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MEASURE VALUED SOLUTIONS FOR STOCHASTIC EVOLUTION EQUATIONS ON HILBERT SPACE AND THEIR FEEDBACK CONTROL

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Abstract

In this paper, we consider a class of semilinear stochastic evolution equations on Hilbert space driven by a stochastic vector measure. The nonlinear terms are assumed to be merely continuous and bounded on bounded sets. We prove the existence of measure valued solutions generalizing some earlier results of the author. As a corollary, an existence result of a measure solution for a forward Kolmogorov equation with unbounded operator valued coefficients is obtained. The main result is further extended to cover Borel measurable drift and diffusion which are assumed to be bounded on bounded sets. Also we consider control problems for these systems and present several results on the existence of optimal feedback controls.

Keywords: stochastic differential equations, Hilbert space, measurable vector fields, finitely additive measure solutions, optimal feedback controls.

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1. Motivation

Let us consider the deterministic evolution equation

$$\begin{aligned} \dot{x} &= Ax + F(x), \; t \geq 0 \\ x(0) &= x_0 \end{aligned} \tag{1.1}$$

in a Hilbert space H where A is the infinitesimal generator of a C_0-semigroup, $S(t)$, $t \geq 0$, on H and $F : H \longrightarrow H$ is a continuous map. It is well known that if H is finite dimensional, the mere continuity of F is good enough to prove the existence of local solutions with possibly finite blow up time. If H is an infinite dimensional Hilbert space continuity no longer guarantees the existence of even local solutions unless the semigroup $S(t), t > 0$, is compact. Because of this, the very notion of solutions required a major generalization to cover continuous as well as discontinuous vector fields [1–5, 11]. Using the general concept of measure solutions one can completely avoid standard assumptions such as the local Lipschitz property and linear growth for both the drift and the diffusion operators as often used in [8]. We are interested in the stochastic system governed by an evolution equation of the form

$$\begin{aligned} dx(t) &= Ax(t)dt + F(x(t))dt + G(x(t-))M(dt),\ t \geq 0 \\ x(0) &= x_0, \end{aligned} \tag{1.2}$$

where A and F are as descrbed above, and $G : H \longrightarrow \mathcal{L}(E, H)$ is a continuous map and M is an E-valued stochastic vector measure defined on the sigma algebra $\mathcal{B}_0$ of Borel subsets of $R_0 \equiv [0, \infty)$.

For simplicity of presentation we have considered both F and G independent of time. However the results presented here can be easily extended to the time varying case without any difficulty.

The rest of the paper is organized as follows. In Section 2, we recall some important facts from analysis sufficient to serve our needs. In Section 3, we present a result on the question of the existence of measure valued solutions and their regularity properties for the system (1.2). In Section 4, we discuss further extensions covering Borel measurable vector fields: F and G. In the final section, we consider control problems and present several existence results.

2. Introduction

Recently the author dealt with the question of the existence of measure valued solutions for semilinear stochastic differential equations with continuous but unbounded nonlinearities driven by a cylindrical Brownian motion [3]. Here we admit Borel measurable, possibly unbounded, vector fields and replace the Brownian motion by a more general stochastic vector measure. Properties of the stochastic vector measure are stated later on.

Radon Nikodyme Property & Lifting

To study the question of existence, we need the characterization of the dual of the Banach space $L_1(I, X)$ where $I \equiv [0, T]$ is a finite interval of the real line and X is a Banach space. It is well known that if both X and its dual X^* satisfy Radon-Nikodym property (RNP) then the dual of $L_1(I, X)$ is given by $L_\infty(I, X^*)$. In case they do not satisfy the RNP, it follows from the theory of "lifting" that the dual of $L_1(I, X)$ is given by $L_\infty^w(I, X^*)$ which consists of w^*-measurable (weak star measurable) functions with values in X^*. For $h \in L_\infty^w(I, X^*)$, define $\parallel h \parallel_{L_\infty^w(I,X^*)} = \alpha_h$ where α_h is the smallest number for which the following inequality,

$$ess - \sup\{| < h(t), x > |, \, t \in I\} \leq \alpha_h \parallel x \parallel_X,$$

holds for all $x \in X$.

Let Z denote any normal topological space and $BC(Z)$ the space of bounded continuous functions on Z with the topology of sup norm, and let $\Sigma_{rba}(Z)$ denote the space of regular bounded finitely additive set functions on Φ_c with a total variation norm where Φ_c denotes the algebra generated by the closed subsets of Z. With respect to these topologies, these are Banach spaces and the dual of $BC(Z)$ is $\Sigma_{rba}(Z)$ [see Dunford and Schwartz 10, Theorem 2, p. 262]. Let $\Pi_{rba}(Z) \subset \Sigma_{rba}(Z)$ denote the class of regular finitely additive probability measures furnished with the relative topology. Since the pair $\{BC(Z), \Sigma_{rba}(Z)\}$ does not satisfy RNP, it follows from the characterization result discussed above that the dual of $L_1(I, BC(Z))$ is given by $L_\infty^w(I, \Sigma_{rba}(Z))$ which is furnished with the weak star topology. Similarly, let $B(Z)$ denote the space of bounded Borel measurable functions on Z with the topology of sup norm. The dual of this space is given by the space of bounded finitely additive measures on Φ_c with a total variation norm. This is denoted by $\Sigma_{ba}(Z)$. We let $\Pi_{ba}(Z) \subset \Sigma_{ba}(Z)$ denote the class of finitely additive probability measures and let $L_\infty^w(I, \Sigma_{ba}(Z))$ denote the dual of the vector space $L_1(I, B(Z))$.

Special Vector Spaces Used in the paper

Now we are prepared to introduce the vector spaces used in the paper. Let H, E be two separable Hilbert spaces and

$$(\Omega, \mathcal{F}, \mathcal{F}_t \uparrow, t \geq 0, P)$$

a complete filtered probability space, $M(J), J \in \mathcal{B}_0$, an E valued $\mathcal{F}_t$ adapted vector measure in the sense that for any $J \in \mathcal{B}_0$ with $J \subset [0,t]$, $M(J)$ or more precisely $e^*(M(J))$ is $\mathcal{F}_t$ measurable for every $e^* \in E^* = E$. For the purpose of this paper, we consider $\mathcal{F}_t \equiv \mathcal{F}_t^M \vee \sigma(x_0)$, where $\mathcal{F}_t^M, \sigma(x_0)$ are the smallest sigma algebras with respect to which the measures M and the initial state x_0 respectively are measurable. Let $I \times \Omega$ be furnished with the predictable σ-field with reference to the filtration $\mathcal{F}_t, t \in I$. Let $M^w_{\infty,2}(I \times \Omega, \Sigma_{rba}(H)) \subset L^w_{\infty,2}(I \times \Omega, \Sigma_{rba}(H))$ denote the vector space of $\Sigma_{rba}(H)$ valued random processes $\{\lambda_t, t \in I\}$, which are $\mathcal{F}_t$-adapted and w^*-measurable in the sense that, for each $\phi \in BC(H)$, $t \longrightarrow \lambda_t(\phi)$ is a bounded $\mathcal{F}_t$ measurable random variable possessing finite second moments. We furnish this space with the w^* topology as before. Clearly, this is the dual of the Banach space

$$M_{1,2}(I \times \Omega, BC(H)) \subset L_{1,2}(I \times \Omega, BC(H))),$$

where the later space is furnished with the natural topology induced by the norm given by

$$\| \varphi \| \equiv \int_I \left(\mathcal{E}\big(\sup\{|\varphi(t,\omega,\xi)|, \xi \in H\}\big)^2 \right)^{1/2} dt.$$

Here we have chosen $X = BC(H))$ and $X^* = \Sigma_{rba}(H))$. For $X = B(H)$ and $X^* = \Sigma_{ba}(H)$, one can verify that $M^w_{\infty,2}(I \times \Omega, \Sigma_{ba}(H))$ is the dual of the Banach space $M_{1,2}(I \times \Omega, B(H))$. We will use both these spaces.

Some Basic properties of M

(M1): $\{M(J), M(K), J \cap K = \emptyset, J, K \in \mathcal{B}_0\}$ are pairwise independent E-valued random variables (vector measures) satisfying $\mathcal{E}\{(M(J),\xi)\} = 0, J \in \mathcal{B}_0, \xi \in E$, where $\mathcal{E}(z) \equiv \int_\Omega z P(d\omega)$.

(M2): There exists a countably additive bounded positive measure π on $\mathcal{B}_0$, denoted by $\pi \in M_c(R_0)$, having bounded total variation on bounded sets, such that for every $\xi, \zeta \in E$,

$$\mathcal{E}\{(M(J),\xi)(M(K),\zeta)\} = (\xi,\zeta)_E\ \pi(J \cap K).$$

Clearly, it follows from this last property that for any $\xi \in E$

$$\mathcal{E}\{(M(J),\xi)^2\} = |\xi|^2_E \pi(J),$$

and that the process N, defined by

$$N(t) \equiv \int_0^t M(ds), t \geq 0,$$

is a square integrable E-valued $\mathcal{F}_t$-martingale. A simple example is given by the stochastic Wiener integral,

$$M(J) \equiv \int_J f(t) dW(t), J \in \mathcal{B}_0$$

where W is the cylindrical Brownian motion on R_0 with values in the Hilbert space E and f is any locally square integrable scalar valued function. In this case $\pi(J) = \int_J |f(t)|^2 dt$. If f is an $\mathcal{F}_t$-adapted square integrable random process, the measure π is given by $\pi(J) = E \int_J |f(t)|^2 dt$. If $f \equiv 1$, π is the Lebesgue measure. In the latter case the system reduces to one driven by cylindrical Brownian motion [3].

