

**A VIABILITY RESULT FOR NONCONVEX
SEMILINEAR FUNCTIONAL
DIFFERENTIAL INCLUSIONS**

VASILE LUPULESCU

*"Constantin Brâncuși" University
Department of Mathematics
Târgu-Jiu, 210152, Romania*

e-mail: vasile@utgjiu.ro

AND

MIHAI NECULA

*"Al. I. Cuza" University of Iași
Faculty of Mathematics
Iași, 700506, Romania*

e-mail: necula@uaic.ro

Abstract

We establish some sufficient conditions in order that a given locally closed subset of a separable Banach space be a viable domain for a semilinear functional differential inclusion, using a tangency condition involving a semigroup generated by a linear operator.

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1. INTRODUCTION

We shall denote by X a separable Banach space and by $\mathcal{C}_\sigma := C([- \sigma, 0], X)$, with $\sigma > 0$, the Banach space of continuous functions from $[- \sigma, 0]$ into X , endowed with the norm $\|\varphi\|_\sigma := \sup\{\|\varphi(s)\|; s \in [- \sigma, 0]\}$. For any function $u : [\tau - \sigma, T] \rightarrow X$ and any $t \in [\tau, T]$ we shall denote by u_t the function defined as follows:

$$u_t : [- \sigma, 0] \rightarrow X, \quad u_t(s) = u(t + s),$$

for every $s \in [- \sigma, 0]$. Clearly, if u is continuous, then $u_t \in \mathcal{C}_\sigma$ for every $t \in [\tau, T]$.

Let K be a given locally closed subset in X and let \mathcal{K}_0 be the following subset of \mathcal{C}_σ :

$$\mathcal{K}_0 := \{\varphi \in \mathcal{C}_\sigma; \varphi(0) \in K\}.$$

We recall that a subset $K \subset X$ is *locally closed* if for each $\xi \in K$ there exists $r > 0$ such that $K \cap B(\xi, r)$ is closed in X , where, as usual, $B(\xi, r)$ denotes the closed ball with center ξ and radius r .

We consider the following functional differential inclusion

$$(1.1) \quad u'(t) \in Au(t) + F(t, u_t), \quad t \in [a, b],$$

where $F : [a, b] \times \mathcal{C}_\sigma \rightarrow 2^X$ is a multifunction with nonempty and closed values and $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of the C_0 -semigroup $S(t) : X \rightarrow X$, $t \geq 0$, and we are interested in finding sufficient conditions in order that K be a *viable domain* for (1.1), i.e. for each $(\tau, \varphi) \in [a, b] \times \mathcal{K}_0$ there exists at least one solution $u : [\tau - \sigma, T] \rightarrow K$ of (1.1) satisfying the initial condition

$$(1.2) \quad u_\tau = \varphi.$$

By a solution to the problem (1.1) and (1.2) we mean a continuous function $u : [\tau - \sigma, T] \rightarrow X$ for which there exists $f \in L^1([\tau, T], X)$ with $f(t) \in F(t, u_t)$ a.e. on $[\tau, T]$ and such that

$$(1.3) \quad u(t) = \begin{cases} \varphi(t - \tau) & \text{for } t \in [\tau - \sigma, \tau), \\ S(t - \tau)\varphi(0) + \int_\tau^t S(t - s)f(s)ds & \text{for } t \in [\tau, T]. \end{cases}$$

The existence of solutions for functional differential equations governed or not by linear and nonlinear operators in Banach spaces has been studied extensively in many papers (see, for example, [4, 9, 10, 15, 22, 23, 26, 28]).

The first viability results for (1.1) in the case $A = 0$ and F single valued have been proved in the papers [20] and [19]. The case when $A = 0$, X is a finite dimensional space and F is upper semicontinuous and with convex compact values has been studied by Haddad ([13, 14]). Haddad's result has been extended by Syam [25] and Gavioli and Malaguti [11] to the infinite dimensional setting. For results, references and applications in this framework we refer to the monographs: [1, 8, 12, 17, 18] and [24]. The case when A is the infinitesimal generator of C_0 -semigroup and F is a continuous single-valued function has been studied by Iacob and Pavel [16].

