n)}) \right)(\omega) \right)
\]

and

\[
\left| g_{s,t}(\omega) - g_{s,t}^{(n)}(\omega) \right| = \text{dist} \left( g_{s,t}(\omega), \left( G_{s,t}(z^{(n)}) \right)(\omega) \right)
\]

on \( R^2_+ \times \Omega \), for each \( n = 1, 2, \ldots \). By virtue of (C3) one gets \( |f_{s,t}(\omega) - f_{s,t}^{(n)}(\omega)| \to 0 \) and \( |g_{s,t}(\omega) - g_{s,t}^{(n)}(\omega)| \to 0 \) a.e., as \( n \to \infty \). Hence by (C1) we can see that a sequence \((u^{(n)})\), defined by \( u^{(n)} = \Phi(f^{(n)}, g^{(n)}) \) weakly converges to \( u \). But \( u^{(n)} \in K(z^{(n)}) \subset C \) for \( n = 1, 2, \ldots \) and \( C \) is weakly closed. Then \( u \in C \), which implies \( K(z) \subset C \). Thus \( z \in \mathcal{K}(-C) \).
Theorem 9. If $F$ and $G$ take on convex values and satisfy $(C_1)$ and $(C_3)$ then stochastic integral inclusion (1) admits a solution.

Proof. Let

$$B = \{(f, g) \in L^2_n \times L^2_n : |f_{s,t}(\omega)| \leq \|F_{s,t}(\omega)\|, |g_{s,t}(\omega)| \leq \|G_{s,t}(\omega)\|\}$$

and put $K = \varphi + \Phi(B)$. It is clear that $K$ is a nonempty convex weakly compact subset of $D$ such that $K(z) \subset K$ for $z \in D$. By virtue of Lemma 8, $K$ is l.s.c. as a set-valued mapping from a paracompact space $K$ considered with its relative topology induced by a weak topology $\sigma(D, D^*)$ on $D$ into a Banach space $(D, \|\cdot\|)$. By (iii) of Theorem 5, $K(z)$ is a closed and convex subset of $D$, for each $z \in K$. Therefore, by Michael’s theorem, there is a continuous selection $k : K \rightarrow D$ for $K$. But $K(K) \subset K$. Then $K$ maps $K$ into itself and is continuous with respect to the relative topology on $K$, defined above. Therefore, by the Schauder and Tikhonov fixed point theorem, there is $z \in K$ such that $z = k(z) \in K(z)$, i.e.,

$$z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v} \text{ a.s.}$$

References


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