3. Existence of measure valued solutions

In recent years, the notion of a generalized solution, which consists of regular finitely additive measure valued functions, has been extensively used in the study of semi linear and quasi linear systems with vector fields which are merely continuous and bounded on bounded sets; see [1–3, 11] and the references therein. The existence of solutions for deterministic systems, such as (1.1), was proved in [1–2, 11] with varying generalities. Our objective here is to prove the existence of measure solutions for the stochastic system (1.2) generalizing a previous result of the author [3]. Since the measure solutions may not be fully supported on the original state space H, it is useful to extend the state space to a compact Hausdorff space containing H as a dense subspace. Since every metric space is a Tychonoff space, H is a Tychonoff space. Hence $\beta H \equiv H^+$, the Stone-Cech compactification of H, is a compact Hausdorff space and consequently bounded continuous functions on H can be extended to continuous functions on H^+. In view of this we shall often use H^+ in place of H and the spaces $M_{1,2}(I \times \Omega, BC(H^+))$ with dual $M^w_{\infty,2}(I \times \Omega, \Sigma_{rba}(H^+)) \supset M^w_{\infty,2}(I \times \Omega, \Pi_{rba}(H^+))$. Here $M^w_{\infty,2}(I \times \Omega, \Pi_{rba}(H^+))$ is the set of all finitely additive probability measure valued processes, a subset of the vector space $M^w_{\infty,2}(I \times \Omega, \Sigma_{rba}(H^+))$. Note that, since H^+ is a compact Hausdorff space, $\Sigma_{rba}(H^+) = \Sigma_{rca}(H^+)$. In view of the fact that the measure solutions of stochastic evolution equations restricted to H are only finitely additive, we prefer to use the notation $\Sigma_{rba}(H^+)$ to emphasize this fact though they are countably additive on H^+.

Without further notice, throughout this paper we use $D\phi$ and $D^2\phi$ to denote the first and second Frechet derivatives of the function ϕ whenever they exist. We denote by Ψ the class of test functions as defined below:

$$\Psi \equiv \left\{\phi \in BC(H) : D\phi, D^2\phi \text{ exist, continuous and bounded on } H\right\}.$$

Define the operators $\mathcal{A}$ $\mathcal{B}$ and $\mathcal{C}$ with domains given by

$$\mathcal{D}(\mathcal{A}) \equiv \{\phi \in \Psi : \mathcal{A}\phi \in BC(H^+)\}$$
$$\mathcal{D}(\mathcal{B}) \equiv \{\phi \in \Psi : D\phi \in D(A^*) \ \&\ \mathcal{B}\phi \in BC(H^+)\},$$

where

$$(3.1)\quad \begin{aligned} (\mathcal{A}\phi)(\xi) &= (1/2)Tr(G^*(D^2\phi)G)(\xi) \\ &\equiv (1/2)Tr(D^2\phi\ GG^*)(\xi), \phi \in \mathcal{D}(\mathcal{A}) \\ \mathcal{B}\phi &= (A^*D\phi(\xi), \xi) + (F(\xi), D\phi(\xi)) \quad \text{for} \quad \phi \in \mathcal{D}(\mathcal{B}) \\ \mathcal{C}\phi(\xi) &\equiv G^*(\xi)D\phi(\xi). \end{aligned}$$

We consider the system

$$(3.2)\qquad dx(t) = Ax(t)dt + F(x(t))dt + G(x(t-))M(dt), x(0) = x_0,$$

and use the notion of measure (generalized) solutions introduced in [3].

Definition 3.1. A measure valued random process

$$\mu \in M^w_{\infty,2}(I \times \Omega, \Pi_{rba}(H^+)) \subset M^w_{\infty,2}(I \times \Omega, \Sigma_{rba}(H^+))$$

is said to be a measure (or generalized) solution of equation (3.2) if for every $\phi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ and $t \in I$, the following equality holds

$$(3.3)\quad \begin{aligned} \mu_t(\phi) &= \phi(x_0) + \int_0^t \mu_s(\mathcal{A}\phi)\ \pi(ds) + \int_0^t \mu_s(\mathcal{B}\phi)\ ds \\ &\quad + \int_0^t < \mu_{s-}(\mathcal{C}\phi), M(ds) >_E \ \ P - a.s. \end{aligned}$$

where

$$\mu_t(\psi) \equiv \int_{H^+} \psi(\xi)\mu_t(d\xi),\ t \in I.$$

Remark 3.2. Note that equation (3.3) can be written in the differential form as follows:

$$d\mu_t(\phi) = \mu_t(\mathcal{A}\phi)\pi(dt) + \mu_t(\mathcal{B}\phi)dt + < \mu_{t-}(\mathcal{C}\phi), M(dt) >$$

with $\mu_0(\phi) = \phi(x_0)$. This is in fact the weak form of the stochastic evolution equation

$$d\mu_t = \mathcal{A}^*\mu_t\pi(dt) + \mathcal{B}^*\mu_t dt + < \mathcal{C}^*\mu_{t-}, M(dt) >_E,\ \mu_0 = \delta_{x_0}, \tag{3.4}$$

on the state space $\Sigma_{rba}(H)$ where $\{\mathcal{A}^*, \mathcal{B}^*, \mathcal{C}^*\}$ are the duals of the operators $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$.

To proceed further we shall need the following

Assumptions.

(A1): there exists a sequence $\{F_n, G_n\}$ with $F_n(x) \in D(A), G_n(x) \in \mathcal{L}(E, D(A))$, for each $x \in H$, and

$F_n(x) \longrightarrow F(x)$ in H uniformly on compact subsets of H

$G_n(x) \longrightarrow G(x)$, strongly in $\mathcal{L}(E, H)$, uniformly on compact subsets of H.

(A2): there exists a pair of sequences of real numbers $\{\alpha_n, \beta_n > 0\}$, possibly $\alpha_n, \beta_n \to \infty$ as $n \to \infty$, such that

$$\| F_n(x) - F_n(y) \| \leq \alpha_n \| x - y \|; \quad \| F_n(x) \| \leq \alpha_n (1 + \| x \|),$$

$$\| G_n(x) - G_n(y) \|_{\mathcal{L}_2(E,H)} \leq \beta_n \| x - y \|; \quad \| G_n(x) \|_{\mathcal{L}_2(E,H)} \leq \beta_n (1 + \| x \|)$$

for all $x, y \in H$; where $\mathcal{L}_2(E, H)$ denotes the Hilbert space of Hilbert-Schmidt operators from E to H.

We note that under the very relaxed assumptions used here, nonlinearities having polynomial growth are also admissible.

The following result generalizes our previous result [3, Theorem 3.2].

Theorem 3.3. *Suppose A is the infinitesimal generator of a C_0-semigroup in H and the maps $F : H \longrightarrow H$, $G : H \longrightarrow \mathcal{L}(E, H)$ are continuous, and bounded on bounded subsets of H, satisfying the approximation properties* (A1) *and* (A2); *and M is a non atomic vector measure satisfying* (M1) *and* (M2). *Then, for every x_0 for which $P\{\omega \in \Omega : |x_0|_H < \infty\} = 1$, the evolution equation* (3.2) *has at least one measure valued solution*

$$\lambda^0 \in M^w_{\infty,2}(I \times \Omega, \Sigma_{rba}(H^+))$$

in the sense of Definition 3.1. *Further, $\lambda^0 \in M^w_{\infty,2}(I \times \Omega, \Pi_{rba}(H^+))$.*

Proof. Since $D(A)$ is dense in H and $x_0 \in H$, a.s (almost surely), there exists a sequence $\{x_{0,n}\} \in D(A)$ such that $x_{0,n} \xrightarrow{s} x_0$ a.s. Consider the Cauchy problem:

$$\begin{aligned} dx(t) &= A_n x(t)dt + F_n(x(t))dt + G_n(x(t-))M(dt), \\ x(0) &= x_{0,n}, \end{aligned} \tag{3.5}$$

where $A_n = nAR(n, A), n \in \rho(A)$, is the Yosida approximation of A. Since for each $n \in N$ and $x \in H$, $F_n(x) \in D(A)$, $G_n(x) : E \mapsto D(A)$, it follows from assumption (A2) that equation (3.5) has a unique strong solution $x_n = \{x_n(t), t \in I\}$ which is $\mathcal{F}_t$-adapted, and for each $n \in N$, $t \in I$, $x_n(t) \in D(A)$ and

$$\sup\{E \parallel x_n(t) \parallel^2_H, t \in I\} < \infty.$$

Since standard assumptions hold, this result follows from classical existence and regularity results given in [8]. Now let $\phi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ with $D\phi$ and $D^2\phi$ having compact supports in H. Since x_n is a strong solution it follows from Ito's formula that, for each $t \in I$

$$\begin{aligned} \phi(x_n(t)) &= \phi(x_{0,n}) + \int_0^t (\mathcal{A}_n\phi)(x_n(s))\pi(ds) + \int_0^t (\mathcal{B}_n\phi)(x_n(s))\, ds \\ &\quad + \int_0^t < (\mathcal{C}_n\phi)(x_n(s-)), M(ds) >_E, \end{aligned} \tag{3.6}$$

where the operators $\{\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n\}$ are as given by (3.1) with $\{A_n, F_n, G_n\}$ replacing the operators $\{A, F, G\}$. Letting $\delta_e(d\xi)$ denote the Dirac measure concentrated at the point $e \in H$, and defining $\lambda^n_t(d\xi) \equiv \delta_{x_n(t)}(d\xi), t \in I$,