There are many methods and techniques in the viability theory, but, generally speaking, the viability criteria fall into two classes: those in which the conditions are given in terms of a classical tangent cone (or Bouligand or Dini or contingent cone) and those in which a proximal normal cone is used. We shall use a tangency condition of the same kind as in [16], accordingly adapted. Also, the construction method for a sequence of approximate solutions to (1.1), defined on an a priori given interval, is closed to the one used by Cârjă and Vrabie [7] and the convergence method is the same that we have used [21].

2. PRELIMINARIES AND THE MAIN RESULT

We assume that the reader is familiar with the basic concepts and results concerning C_0 -semigroups, we refer to Vrabie [27] for details.

Let the Banach space X be endowed, with the σ -field $\mathcal{B}(X)$ of Borel subsets and let $\mathcal{J} = [a, b)$ be endowed with the Lebesgue measure and the σ -field $\mathcal{L}(\mathcal{J})$ of Lebesgue measurable subsets.

For nonempty subsets A, B of X and $a \in A$, we denote

$$d(a, B) = \inf\{\|a - b\|; b \in B\}, \quad d(A, B) = \sup\{d(a, B); a \in A\},$$

and by

$$d_{HP}(A, B) = \max\{d(A, B), d(B, A)\}$$

we denote the Hausdorff-Pompeiu distance between A and B .

Let us introduce the following hypotheses which we shall use throughout this paper.

(H_0) X is a separable Banach space, $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of the C_0 -semigroup $\{S(t); t \geq 0\}$, K is a locally closed subset in X and $F : \mathcal{J} \times \mathcal{K}_0 \rightarrow 2^X$ is a multifunction with nonempty and closed values;

(H_1) For each $(\tau, \varphi) \in \mathcal{J} \times \mathcal{K}_0$ there exist $\rho > 0$, $r > 0$ and an integrable function $\chi \in L^1([\tau, \tau + \rho], \mathbb{R}_+)$ such that

$$(2.1) \quad \sup\{|F(t, \psi)|; \psi \in \mathcal{K}_0 \times B_\sigma(\varphi, r)\} \leq \chi(t)$$

a.e on $[\tau, \tau + \rho]$, where $|F(t, \varphi)| := \sup\{\|y\|; y \in F(t, \psi)\}$ and

$$B_\sigma(\varphi, r) := \{\psi \in \mathcal{C}_\sigma; \|\psi - \varphi\|_\sigma \leq r\};$$

(H_2) For each $(\tau, \varphi) \in \mathcal{J} \times \mathcal{K}_0$ there exist $\rho > 0$, $r > 0$, $\mu \in L^1([\tau, \tau + \rho], \mathbb{R}_+)$ and a negligible subset $\mathcal{Z} \subset [\tau, \tau + \rho]$ such that

$$(2.2) \quad d_{HP}(F(t, \varphi_1), F(t, \varphi_2)) \leq \mu(t)\|\varphi_1 - \varphi_2\|_\sigma$$

for every $t \in [\tau, \tau + \rho] \setminus \mathcal{Z}$ and every $\varphi_1, \varphi_2 \in \mathcal{K}_0 \times B_\sigma(\varphi, r)$;

(H_3) For each $\varphi \in \mathcal{K}_0$ the multifunction $F(\cdot, \varphi) : \mathcal{J} \rightarrow 2^X$ is measurable;

(H_4) For every $(\tau, \varphi) \in \mathcal{J} \times \mathcal{K}_0$ the following tangential condition holds:

$$\liminf_{h \downarrow 0} \frac{1}{h} d(S(h)\varphi(0) + \int_\tau^{\tau+h} S(\tau + h - s)F(s, \varphi)ds, K) = 0.$$

Here the integral is in the sense of Aumann [2].

We are now ready to state the main result of this paper.

Theorem 2.1. *If the assumptions (H_0)–(H_4) are satisfied, then K is a viable domain for (1.1).*

In order to prove our theorem we need the following technical result, concerning a measurable multifunction in Banach spaces, established by Q.I. Zhu [29].

Theorem 2.2. *Let X be a separable Banach space, $\psi : [a, b) \rightarrow X$ a measurable function and $G(\cdot) : [a, b) \rightarrow 2^X$ a measurable multifunction with nonempty and closed values. Then for any positive measurable function $\nu : [a, b) \rightarrow \mathbb{R}_+$ there exists a measurable selection $g(\cdot) \in G(\cdot)$ such that*

$$\|g(t) - \psi(t)\| \leq d(\psi(t), G(t)) + \nu(t)$$

a.e. on $[a, b)$.