$\lambda_0^n(d\xi) \equiv \delta_{x_{0,n}}(d\xi)$, and using the notation of Definition 3.1 we can rewrite (3.6) as

$$
\begin{aligned}
\lambda_t^n(\phi) &= \lambda_0^n(\phi) + \int_0^t \lambda_s^n(\mathcal{A}_n\phi)\ \pi(ds) + \int_0^t \lambda_s^n(\mathcal{B}_n\phi)\ ds \\
&\quad + \int_0^t < \lambda_{s-}^n(\mathcal{C}_n\phi), M(ds) >_E \ \ P - a.s.
\end{aligned}
\tag{3.7}
$$

Notice that, for each integer n, the functional ℓ_n, given by

$$\ell_n(\psi) \equiv \mathcal{E}\int_{I\times H^+} \psi(t,\xi)\lambda_t^n(d\xi)dt \equiv \int_{I\times\Omega\times H^+} \psi(t,\omega,\xi)\ \lambda_{t,\omega}^n(d\xi)\ dtdP,$$

is a well defined bounded linear functional on $M_{1,2}(I\times\Omega, BC(H^+))$ and that

$$|\ell_n(\psi)| \le \parallel \psi \parallel_{M_{1,2}(I\times\Omega,BC(H^+))}, \quad \text{for all } n \in N.$$

Thus the family of linear functionals $\{\ell_n\}$ is contained in a bounded subset of the dual of $M_{1,2}(I\Omega, BC(H^+))$. Equivalently, the family $\{\lambda^n\}$ is contained in $M_{\infty,2}^w(I\times\Omega, \Pi_{rba}(H^+))$ which is a bounded subset of $M_{\infty,2}^w(I\times\Omega, \Sigma_{rba}(H^+))$. Hence, by Alaoglu's theorem, there exists a generalized subsequence (subnet) of the sequence (net) $\{\lambda^n\}$, relabeled as $\{\lambda^n\}$, and a $\lambda^0 \in M_{\infty,2}^w(I\times\Omega, \Sigma_{rba}(H^+))$, so that $\lambda^n \xrightarrow{w^*} \lambda^0$. We show that λ^0 is a measure (generalized) solution of equation (3.2) in the sense of Definition 3.1. Define

$$
\begin{aligned}
\psi_{1,n}(\xi) &\equiv (1/2)Tr\big(G_n^*(D^2\phi)G_n)(\xi)\big) = (1/2)Tr\big(((D^2\phi)G_nG_n^*)(\xi)\big) \\
\psi_1(\xi) &\equiv (1/2)Tr\big((G^*(D^2\phi)G)(\xi)\big) = (1/2)Tr\big(((D^2\phi)GG^*)(\xi)\big).
\end{aligned}
$$

It is clear that if $G_n(x) \longrightarrow G(x)$ strongly in $\mathcal{L}(E,H)$ uniformly on compact subsets of H, so does $G_n^*(x) \longrightarrow G^*(x)$ strongly in $\mathcal{L}(H,E)$ uniformly on compact subsets of H. Since $D^2\phi$ has a compact support, and, for each $\phi \in D(\mathcal{A})$, we have $\psi_{1,n}, \psi_1 \in BC(H)$, it follows from assumption (A1) that $\psi_{1,n} \longrightarrow \psi_1$ uniformly on H, that is, $\psi_{1,n} \xrightarrow{s} \psi_1$ in $BC(H^+)$. Combining this with the fact that the measure π has a bounded variation on bounded sets, it follows from the weak* convergence of λ^n to λ^o that, for any $z \in L_2(\Omega, \mathcal{F}, P) = L_2(\Omega)$, and $t \in I$, we have

$$\int_{\Omega\times[0,t]} z\ \lambda_s^n(\psi_{1,n})\pi(ds)dP \longrightarrow \int_{\Omega\times[0,t]} z\ \lambda_s^0(\psi_1)\pi(ds)dP. \tag{3.8}$$

Define

$$\psi_{2,n}(\xi) \equiv (A_n^*(D\phi)(\xi), \xi) \text{ and } \psi_2(\xi) \equiv (A^*(D\phi)(\xi), \xi).$$

Since $A_n \longrightarrow A$ on $D(A)$ in the strong operator topology and, for $\phi \in D(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$, $D\phi(x) \in D(A^*)$, and further, by our choice of ϕ, $D\phi$ is continuous having a compact support, we can deduce that $\psi_{2,n} \longrightarrow \psi_2$ uniformly on H; that is, $\psi_{2,n} \xrightarrow{s} \psi_2$ in $BC(H^+)$. Hence, again we have

$$\int_{\Omega\times[0,t]} z\ \lambda_s^n(\psi_{2,n})dsdP \longrightarrow \int_{\Omega\times[0,t]} z\ \lambda_s^o(\psi_2)dsdP. \tag{3.9}$$

Similarly, define

$$\psi_{3,n}(\xi) \equiv (F_n(\xi), D\phi(\xi)) \text{ and } \psi_3(\xi) \equiv (F(\xi), D\phi(\xi))$$
$$\psi_{4,n}(\xi) \equiv G_n^*(\xi)D\phi(\xi) \text{ and } \psi_4(\xi) \equiv G^*(\xi)D\phi(\xi).$$

Again, since $\phi \in D(\mathcal{A}) \cap D(\mathcal{B})$ and $D\phi$ has a compact support and by our assumption $F_n \longrightarrow F$ uniformly on compact subsets of H, it follows that $\psi_{3,n} \xrightarrow{s} \psi_3$ in the topology of $BC(H^+)$. Thus, we have

$$\int_{\Omega\times[0,t]} z\ \lambda_s^n(\psi_{3,n})dsdP \longrightarrow \int_{\Omega\times[0,t]} z\ \lambda_s^o(\psi_3)dsdP \tag{3.10}$$

for every $z \in L_2(\Omega)$. Combining (3.8)–(3.10) we conclude that, for every $z \in L_2(\Omega)$ and $\phi \in D(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ with $D\phi, D^2\phi$ having compact supports,

$$\int_{\Omega\times[0,t]} z\ \lambda_s^n(\mathcal{A}_n\phi)\ \pi(ds)dP \longrightarrow \int_{\Omega\times[0,t]} z\ \lambda_s^0(\mathcal{A}\phi)\ \pi(ds)dP \tag{3.11}$$

$$(3.12) \qquad \int_{\Omega\times[0,t]} z\ \lambda_s^n(\mathcal{B}_n\phi)\ dsdP \longrightarrow \int_{\Omega\times[0,t]} z\ \lambda_s^0(\mathcal{B}\phi)\ dsdP.$$

Since $x_{0,n} \stackrel{s}{\longrightarrow} x_0$ $a.s$ and $\phi \in BC(H^+)$, we have $\phi(x_{0,n}) \longrightarrow \phi(x_0)$ $a.s.$ Then, by Lebesgue dominated convergence theorem, for every $z \in L_2(\Omega)$ we have

$$(3.13) \qquad \int_\Omega z\ \phi(x_{0,n})dP \longrightarrow \int_\Omega z\ \phi(x_0)dP \equiv \int_\Omega z\ \lambda_0(\phi)dP$$

where $\lambda_0(\phi) \equiv \int_H \phi(\xi)\delta_{x_0}(d\xi)$. Recall that the vector measure M induces a square integrable E-valued $\mathcal{F}_t$ martingale denoted by $\{N(t) \equiv \int_0^t M(ds),\ t \geq 0\}$ with the quadratic variation given by the measure $\int_0^t \pi(ds) = \pi([0,t])$. Considering the stochastic integral in (3.7), since $D\phi$ is continuous having a compact support and G_n is continuous and bounded on bounded sets, we have $\mathcal{C}_n\phi \in BC(H^+, E)$. Thus, π being a bounded positive measure having a bounded total variation, it follows from this that

$$\mathcal{E}\int_I \parallel (\mathcal{C}_n\phi)(x_n(s-)) \parallel_E^2 \pi(ds) < \infty$$

for each $n \in N$. Hence the last (stochastic) integral in (3.7) is a well defined square integrable $\mathcal{F}_t$ martingale. Our objective is to show that for any $z \in L_2(\Omega)$ we have

$$(3.14) \qquad \begin{aligned} &\mathcal{E}\Big\{ z \int_0^t < \lambda_{s-}^n(\mathcal{C}_n\phi), M(ds) >_E \Big\} \stackrel{n\to\infty}{\longrightarrow} \\ &\mathcal{E}\Big\{ z \int_0^t < \lambda_{s-}^o(\mathcal{C}\phi), M(ds) >_E \Big\},\ t \in I. \end{aligned}$$

This can be proved using well-known properties of iterated conditional expectations following similar arguments as in [3]. Consider the expression on the left of (3.14). For $z \in L_2(\Omega)$, it follows from the properties of conditional expectation and the martingale theory that

$$
\begin{aligned}
&\mathcal{E}\Big(z \int_0^t < \lambda^n_{s-}(\mathcal{C}_n\phi), dN(s) >_E\Big) \\
&= \mathcal{E}\Big(z_t \int_0^t < \lambda^n_{s-}(\mathcal{C}_n\phi), dN(s) >_E\Big)
\end{aligned} \tag{3.15}
$$

where $z_t \equiv E\{z|\mathcal{F}_t\}$ is a square integrable $\mathcal{F}_t$ martingale. Hence there exists an $\mathcal{F}_t$-adapted processes $\eta(t), t \geq 0$, with values in E and an $\mathcal{F}_0$ measurable random variable $z_0 \in L_2(\Omega)$ such that

$$
\mathcal{E} \int_I \| \eta(t) \|_E^2 \, \pi(dt) < \infty,
$$

and that

$$
z_t = z_0 + \int_0^t < \eta(s), dN(s) >_E . \tag{3.16}
$$

Thus

$$
\begin{aligned}
&\mathcal{E}\Big(z_t \int_0^t < \lambda^n_{s-}(\mathcal{C}_n\phi), dN(s) >_E\Big) \\
&= \mathcal{E}\Big(\int_0^t < \eta(s), \lambda^n_{s-}(\mathcal{C}_n\phi) >_E \pi(ds)\Big).
\end{aligned} \tag{3.17}
$$

Since $D\phi$ has a compact support, $\mathcal{C}_n\phi \longrightarrow \mathcal{C}\phi$ in the topology of $BC(H^+, E)$ and hence

$$
< \eta(t), (\mathcal{C}_n\phi)(t,\xi) >_E \xrightarrow{s} < \eta(t), (\mathcal{C}\phi)(t,\xi) >_E \ \text{ in } BC(H^+) \ \ \pi \times P - a.e.
$$

In fact, due to square integrability of η with respect to the measure $\pi \times P$ on the predictable sigma field and the boundedness of the sequence $\{\mathcal{C}_n\phi\}$, it follows from the dominated convergence theorem that

$$
< \eta, \mathcal{C}_n\phi >_E \xrightarrow{s} < \eta, \mathcal{C}\phi >_E,
$$

in the topology of $M_{1,2}(I \times \Omega, BC(H^+))$ as $n \to \infty$. Using this result (strong convergence) and the fact that $\lambda^n \xrightarrow{w^*} \lambda^0$ in $M^w_{\infty,2}(I \times \Omega, \Sigma_{rba}(H^+))$, we conclude from duality of the two spaces involved that, for each $t \in I$,

$$
\begin{aligned}
\mathcal{E}\Big(\int_0^t < \eta(s), \lambda^n_{s-}(\mathcal{C}_n\phi) >_E \pi(ds)\Big) &= \mathcal{E}\Big(\int_0^t \lambda^n_{s-}\big(< \eta(s), \mathcal{C}_n\phi >_E\big)\pi(ds)\Big) \\
&\longrightarrow \mathcal{E}\Big(\int_0^t \lambda^o_{s-}\big(< \eta(s), (\mathcal{C}\phi) >\big)\pi(ds)\Big) \\
&= \mathcal{E}\Big(\int_0^t < \eta(s), \lambda^o_{s-}(\mathcal{C}\phi) > \pi(ds)\Big) \\
&= \mathcal{E}\Big(z_t \int_0^t < \lambda^o_{s-}(\mathcal{C}\phi), dN(s) >\Big) \\
&= \mathcal{E}\Big(z \int_0^t < \lambda^o_{s-}(\mathcal{C}\phi), dN(s) >\Big) \\
&= \mathcal{E}\Big(z \int_0^t < \lambda^o_{s-}(\mathcal{C}\phi), M(ds) >\Big).
\end{aligned} \tag{3.18}
$$

The second line follows from the convergence properties just stated, the third line is obvious and the fourth and the fifth lines follow from the martingale property of $z_t, t \geq 0$, and its representation (3.16); and the sixth line follows from the definition of the martingale N generated by the vector measure M. Thus (3.14) follows from (3.15)–(3.18). Now multiplying both sides of equation (3.7) by an arbitrary $z \in L_2(\Omega)$ and taking the limit of the expected values, it follows from (3.11), (3.12), (3.13) and (3.14) that

$$
\begin{aligned}
\mathcal{E}(z\lambda^o_t(\phi)) &= \mathcal{E}(z\lambda^o_0(\phi)) + \mathcal{E}\Big(z\int_0^t \lambda^o_s(\mathcal{A}\phi)\ \pi(ds)\Big) \\
&\quad + \mathcal{E}\Big(z\int_0^t \lambda^o_s(\mathcal{B}\phi)\ ds\Big) + \mathcal{E}\Big(z\int_0^t < \lambda^o_{s-}(\mathcal{C}\phi), M(ds) >_E\Big).
\end{aligned} \tag{3.19}
$$

Since this holds for arbitrary $z \in L_2(\Omega)$, we have, for each $t \in I$,

$$
\begin{aligned}
\lambda^o_t(\phi) &= \lambda^o_0(\phi) + \int_0^t \lambda^o_s(\mathcal{A}\phi)\ \pi(ds) + \int_0^t \lambda^o_s(\mathcal{B}\phi)\ ds \\
&\quad + \int_0^t < \lambda^o_{s-}(\mathcal{C}\phi), M(ds) >_E \quad P - a.s.
\end{aligned} \tag{3.20}
$$

By virtue of the fact that $\lambda^o \in M^w_{\infty,2}(I \times \Omega, \Sigma_{rba}(H^+))$, it is evident that for each $\phi \in D(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$, $\lambda^o_t(\mathcal{A}\phi), \lambda^o_t(\mathcal{B}\phi), \lambda^o_{t-}(\mathcal{C}\phi)$ are well defined $\mathcal{F}_t$ adapted processes and that $\lambda^o(\mathcal{A}\phi) \in L_1(\pi, L_2(\Omega)), \lambda^o(\mathcal{B}\phi) \in L_1(I, L_2(\Omega)), \lambda^o(\mathcal{C}\phi) \in L_1(I, L_2(\Omega, E))$. Thus equation (3.20) holds for all $\phi \in D(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ and not only for those having first and second Frechet differentials with compact supports. Hence λ^o is a measure valued or generalized solution of equation (3.2) in the sense of Definition 3.1. The proof of the last assertion of the theorem follows from the fact that positivity is preserved under weak star convergence. Thus $\lambda^o \in M^w_{\infty,2}(I \times \Omega, \Pi_{rba}(H^+))$. This completes the proof.

Remark 3.4. It is clear from the above result that for generalized (measure) solutions it is only required that F, G are continuous and bounded on bounded sets. Thus these maps may have polynomial growth [3]. In contrast, for standard mild solutions it is usually assumed that F, G are locally Lipschitz having at most linear growth [7, 8]. Our result provides a stochastic finitely additive regular measure valued process as the solution. However, it is countably additive on the compact Hausdorff space H^+ containing the original state space H as a dense subspace.

The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.5. *Consider the forward Kolmogorov equation,*

$$\begin{aligned} d\vartheta_t &= \mathcal{A}^*\vartheta_t\pi(dt) + \mathcal{B}^*\vartheta_t dt \\ \vartheta(0) &= \nu_0, \end{aligned} \tag{3.21}$$

with $\mathcal{A}^*$, $\mathcal{B}^*$ denoting the duals of the operators $\mathcal{A}$, $\mathcal{B}$ respectively. Suppose $\{A, F, G, M, \pi\}$ satisfy the assumptions of Theorem 3.3. Then, for each $\nu_0 \in \Pi_{rba}(H)$, equation (3.21) has at least one weak solution $\nu \in L^w_\infty(I, \Pi_{rba}(H^+)) \subset L^w_\infty(I, \Sigma_{rba}(H^+))$ in the sense that for each $\phi \in D(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ the following equality holds

$$\nu_t(\phi) = \nu_0(\phi) + \int_0^t \nu_s(\mathcal{A}\phi)\ \pi(ds) + \int_0^t \nu_s(\mathcal{B}\phi)ds, t \in I. \tag{3.22}$$

Proof. The proof is quite similar to that of Corollary 3.4 of [3, p. 85], and hence we present only a brief outline. Since H is separable and $\nu_0 \in \Pi_{rba}(H)$ there exists a random variable x_0 taking values in H P-a.s (possibly on a Skorokhod extension) such that $\mathcal{L}(x_0) = \nu_0$ and for each $\phi \in BC(H)$,

$$\mathcal{E}\phi(x_0) = \mathcal{E} \int_{H^+} \phi(\xi)\delta_{x_0}(d\xi) = \int_{H^+} \phi(\xi)\nu_0(d\xi).$$

Here we have used ϕ itself to denote its extension from H to H^+. Using x_0 defined above as the initial state, it follows from Theorem 3.3 that equation 3.2 has at least one generalized solution $\lambda^0 \in M^w_{\infty,2}(I \times \Omega, \Pi_{rba}(H^+))$ satisfying equation (3.20) for each $\phi \in D(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$. Then the map

$$\psi \longrightarrow \mathcal{E}\left(\int_I \lambda^0_t(\psi)dt\right)$$

is a continuous linear functional on $L_1(I, BC(H^+))$. Hence, by duality provided by the theory of lifting and the fact that $\lambda^o \in M^w_{\infty,2}(I\times\Omega, \Pi_{rba}(H^+))$, there exists a unique $\mu \in L^w_\infty(I, \Pi_{rba}(H^+))$ so that

$$\mathcal{E}\left(\int_I \lambda^0_t(\psi)dt\right) = <\mu, \psi> \equiv \int_I \mu_t(\psi)dt. \tag{3.23}$$

Now following similar arguments as in [3, Corollary 3.4], one can justify that μ satisfies the identity (3.22) with $\mu_0 = \nu_0$. This completes our brief outline of the proof that equation (3.21) has a weak solution in the sense of identity (definition) (3.22). ∎

Remark 3.6. Note that Corollary 3.5 proves the existence of (measure) solutions for Kolmogorov equation (3.21) with unbounded coefficients. This generalizes similar results of Cerrai [6] for parabolic and elliptic equations on finite dimensional spaces.