In what follows, we recall a general principle on ordered sets due to Brézis and Browder [3]. It will be used in the next section in order to obtain some "maximal" elements in an ordered set.

Theorem 2.3. *Let \preceq be a given preorder on the nonempty set M and let $\mathcal{S} : M \rightarrow \mathbb{R} \cup \{+\infty\}$ be an increasing function. Suppose that each increasing sequence in M is majorated in M . Then, for each $\xi_0 \in M$, there exists $\bar{\xi} \in M$ with $\xi_0 \preceq \bar{\xi}$ such that $\bar{\xi} \preceq \xi$ implies $\mathcal{S}(\bar{\xi}) = \mathcal{S}(\xi)$.*

In the paper by Brézis and Browder [3], the function \mathcal{S} is supposed to be finite and bounded from above, but, as remarked in [6], this restriction can be removed by replacing the function \mathcal{S} by $\xi \rightarrow \arctan \mathcal{S}(\xi)$.

Finally, let u be a function defined on the interval \mathcal{J} of \mathbb{R} with values into X . For some $\delta > 0$, we denote by $\omega(u, \mathcal{J}_0, \delta)$ the *modulus of continuity* of u on the subinterval $\mathcal{J}_0 \subset \mathcal{J}$, defined by

$$\omega(u, \mathcal{J}_0, \delta) = \sup\{\|u(t) - u(s)\|; t, s \in \mathcal{J}_0, |t - s| \leq \delta\}.$$

It is easy to see that $\omega(u, \cdot, \delta)$ and $\omega(u, \mathcal{J}_0, \cdot)$ are increasing functions and that u is uniformly continuous on \mathcal{J}_0 if and only if $\lim_{\delta \downarrow 0} \omega(u, \mathcal{J}_0, \delta) = 0$.

3. PROOF OF THE MAIN RESULT

We shall show that the tangential condition (H_4) along with Brézis-Browder Ordering Principle, i.e. Theorem 2.3 above, imply that for each initial point $(\tau, \varphi) \in \mathcal{J} \times \mathcal{K}_0$, there exist $T \in (\tau, b)$ and one sequence $u^n : [\tau - \sigma, T] \rightarrow X$ of "approximate solutions" of (1.1) such that $(u^n)_n$ converges uniformly to a solution $u : [\tau - \sigma, T] \rightarrow K$ of (1.1) satisfying (1.2).

We assume that the hypotheses (H_0) – (H_4) are satisfied and we begin by fixing an arbitrary initial data $(\tau, \varphi) \in \mathcal{J} \times \mathcal{K}_0$. Since the hypotheses (H_1)

and (H_2) have a local character and K is locally closed we can choose $r > 0$, $\rho \in (0, b - \tau)$, χ and μ in $L^1([\tau, \tau + \rho], \mathbb{R}_+)$ such that $K \cap B(\varphi(0), r)$ is closed in X and the relations (2.1) and (2.2) are satisfied on $[\tau, \tau + \rho] \times B_\sigma(\varphi, r)$. We emphasize that this choice of r, ρ, χ and μ will be kept unmodified until the end of this proof.

Remark 3.1. The following statements hold:

- (i) If $\alpha \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ then $\alpha(0) \in K \cap B(\varphi(0), r)$,
- (ii) If $K \cap B(\varphi(0), r)$ is closed in X then $\mathcal{K}_0 \cap B_\sigma(\varphi, r)$ is closed in C_σ .

Indeed, the first statement is obvious. For the second, let us assume that $K \cap B(\varphi(0), r)$ is closed and let us consider a sequence $(\alpha_n)_n$ in $\mathcal{K}_0 \cap B_\sigma(\varphi, r)$ that is convergent (in the norm $\|\cdot\|_\sigma$) to $\alpha \in \mathcal{C}_\sigma$. It readily follows that $\alpha \in B_\sigma(\varphi, r)$, $\alpha_n(0) \rightarrow \alpha(0)$ and $\alpha_n(0) \in K \cap B(\varphi(0), r)$, therefore, since $K \cap B(\varphi(0), r)$ is closed, we obtain that $\alpha(0) \in K$ and thus $\alpha \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$.