So far we have not discussed the question of uniqueness of solutions. In [4] the uniqueness of measure solution was proved using spectral properties of the operator $\mathcal{A}$. A direct proof based on a semigroup approach was given in [5]. Using a similar technique as in [5], we can prove the uniqueness of (weak) solution of equation (3.21) as presented below.

Corollary 3.7 (Uniqeness)**.** *Suppose the assumptions of Corollary* 3.5 *hold and that* $D(\mathcal{A}) \cap D(\mathcal{B})$ *is dense in* $BC(H)$. *Then the solution (weak solution) of the evolution equation* (3.21) *is unique.*

Proof. We prove the uniqueness of (weak) solution by use of a similar technique as in the general semigroup theory; see also [5]. Accordingly, it suffices to demonstrate that for a given $\pi \in M_c(I)$, the pair $\{\mathcal{A}^*, \mathcal{B}^*\}$ determines a unique evolution operator $\{U^*(t,s), 0 \le s \le t < \infty\}$ on $\Sigma_{rba}(H)$. This will guarantee the uniqueness of weak solution (whenever it exists) of equation (3.21) having the representation,

$$\mu_t = U^*(t,0)\nu_0,\ t \geq 0.$$

By Corollary 3.5, for each given initial measure, equation (3.21) has at least one weak solution. For the given quadratic variation measure $\pi \in M_c(I)$, associated with the martingale measure M, suppose the pair $\{\mathcal{A}_i^*, \mathcal{B}_i^*\}$, $i = 1,2$, generates the evolution operator $U_i^*(t,s), 0 \leq s \leq t < \infty$, $i = 1,2$. For $\varphi \in D(\mathcal{A}_i) \cap D(\mathcal{B}_i)$ and $\nu \in D(\mathcal{A}_i^*) \cap D(\mathcal{B}_i^*)$, define the function,

$$h(r) \equiv < U_2^*(t,r)U_1^*(r,s)\nu, \varphi >= \nu(U_1(r,s)U_2(t,r)\varphi), r \in [s,t].$$

It is a well-known fact that the infinitesimal generators commute with their corresponding evolution operators on their domain. Using this fact, it is easy to verify that, for $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{B}_1 = \mathcal{B}_2$, the variation of h on $[s,t]$ is zero. Hence h is constant on $[s,t]$ and so $h(t) = h(s)$ implying $\nu(U_1(t,s)\varphi) = \nu(U_2(t,s)\varphi)$. This holds for all $\varphi \in D(\mathcal{A}_i) \cap D(\mathcal{B}_i)$ and $\nu \in D(\mathcal{A}_i^*) \cap D(\mathcal{B}_i^*)$. Since $D(\mathcal{A}_i) \cap D(\mathcal{B}_i)$ is dense in $BC(H)$, and ν is arbitrary, we have $U_1^*(t,s) = U_2^*(t,s)$ for $0 \leq s \leq t < \infty$ proving uniqueness. ∎

Remark 3.8. Using the unique transition operator corresponding to the Kolmogorov equation (3.21), as stated in Corollary 3.7, and the variation of constants formula, we can prove the uniqueness of weak solution of the stochastic evolution equation (3.4) on $\Sigma_{rba}(H^+)$. This is done by using the corresponding Volterra type functional equation,

$$\mu_t = U^*(t,0)\mu_0 + \int_0^t U^*(t,s)(C^*\mu_{s-})M(ds), t \in I.$$

Remark 3.9. If the martingale measure M is nonatomic and the associated quadratic variation measure π is absolutely continuous with respect to the Lebesgue measure, it follows from expression (3.20) that the measure solution $t \longrightarrow \lambda_t^o$ is weak star continuous.

4. Extension to measurable vector fields

In some situations F and G may not be even continuous. However, assuming that they are bounded Borel measurable, it is possible to prove the existence results similar to those of deterministic evolutions [4]. We present here a result analogous to that of theorem 3.3 with the major exception that in the present case the measure solutions are no longer regular. They are bounded finitely additive measure valued processes.

Theorem 4.1. *Consider the system* (1.2). *Suppose* $\{A, M\}$ *satisfy the assumptions of Theorem* 3.3 *and that* $F : H \longrightarrow H$ *and* $G : H \longrightarrow \mathcal{L}(E, H)$ *are Borel measurable maps bounded on bounded sets. Then, for every* x_0 *for which* $P\{\omega \in \Omega : |x_0|_H < \infty\} = 1$, *statistically independent of* M, *the evolution equation* (1.2) *has a unique measure or generalized solution* $\lambda^0 \in M^w_{\infty,2}(I \times \Omega, \Sigma_{ba}(H^+))$. *Further,* $\lambda^0 \in M^w_{\infty,2}(I \times \Omega, \Pi_{ba}(H^+))$.

Proof. First suppose that $\{F, G\}$ are bounded Borel measurable maps (i.e. uniformly bounded on H). Then it follows from Proposition 3.2 of [4] that the pair $\{F, G\}$ has an approximating sequence$\{F_n, G_n\}$ satisfying (A1) and (A2). Given this fact, the proof is almost identical to that of Theorem 3.3 with the replacement of $\Sigma_{rba}(H^+)$ by $\Sigma_{ba}(H^+)$. In other words, for uniformly bounded Borel measurable maps F and G, the system (1.2) has a unique measure solution. For the unbounded case, define the composition maps

$$F_\gamma \equiv FoR_\gamma, G_\gamma \equiv GoR_\gamma, \gamma \in R_0$$

where R_γ is the retraction of the ball $B_\gamma \equiv \{x \in H : \| x \|_H < \gamma\}$. Clearly, these are (uniformly) bounded Borel measurable maps and it follows from the preceding result that the system (1.2) with $\{F, G\}$ replaced by $\{F_\gamma, G_\gamma\}$ has a unique measure solution $\lambda^\gamma \in M^w_{\infty,2}(I \times \Omega, \Sigma_{ba}(H^+))$. Then following a similar limiting process as in [4, Theorem 3.3], one can show that the net $\{\lambda^\gamma, \gamma \in R_0\}$ has a weak star convergent subnet converging to an element $\lambda^o \in M^w_{\infty,2}(I \times \Omega, \Sigma_{ba}(H^+))$ which is the unique measure solution of equation (1.2) in the sense of Definition 3.1. ∎

Remark. Our results can be extended to include vector measure M with atoms and hence jump processes under some additional assumptions on G. In this case the operator $\mathcal{C}$ is given by

$$(\mathcal{C}(t)\varphi)(\xi) \equiv \int_0^1 G^*(\xi) D\varphi(\xi + \theta G(\xi) M(\{t\})) d\theta.$$

If t is not an atom, this operator reduces to the one given by (3.1).

5. Optimal feedback controls

Consider the control system

$$\begin{aligned} dx(t) &= Ax(t)dt + F(x(t))dt + \Gamma(x(t))\; u(t,x(t))\; dt \\ &\quad + G(x(t-))M(dt) \\ x(0) &= x_0, \end{aligned} \tag{5.1}$$

where $\Gamma : H \longrightarrow \mathcal{L}(\Xi, H)$ is a bounded Borel measurable map with Ξ being another separable Hilbert space and $u : I \times H \longrightarrow \Xi$ is any bounded Borel measurable function representing the control. Let $BM(I\times H, \Xi)$ denote the class of bounded Borel measurable functions from $I \times H$ to Ξ. Furnished with the uniform norm topology,

$$\parallel u \parallel \equiv \sup\{|u(t,x)|_{\Xi}, (t,x) \in I \times H\},$$

it is a Banach space. For admissible controls, we use a weaker topology and introduce the following class of functions. Let U be a closed bounded (possibly convex) subset of Ξ and

$$\mathcal{U} \equiv \big\{u \in BM(I \times H, \Xi) : u(t,x) \in U \;\forall\; (t,x) \in I \times H\big\}. \tag{5.2}$$

On $BM(I \times H, \Xi)$, we introduce the topology of weak convergence in Ξ uniformly on compact subsets of $I \times H$ and denote this topology by τ_{wu}. In other words, a sequence $\{u_n\} \subset BM(I \times H, \Xi)$ is said to converge to $u_0 \in BM(I \times H, \Xi)$ in the topology τ_{wu} if, for every $v \in \Xi$,

$$(u_n(t,x), v)_{\Xi} \longrightarrow (u_0(t,x), v)_{\Xi}$$

uniformly in (t,x) on compact subsets $K \subset I \times H$. We assume that $\mathcal{U}$ has been furnished with the relative τ_{wu} topology. Let $\mathcal{U}_{ad} \subset \mathcal{U}$ be any τ_{wu} compact set and choose this set for admissible controls.

In view of system (5.1), we consider the Lagrange problem $P1$: Find $u^o \in \mathcal{U}_{ad}$ that minimizes the cost functional

$$J(u) \equiv \mathcal{E} \int_0^T \ell(t, x(t))dt, \tag{5.3}$$

with ℓ being any real valued Borel measurable function on $I \times H$ which is bounded on bounded sets. Since, under the general assumptions of Theorem 3.3 and Theorem 4.1, the control system (5.1) may have no path wise solution but has a measure solution, the control problem as stated above must be reformulated in terms of measure solutions. For this purpose we introduce the operator $\mathcal{B}_u$ associated with the control u as follows. For each $u \in \mathcal{U}_{ad}$, define

$$(\mathcal{B}_u\phi)(t,\xi) \equiv (u(t,\xi), \Gamma^*(\xi)D\phi(\xi))_\Xi, (t,\xi) \in I \times H,$$

where $\Gamma^*(\xi) \in \mathcal{L}(H,\Xi)$ is the adjoint of the operator $\Gamma(\xi)$. Clearly the operator $\mathcal{B}_u$ is well defined on $D(\mathcal{A}) \cap D(\mathcal{B})$. Then the correct formulation of the original control problem is given by $(P1)$: find $u^o \in \mathcal{U}_{ad}$ that minimizes the functional

$$J(u) \equiv \mathcal{E}\int_0^T \int_H \ell(t,\xi)\lambda_t^u(d\xi)dt \equiv \mathcal{E}\int_0^T \hat{\ell}(t,\lambda_t^u)dt \tag{5.4}$$

where λ^u is the (weak) solution of equation

$$\begin{aligned} &d\lambda_t = \mathcal{A}^*\lambda_t\pi(dt) + \mathcal{B}^*\lambda_t dt + \mathcal{B}_u^*\lambda_t dt + < \mathcal{C}^*\lambda_{t-}, M(dt) >_E, \\ &\lambda_0 = \delta_{x_0}. \end{aligned} \tag{5.5}$$

Other control problems are considered later. We need the following result on the continuous dependence of solutions on control.

Lemma 5.1. *Consider the system* (5.5) *with admissible controls $\mathcal{U}_{ad}$ as defined above, and suppose the assumptions of Theorem* 4.1 *hold and that $\Gamma : H \longrightarrow \mathcal{L}(\Xi, H)$ is a bounded Borel measurable map. Then for every $u \in \mathcal{U}_{ad}$ the system* (5.5) *has a unique weak solution $\lambda^u \in M^w_{\infty,2}(I\times\Omega, \Pi_{ba}(H^+))$ and further, the control to solution map $u \longrightarrow \lambda^u$ from $\mathcal{U}_{ad}$ to $M^w_{\infty,2}(I \times \Omega, \Sigma_{ba}(H^+))$ is (sequentially) continuous with respect to the topologies τ_{wu} on $\mathcal{U}_{ad}$ and weak star topology on $M^w_{\infty,2}(I \times \Omega, \Sigma_{ba}(H^+))$.*

Proof. The existence of solution follows from Theorem 4.1 with the operator $\mathcal{B}$ replaced by the sum $\mathcal{B} + \mathcal{B}_u$, and uniqueness follows from Corollary 3.7. We prove continuity. Let $\{u^n, u^o\} \subset \mathcal{U}_{ad}$ and $u^n \xrightarrow{\tau_{wu}} u^o$ and suppose $\{\lambda^n, \lambda^o\} \subset M^w_{\infty,2}(I \times \Omega, \Pi_{ba}(H^+))$ denote the associated weak solutions of (5.5). Then clearly, the difference $\mu^n \equiv \lambda^n - \lambda^o \in M^w_{\infty,2}(I \times \Omega, \Sigma_{ba}(H^+))$ is a sequence satisfying the identity

$$
\begin{aligned}
\mu_t^n(\varphi) &= \int_0^t \mu_s^n(\mathcal{A}\varphi)\pi(ds) + \int_0^t \mu_s^n\big((\mathcal{B}+\mathcal{B}_{u^o})\varphi\big)ds \\
&+ \int_0^t < \mu_{s-}^n(\mathcal{C}\varphi), M(ds) > + \int_0^t \lambda_s^n\big((\mathcal{B}_{u^n}-\mathcal{B}_{u^o})\varphi\big)ds
\end{aligned} \tag{5.6}
$$

with $\mu_0^n = \mu_{0-}^n = 0$. Since the set $\{\lambda^n, \lambda^o\}$ is contained in $M_{\infty,2}^w(I \times \Omega, \Pi_{ba}(H^+))$, the difference sequence $\{\mu^n\}$ with the initial condition $\mu_0^n = 0$, is contained in a bounded subset of $M_{\infty,2}^w(I \times \Omega, \Sigma_{ba}(H^+))$. Thus by virtue of Alaoglu's theorem, both these sequences have w^* convergent generalized subsequences or subnets which we relabel as the original sequence. Let λ^* denote the weak star limit of λ^n and μ^* the limit of μ^n. Since u^n converges to u^o in τ_{wu} topology, and $D\varphi$ has a compact support and Γ is a uniformly bounded Borel measurable operator valued function, it follows from the dominated convergence theorem that, for any $z \in L_2(\Omega)$,

$$
z((\mathcal{B}_{u^n} - \mathcal{B}_{u^o})\varphi) \equiv z(u^n - u^o, \Gamma^* D\varphi)_\Xi \stackrel{s}{\longrightarrow} 0 \ \text{ in } M_{1,2}(I \times \Omega, B(H^+)).
$$

Combining this with the fact that $\lambda^n \stackrel{w^*}{\longrightarrow} \lambda^*$, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \mathcal{E}\Big(z \int_0^t \lambda_s^n((\mathcal{B}_{u^n} - \mathcal{B}_{u^o})\varphi)ds\Big) \\
&= \lim_{n\to\infty} \mathcal{E}\Big(\int_0^t \lambda_s^n\big(z((\mathcal{B}_{u^n} - \mathcal{B}_{u^o})\varphi)\big)ds\Big) = 0, \ t \in I.
\end{aligned}
$$

Then multiplying (5.6) by $z \in L_2(\Omega)$ and taking the expectation on both sides and following similar limit arguments as in the proof of Theorem 3.3, we obtain

$$
\begin{aligned}
\mu_t^*(\varphi) &= \int_0^t \mu_s^*(\mathcal{A}\varphi)\pi(ds) + \int_0^t \mu_s^*\big((\mathcal{B}+\mathcal{B}_{u^o})\varphi\big)ds \\
&+ \int_0^t < \mu_{s-}^*(\mathcal{C}\varphi), M(ds) > ds, t \in I,
\end{aligned} \tag{5.7}
$$

for all $\varphi \in D(\mathcal{A}) \cap D(\mathcal{B})$. This is a homogeneous linear Volterra type functional equation for μ^*, and hence, following the same procedure as in [5], we find that $\mu^* = 0$. In other words, the weak star limit λ^* coincides with λ^o, the weak solution corresponding to the limit control u^o. This proves the continuity as stated. ■

Now we consider the control problem $P1$.

Theorem 5.2. *Consider the system* (5.5) *and the Lagrange problem* (5.4) *with admissible controls* $\mathcal{U}_{ad}$. *Suppose the assumptions of Lemma* 5.1 *hold and that* ℓ *is a Borel measurable real valued function defined on* $I \times H$ *and bounded on bounded sets and that there exists a function* $\ell_0 \in L_1(I)$ *such that*

$$\ell(t,\xi) \geq \ell_0(t) \ \forall\ \xi \in H.$$

Then there exists an optimal control for the problem $P1$.

Proof. Since ℓ is bounded from below by an integrable function ℓ_0, we have

$$J(u) \equiv \mathcal{E}\left(\int_0^T \hat{\ell}(t,\lambda_t^u)dt\right) > -\infty,\ \forall\ u \in \mathcal{U}_{ad}. \tag{5.8}$$

Clearly, if $J(u) = +\infty$ for all $u \in \mathcal{U}_{ad}$, there is nothing to prove. So suppose there are controls for which $J(u)$ is finite. Define $\inf\{J(u), u \in \mathcal{U}_{ad}\} = m$, and let $\{u^n\} \subset \mathcal{U}_{ad}$ be a minimizing sequence. Since $\mathcal{U}_{ad}$ is τ_{wu} compact, there exists a generalized subsequence (subnet), relabeled as the original sequence, and a control $u^o \in \mathcal{U}_{ad}$ such that $u^n \xrightarrow{\tau_{wu}} u^o$. Then by virtue of Lemma 5.1, along a further subnet if necessary, we have $\lambda^{u^n} \xrightarrow{w^*} \lambda^{u^o}$. Note that the functional (5.8) is linear in λ^u and bounded (since $\{u^n\}$ is a minimizing sequence) and hence continuous along the minimizing sequence $\{\lambda^{u^n}\}$. Thus $\lim_{n\to\infty} J(u^n) = J(u^o) = m$ and u^o is the optimal control. ■

Next we consider the control problem $P2$:

$$J(u) \equiv \mathcal{E}\int_{I\times H}\{\ell(t,\xi) + \rho(\xi)|u(t,\xi)|_\Xi\}\lambda_t^u(d\xi)dt \longrightarrow \inf, \tag{5.9}$$

where ρ is a positive bounded Borel measurable function on H with compact support and λ^u is the weak solution of the stochastic evolution equation (5.5) corresponding to control u.