Since $\{S(t); t \geq 0\}$ is a C_0 -semigroup, there exist $M_0 \geq 1$ and $\omega_0 \geq 0$ such that $\|S(t)\xi\| \leq M_0 e^{\omega_0 t} \|\xi\|$ for every $t \geq 0$ and for every $\xi \in X$. We define

$$(3.1) \quad M = M_0 e^{\omega_0 \rho}$$

and we have $\|S(t - \tau)\xi\| \leq M \|\xi\|$ for every $t \in [\tau, \tau + \rho]$ and every $\xi \in X$.

We shall define the "approximate solution" concept.

Definition 3.1. Let $\varepsilon \in (0, 1)$, $\nu \in (\tau, \tau + \rho]$ and $\psi \in L^1([\tau, \tau + \rho], X)$ be arbitrarily fixed.

We shall denote by (θ, β, g, f, u) a 5-tuple composed of the measurable functions $\theta : [\tau, \nu] \rightarrow [\tau, \nu]$, $\beta : \Delta_\nu = \{(t, s); \tau \leq s < t \leq \nu\} \rightarrow [0, \nu - \tau]$, $g \in L^\infty([\tau, \nu], X)$, $f \in L^1([\tau, \nu], X)$ and by the function $u : [\tau - \sigma, \nu] \rightarrow X$ defined by

$$(3.2) \quad u(t) = \begin{cases} \varphi(t - \tau) & \text{for } t \in [\tau - \sigma, \tau), \\ S(t - \tau)\varphi(0) + \int_\tau^t S(t - s)f(s)ds + \int_\tau^t S(\beta(t, s))g(s)ds, & t \in [\tau, \nu]. \end{cases}$$

The 5-tuple (θ, β, g, f, u) will be called an (ε, ψ) -approximate solution of (1.1) and (1.2) on $[\tau - \sigma, \nu]$ if the following conditions are satisfied:

- (A₁) $u_{\theta(t)} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ for every $t \in [\tau, \nu]$;
- (A₂) $0 \leq t - \theta(t)$ for every $t \in [\tau, \nu]$;
- (A₃) $\beta(t, s) \leq t - \tau$ for $\tau \leq s < t \leq \nu$ and $t \rightarrow \beta(t, s)$ is nonexpansive on $(s, \nu]$;
- (A₄) $\|g(t)\| \leq \varepsilon$ a.e. on $[\tau, \nu]$;
- (A₅) $f(t) \in F(t, u_{\theta(t)})$ a.e. on $[\tau, \nu]$;
- (A₆) $\|f(t) - \psi(t)\| \leq d(\psi(t), F(t, u_{\theta(t)})) + \varepsilon\mu(t)$ a.e. on $[\tau, \nu]$;
- (A₇) $\|u_t - u_{\theta(t)}\|_\sigma \leq \varepsilon$ for every $t \in [\tau, \nu]$;
- (A₈) $u_\nu \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$.

Remark 3.2. We emphasize that although the function u is uniquely determined by β , g and f , for the sake of simplicity, we preferred to consider it as a component of (θ, β, g, f, u) .

Remark 3.3. Let $\nu \in (\tau, \tau + \rho]$, $\beta : \Delta_\nu \rightarrow [0, \nu - \tau]$, $g \in L^\infty([\tau, \nu], X)$, $f \in L^1([\tau, \nu], X)$ be given and $u : [\tau - \sigma, \nu] \rightarrow X$ be defined by (3.2). If β satisfies (A₃) then, using Lebesgue's Theorem, we deduce that the function u is continuous on the whole interval $[\tau - \sigma, \nu]$ and so $u_t \in C_\sigma$, for every $t \in [\tau, \nu]$.

In the sequel we define the operator solution to the problem (1.1) and (1.2), $Qf : L^1([\tau, \nu], X) \rightarrow C([\tau - \sigma, \nu], X)$, by

$$(3.3) \quad (Qf)(t) = \begin{cases} \varphi(t - \tau) & \text{for } t \in [\tau - \sigma, \tau], \\ S(t - \tau)\varphi(0) + \int_\tau^t S(t - s)f(s)ds & \text{for } t \in [\tau, \nu]. \end{cases}$$

We notice that u is a solution of (1.1) and (1.2) on $[\tau - \sigma, T]$ if there exists $f \in L^1([\tau, T], X)$ such that $u = Qf$ and $f(t) \in F(t, u_t)$ a.e. on $[\tau, T]$.