Theorem 5.3. *Consider the Lagrange problem* P2 *with the objective functional* (5.9) *subject to the dynamic constraint described by the system* (5.5) *with admissible* $\mathcal{U}_{ad}$. *Suppose* ℓ *satisfies the conditions as stated in Theorem* 5.2, *and* ρ *is any real valued nonnegative bounded Borel measurable function on* H *having a compact support. Then there exists an optimal control for the problem* P2.

Proof. Again by virtue of the assumption on ℓ, we have $J(u) > -\infty$. If $J(u) \equiv +\infty$ for all $u \in \mathcal{U}_{ad}$ there is nothing to prove. So we may assume the contrary. Let $\{u^n\}$ be a minimizing sequence so that

$$\lim_{n\to\infty} J(u^n) = \inf\{J(u), u \in \mathcal{U}_{ad}\} \equiv \tilde{m}.$$

We show that the second term of the objective functional (5.9), denoted by J_2, is τ_{wu} lower semi continuous on $\mathcal{U}_{ad}$. Since $\mathcal{U}_{ad}$ is τ_{wu} compact, the sequence $\{u^n\}$ contains a generalized subsequence, relabeled as the original sequence, which converges in τ_{wu} topology to an element $u^o \in \mathcal{U}_{ad}$. Consider the value of J_2 at u^o,

$$J_2(u^o) \equiv \mathcal{E}\int_{I\times H} \rho(\xi)|u^o(t,\xi)|_{\Xi}\ \lambda_t^{u^o}(d\xi)dt. \tag{5.10}$$

Since $u^o(t,\xi)$ is a Ξ valued bounded (Borel) measurable function, by Riesz theorem there exists a $B_1(\Xi)$ valued bounded (Borel) measurable function η^o on $I \times H$ such that

$$|u^o(t,\xi)|_{\Xi} = (u^o(t,\xi), \eta^o(t,\xi))_{\Xi},\ \forall (t,\xi) \in I \times H.$$

In fact, one can take $\eta^o(t,\xi) = u^o(t,\xi)/|u^o(t,\xi)|_{\Xi}$ with the convention that $0/0 \equiv 0$. Hence (5.10) can be written as

$$\begin{aligned} J_2(u^o) &\equiv \mathcal{E}\int_{I\times H} \rho(\xi)(u^o(t,\xi),\eta^o(t,\xi))_{\Xi}\ \lambda_t^{u^o}(d\xi)dt \\ &= \mathcal{E}\int_{I\times H} \rho(\xi)(u^o(t,\xi),\eta^o(t,\xi))_{\Xi}\big(\lambda_t^{u^o} - \lambda_t^{u^n}\big)(d\xi)dt \\ &\quad + \mathcal{E}\int_{I\times H} \rho(\xi)(u^o(t,\xi) - u^n(t,\xi),\eta^o(t,\xi))\ \lambda_t^{u^n}(d\xi)dt \\ &\quad + \mathcal{E}\int_{I\times H} \rho(\xi)(u^n(t,\xi),\eta^o(t,\xi))_{\Xi}\ \lambda_t^{u^n}(d\xi)dt. \end{aligned} \tag{5.11}$$

By virtue of Lemma 5.1, the first term of (5.11) converges to zero as $n \to \infty$. Since ρ has a compact support and $u^n \xrightarrow{\tau_{wu}} u^o$, it is clear that $\rho(u^0 - u^n, \eta^0)_\Xi \longrightarrow 0$ uniformly on $I \times H$, hence strongly in $M_{1,2}(I \times \Omega, BC(H))$, while λ^{u^n} converges to λ^{u^o} in the weak star topology of $M^w_{\infty,2}(I\times\Omega, \Sigma_{rba}(H))$. Thus the second term of (5.11) also converges to zero as $n \to \infty$. Clearly, it follows from positivity of both ρ and the measure solutions, and the fact that $|\eta^o(t,\xi)|_\Xi \equiv 1$, that the third term is majorized by $J_2(u^n)$. From these facts one can easily verify that

$$J_2(u^o) \leq \liminf_{n\to\infty} J_2(u^n). \tag{5.12}$$

Thus J_2 is τ_{wu} lower semicontinuous. It was already seen in Theorem 5.2 that the first term of the cost functional is continuous. Thus $u \longrightarrow J(u)$ is τ_{wu} lower semi continuous. Since $\{u^n\}$ is a minimizing sequence with limit $u^o \in \mathcal{U}_{ad}$, it follows from τ_{wu} lower semi continuity that

$$\tilde{m} \leq J(u^o) \leq \liminf J(u^n) \leq \lim J(u^n) = \inf\{J(u), u \in \mathcal{U}_{ad}\} \equiv \tilde{m}.$$

Hence u^o is the optimal control proving existence. ■

Remark. The assumption of compactness of the support of ρ seems to be restrictive. It can be partially remedied by choosing an increasing sequence of compact subsets $\{C_k\}$ of H, converging to H, and a sequence of $\{\rho_k\}$ with support C_k approximating any bounded Borel measurable function ρ. Then in the limit one has the optimal control associated with the original weight function ρ.

Another interesting control problem, identified as $P3$, consists of maximizing the functional:

$$J(u) = f(\mathcal{E}\lambda^u_{t_1}(\varphi_1), \cdots, \mathcal{E}\lambda^u_{t_d}(\varphi_d))\} \longrightarrow \sup \tag{5.11}$$

where $f : R^d \longrightarrow R$ is a function, and $\{\varphi_i\} \in B(H)$ is a finite set of bounded real valued Borel measurable functions on H.

Theorem 5.4. *Consider the system* (5.5) *with admissible controls* $\mathcal{U}_{ad} \subset \mathcal{U}$ *as defined by* (5.2) *and suppose the assumptions of Lemma* 5.1 *hold. Further, suppose the stochastic vector measure* M *is nonatomic and the associated* *quadratic variation measure π is absolutely continuous with respect to the Lebesgue measure and the function f is upper semicontinuous from R^d to R and $\{\varphi_i\} \in B(H)$ are real valued bounded Borel measurable functions. Then the Problem* P3 *has a solution.*

Proof. Since M is nonatomic and π is absolutely continuous with respect to the Lebesgue measure, it follows from the integral expression (3.3) or (3.20) that $t \longrightarrow \lambda_t^u$ is weak star continuous P-a.s. Hence $t \longrightarrow \lambda_t^u(\varphi)$ is continuous P-a.s for each $\varphi \in B(H)$. Thus by Fubini's theorem, $\lambda_{t_i}^u(\varphi_i)$ is a well defined integrable random variable for all $i = 1, 2, \cdots, d$. Let $\{u^n\} \in \mathcal{U}_{ad}$ be a maximizing sequence. Since $\mathcal{U}_{ad}$ is τ_{wu} compact there exists a $u^o \in \mathcal{U}_{ad}$ such that, along a subsequence if necessary, $u^n \xrightarrow{\tau_{wu}} u^o$. Hence it follows from Lemma 5.1 that

$$\mathcal{E}\lambda_{t_i}^{u^n}(\varphi_i) \longrightarrow \mathcal{E}\lambda_{t_i}^{u^o}(\varphi_i), i = 1, 2, \cdots, d.$$

Thus, by upper semicontinuity of f, we have

$$\limsup_{n\to\infty} f(\mathcal{E}\lambda_{t_1}^{u^n}(\varphi_1), \cdots, \mathcal{E}\lambda_{t_d}^{u^n}(\varphi_d)) \leq f(\mathcal{E}\lambda_{t_1}^{u^o}(\varphi_1), \cdots, \mathcal{E}\lambda_{t_d}^{u^o}(\varphi_d)).$$

Hence J is upper semicontinuous with respect to τ_{wu} topology, that is,

$$\limsup_{n\to\infty} J(u^n) \leq J(u^o).$$

Since $\{u^n\}$ is a maximizing sequence and $u^o \in \mathcal{U}_{ad}$, it follows from this that

$$m^o \equiv \sup_{u\in\mathcal{U}_{ad}} J(u) = \lim_{n\to\infty} J(u^n) \leq \limsup_{n\to\infty} J(u^n) \leq J(u^o) \leq m^o.$$

This proves that u^o is a maximizing control and hence follows the existence of an optimal control. ■

A fourth interesting problem, identified as $P4$, can be stated as follows. Let $\Psi \in B(H)$ and $g \in C_b(R)$ be given. The problem is to find a control that minimizes (maximizes) the functional

$$J(u) \equiv \mathcal{E}g(\lambda_T^u(\Psi)). \tag{5.12}$$

Theorem 5.5. *Consider the system* (5.5) *and the objective functional* (5.12) *with admissible controls* $\mathcal{U}_{ad} \subset \mathcal{U}$ *as defined by* (5.2) *and suppose the assumptions of Lemma* 5.1 *hold. Further, suppose the martingale measure* M *is nonatomic and the associated quadratic variation measure* π *is absolutely continuous with respect to the Lebesgue measure and the function* $g \in C_b(R)$ *and* $\Psi \in B(H)$. *Then the Problem* P4 *has a solution.*