Remark 3.4. Let $\nu \in (\tau, \tau + \rho]$, $\beta : \Delta_\nu \rightarrow [0, \nu - \tau]$, $g \in L^\infty([\tau, \nu], X)$, $f \in L^1([\tau, \nu], X)$ be given and $u : [\tau - \sigma, \nu] \rightarrow X$ be defined by (3.2). If $\|f(t)\| \leq \chi(t)$ and $\|g(t)\| \leq 1$ a.e. on $[\tau, \nu]$, then we have

$$\begin{aligned}
(3.4) \quad \|u_\nu - u_\tau\|_\sigma &\leq \omega(\varphi, [-\sigma, 0], \nu - \tau) + M \sup_{0 \leq h \leq \nu - \tau} \|S(h)\varphi(0) - \varphi(0)\| \\
&\quad + 2M \int_\tau^\nu \chi(s)ds + 2M(\nu - \tau).
\end{aligned}$$

Indeed, for every $t, s \in [\tau, \nu]$ we have

$$\begin{aligned}
\|u_t - u_s\|_\sigma &= \sup_{\alpha \in [-\sigma, 0]} \|u_t(\alpha) - u_s(\alpha)\| \\
&= \sup_{\alpha \in [-\sigma, 0]} \|u(t + \alpha) - u(s + \alpha)\| \leq \omega(u, [\tau - \sigma, \nu], |t - s|) \\
&\leq \omega(u, [\tau - \sigma, \tau], |t - s|) + \omega(u, [\tau, \nu], |t - s|).
\end{aligned}$$

Since $u_\tau = \varphi$ we get $\omega(u, [\tau - \sigma, \tau], |t - s|) = \omega(\varphi, [-\sigma, 0], |t - s|)$ and so

$$\|u_t - u_s\|_\sigma \leq \omega(\varphi, [-\sigma, 0], |t - s|) + \omega(u, [\tau, \nu], |t - s|).$$

From the definition of u on $[\tau, \nu]$ we obtain

$$\omega(u, [\tau, \nu], \delta) \leq \omega(Qf, [\tau, \nu], \delta) + 2M(\nu - \tau)\|g\|_\infty$$

and therefore we have

$$(3.5) \quad \|u_t - u_s\|_\sigma \leq \omega(\varphi, [-\sigma, 0], |t - s|) + \omega(Qf, [\tau, \nu], |t - s|) + 2M(\nu - \tau)\|g\|_\infty,$$

for every $t, s \in [\tau, \nu]$. Consequently, using the estimate

$$\omega(Qf, [\tau, \nu], \delta) \leq M \sup_{0 \leq h \leq \delta} \|S(h)\varphi(0) - \varphi(0)\| + 2M \int_\tau^\nu \chi(s)ds,$$

we get (3.4).

Remark 3.5. Let us consider $\bar{f} \in L^1([\tau, \bar{\nu}], X)$, $\nu \in (\tau, \bar{\nu})$ and $f = \bar{f}|_{[\tau, \nu]}$. Since $(Qf)(t)$ depends only on the values of f on the interval $[\tau, \nu]$, we deduce that $Qf = (Q\bar{f})|_{[\tau, \nu]}$ and therefore

$$\omega(Qf, [\tau, \nu], \delta) = \omega(Q\bar{f}, [\tau, \nu], \delta) \leq \omega(Q\bar{f}, [\tau, \bar{\nu}], \delta), \text{ for every } \delta \geq 0.$$

In the next lemma we show how to choose $T \in (\tau, \tau + \rho]$ and how to construct, for every $\varepsilon \in (0, 1)$ and every $\psi \in L^\infty([\tau, \tau + \rho], X)$, an (ε, ψ) -approximate solution on $[\tau - \sigma, T]$.

Lemma 3.1. *Assume that the hypotheses (H_0) – (H_4) are satisfied. Then there exists $T \in (\tau, \tau + \rho]$ with $\int_\tau^T \mu(s)ds \leq 1/2$ such that for every $\varepsilon \in (0, 1)$ and every $\psi \in L^\infty([\tau, \tau + \rho], X)$ the problem (1.1) and (1.2) have at least one (ε, ψ) -approximate solution on $[\tau - \sigma, T]$.*