Proof. Since M is non-atomic and π is absolutely continuous with respect to the Lebesgue measure, it follows from (3.20) that $t \longrightarrow \lambda_t^u$ is weak star continuous $P - a.s.$ Hence, for $\Psi \in B(H)$, $\{\lambda_T^u(\Psi), u \in \mathcal{U}_{ad}\}$ is a family of well defined real valued random variables. Without loss of generality, we may assume that $\| \Psi \| \leq 1$. Clearly, then

$$|\lambda_T^u(\Psi)| \leq 1, P - a.s, \forall\ u \in \mathcal{U}_{ad}.$$

For each $u \in \mathcal{U}_{ad}$, define the probability measure μ^u on $\mathcal{B}(R)$, the Borel sets of R, by setting

$$\mu^u(\Gamma) = P\{\lambda_T^u(\Psi) \in \Gamma\}$$

for $\Gamma \in \mathcal{B}(R)$. Then the expression (5.12) is equivalent to

$$J(u) = \int_R g(\zeta)\mu^u(d\zeta) \equiv L(\mu^u). \tag{5.13}$$

Since $g \in C_b(R)$, it is clear that L is a continuous linear functional on the space of probability measures $\mathcal{M}_1(R)$. Note that the family of probability measures,

$$\mathcal{M}_0 \equiv \{\mu^u, u \in \mathcal{U}_{ad}\},$$

is contained in $\mathcal{M}_1$ and has a compact support given by the interval $[-1, +1]$, and therefore it is uniformly tight and hence a relatively weakly compact subset of $\mathcal{M}_1$. Using the assumption of τ_{wu} compactness of the set $\mathcal{U}_{ad}$, one can prove that $\mathcal{M}_0$ is also weakly closed. Thus $\mathcal{M}_0$ is a weakly compact subset of $\mathcal{M}_1(R)$. Therefore the linear functional L, which is continuous and hence weak star continuous, attains its minimum (maximum) on the set $\mathcal{M}_0$ and consequently J attains its minimum (maximum) on $\mathcal{U}_{ad}$. This proves the existence of an optimal control for the problem $P4$. ■

A Special Case

Suppose $\sigma \in \mathcal{B}(H)$ is the target set and we want to maximize the expected value of concentration of mass on σ. Then we take $\Psi = \chi_\sigma$, and $g \in C_b^+(R)$ a (any) monotone increasing function. According to the above result, there exists a control $u^o \in \mathcal{U}_{ad}$ such that

$$\mathcal{E}\{g(\lambda_T^{u^o}(\sigma))\} \geq \mathcal{E}\{g(\lambda_T^u(\sigma))\}, \ \forall \ u \in \mathcal{U}_{ad}.$$

The obstacle avoidance problem can be treated similarly.

A problem closely related to (P4) is the problem (P5) given by

$$J(u) = \mathcal{E}\Big\{g(\lambda_{t_1}^u(\varphi_1), \cdots, \lambda_{t_d}^u(\varphi_d))\Big\} \longrightarrow \inf. \tag{5.14}$$

Here $g : R^d \longrightarrow [0, \infty)$, $\{t_i \in I, \varphi_i \in B(H)\}$, and $d \in N$. Using a similar procedure as in Theorem 5.5 one can prove the following result.

Theorem 5.6. *Consider the system* (5.5) *and the objective functional* (5.14) *with admissible controls* $\mathcal{U}_{ad}$, $g \in C_b(R^d), \varphi_i \in B(H), t_i \in I$, *and suppose* $\{M, \pi\}$ *satisfy the assumptions of Theorem* 5.5. *Then the Problem* P5 (5.14) *has a solution.*

Proof. The proof is very similar to that of Theorem 5.5. Here $J(u)$ is given by

$$J(u) = \int_{R^d} g(z)\mu^u(dz)$$

with the measure μ^u defined by

$$\mu^u(\sigma) \equiv P\{(\lambda_{t_1}^u(\varphi_1), \lambda_{t_2}^u(\varphi_2), \cdots, \lambda_{t_d}^u(\varphi_d)) \in \sigma\},$$

for $\sigma \in \mathcal{B}(R^d)$ and $\mathcal{M}_0 \equiv \{\mu^u : u \in \mathcal{U}_{ad}\} \subset \mathcal{M}_1(R^d)$. Again the assertion follows from the facts that the family $\mathcal{M}_0(R^d)$ is tight and weakly closed and $g \in C_b(R^d)$. ■

Another interesting control problem is to find a control $u \in \mathcal{U}_{ad}$ that maximizes the functional

$$J(u) \equiv \mathcal{E} \int_I g(t, \lambda_t^u(G(t))dt \longrightarrow \max, \tag{5.15}$$

subject to the dynamics (5.5) with $\lambda = \lambda^u$, where $t \longrightarrow G(t)$ is a multifunction with values in $c(H)$(nonempty closed subsets of H) and continuous (in the sense of the Hausdorf metric) and $g : I \times [0, \infty) \longrightarrow [0, \infty)$ is a nonnegative continuous function and nondecreasing in the second argument. Clearly, this problem is equivalent to tracking a diffuse moving target. We call this problem P6.

Theorem 5.7. *Consider the system* (5.5) *with admissible controls* $\mathcal{U}_{ad}$ *and objective functional* (5.15) *with* $\{M, \pi\}$ *satisfying the assumptions of Theorem* 5.5. *Let* $t \rightarrow G(t)$ *be a multifunction with values which are nonempty closed subsets of* H *and continuous in the Hausdorf metric. The function* $g : I \times [0, \infty) \longrightarrow [0, \infty)$ *is nonnegative, continuous and bounded on bounded sets and nondecreasing in the second argument. Then problem* P6 *has a solution.*

Proof. Since for all $t \in I$ and $u \in \mathcal{U}_{ad}$, $\lambda_t^u(G(t)) \in [0, 1] \equiv K$ with probability one, the objective functional (5.15) is equivalent to the following functional

$$J(u) = \int_{I \times K} g(t, z)\mu_t^u(dz)dt \equiv L_g(\mu^u) \tag{5.16}$$

where the measure valued function μ^u is given by

$$\mu_t^u(\sigma) \equiv P\{\lambda_t^u(G(t)) \in \sigma\}, \sigma \in \mathcal{B}_1 \tag{5.17}$$

with $\mathcal{B}_1$ denoting the sigma algebra of Borel subsets of the unit interval $K \equiv [0, 1]$. Let $\mathcal{M}_1(K)$ denote the space of probability measures on $\mathcal{B}_1$ endowed with the standard weak topology. Since the weak topology is metrizable (for example, Prohorov metric), and K is compact, $\mathcal{M}_1(K)$ is a compact metric space with a metric, say, d. Consider the topological space $C(I, \mathcal{M}_1)$ with the topology of uniform convergence on I induced by the metric,

$$D(\mu, \nu) \equiv \sup\{d(\mu_t, \nu_t), t \in I\}.$$

Since $\mathcal{M}_1(K)$ is a complete metric space, so also is $(C(I, \mathcal{M}_1(K)), D)$. Now consider the set

$$C_0 \equiv \{\mu^u, u \in \mathcal{U}_{ad}\} \tag{5.18}$$

where μ^u is given by (5.17). Since $t \longrightarrow \lambda_t^u$ is weak star continuous P-a.s and $t \longrightarrow G(t)$ is continuous in the Hausdorf metric, we have $t \longrightarrow \mu_t^u$ continuous with respect to the topology induced by the metric d. Using the integral expression (3.3) or (3.20) with the operator $\mathcal{B}$ replaced by $\mathcal{B}+\mathcal{B}_u$, it follows from nonatomicity of the measure M and absolute continuity of the measure π with respect to the Lebesgue measure that, for each $t \in (0,T]$ and $\varphi \in D(\mathcal{A}) \cap D(\mathcal{B})$,

$$P\Big\{\lim_{h\to 0}(\lambda_{t+h}^u(\varphi) - \lambda_t^u(\varphi)) = 0\Big\} = 1,$$

uniformly with respect to $u \in \mathcal{U}_{ad}$. Thus the set C_0 as given by (5.18), is an equicontinuous subset of $C(I, \mathcal{M}_1)$. Since $\mathcal{M}_1(K)$ is compact, it is obvious that each t-section of C_0 given by $C_0(t) \equiv \{\mu_t, \mu \in C_0\}$ is relatively compact in $\mathcal{M}_1(K)$. Thus by Ascoli-Arzella theorem, C_0 is a relatively compact subset of $C(I, \mathcal{M}_1(K))$. Further, it follows from sequential continuity (see Lemma 5.1) and τ_{wu} compactness of $\mathcal{U}_{ad}$ that C_0 is sequentially closed. These imply that the set C_0 is sequentially compact. Thus the functional L_g, defined by (5.16), being linear and continuous, attains both its minimum and maximum on the compact set C_0. Hence there exists a $\mu^o \in C_0$ or equivalently a control law $u^o \in \mathcal{U}_{ad}$ at which L_g attains its maximum. This proves the existence of an optimal control for the problem P6. ■

Remark. For the construction of optimal controls, it is essential to develop necessary and sufficient conditions for optimality. This will be considered in a forthcoming paper.

Open Problem. It would be interesting to admit π which may contain atoms. In other words it will allow inclusion of continuous as well as jump processes for the martingale measure M.

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