Proof. We fix $T \in (\tau, \tau + \rho]$ such that

$$(3.6) \quad \begin{aligned} & \omega(\varphi, [-\sigma, 0], T - \tau) + M \sup_{0 \leq h \leq T - \tau} \|S(h)\varphi(0) - \varphi(0)\| \\ & + 2M \int_\tau^T \chi(s)ds + 2M(T - \tau) \leq r \end{aligned}$$

and

$$(3.7) \quad \int_\tau^T \mu(s)ds \leq 1/2.$$

We denote by \mathcal{M}_T the set of all (ε, ψ) -approximate solutions (θ, β, g, f, u) on $[\tau - \sigma, \nu] \subset [\tau - \sigma, T]$ and we begin by proving that \mathcal{M}_T is a nonempty set. Applying Theorem 2.2 to $G(\cdot) = F(\cdot, \varphi)$ on $[\tau, T]$ we obtain that there exists a measurable selection $\bar{f} : [\tau, T] \rightarrow X$ such that $\bar{f}(t) \in F(t, \varphi)$ a.e. on $[\tau, T]$ and

$$\|\bar{f}(t) - \psi(t)\| \leq d(\psi(t), F(t, \varphi)) + \varepsilon \mu(t) \text{ a.e. on } [\tau, T].$$

Moreover, from (H_1) we obtain that $\|\bar{f}(t)\| \leq \chi(t)$ a.e. on $[\tau, T]$ and therefore $\bar{f} \in L^1([\tau, T], X)$. Using the tangential condition (H_4) for $(\tau, \varphi) \in \mathcal{I} \times \mathcal{K}_0$ we obtain that there exist $(h_n)_n$ in \mathbb{R}_+ with $h_n \downarrow 0$ and $(q_n)_n$ in X with $q_n \rightarrow 0$ such that

$$(3.8) \quad S(h_n)\varphi(0) + \int_\tau^{\tau+h_n} S(\tau + h_n - s)\bar{f}(s)ds + h_n q_n \in K,$$

for every $n \in \mathbb{N}$. We can fix $n_0 \in \mathbb{N}$ such that $h_{n_0} \in (0, T - \tau]$ and $\|q_{n_0}\| \leq \varepsilon$. This choice is possible because

$$\lim_{\delta \downarrow 0} \omega(\varphi, [-\sigma, 0], \delta) = 0 \text{ and } \lim_{\delta \downarrow 0} \omega(Q\bar{f}, [\tau, T], \delta) = 0.$$

For n_0 fixed as above, we define: $\nu_0 := \tau + h_{n_0}$, $\theta(t) := \tau$ for every $t \in [\tau, \nu_0]$, $\beta(t, s) = 0$ for every $(t, s) \in \Delta_{\nu_0}$, $g(t) := q_{n_0}$ and $f(t) := \bar{f}(t)$ a.e. on $[\tau, \nu_0]$ and we show that (θ, β, g, f, u) , with u defined by (3.2), is an (ε, ψ) -approximate solution on $[\tau - \sigma, \nu_0] \subset [\tau - \sigma, T]$.

It is easy to see that the conditions (A_1) – (A_6) are fulfilled.

Let us verify the conditions (A_7) and (A_8) . Using (3.5), Remark 3.5 and our choice for h_{n_0} , we obtain that

$$\begin{aligned} \|u_t - u_{\theta(t)}\|_\sigma &= \|u_t - u_\tau\|_\sigma \leq \omega(\varphi, [-\sigma, 0], t - \tau) \\ &+ \omega(Qf, [\tau, \nu_0], t - \tau) + 2M(\nu_0 - \tau)\|g\|_\sigma \leq \omega(\varphi, [-\sigma, 0], h_{n_0}) \\ &+ \omega(Q\bar{f}, [\tau, T], h_{n_0}) + 2M(T - \tau)\|g\|_\sigma \leq \varepsilon \end{aligned}$$

for every $t \in [\tau, \nu_0]$ and so (A_7) is fulfilled. Furthermore, from (A_1) , (A_4) and (A_5) we get $\|f(t)\| \leq \chi(t)$ and $\|g(t)\| \leq \varepsilon \leq 1$ a.e. on $[\tau, \nu_0]$ and therefore, using (3.4) and (3.6), we have

$$\|u_{\nu_0} - \varphi\|_\sigma = \|u_{\nu_0} - u_\tau\|_\sigma \leq r,$$

hence $u_{\nu_0} \in B_\sigma(\varphi, r)$. Since by (3.3) and (3.8) we have

$$u_{\nu_0}(0) = u(\nu_0) = S(h_{n_0})\varphi(0) + \int_\tau^{\tau+h_{n_0}} S(\tau + h_{n_0} - s)\bar{f}(s)ds + h_{n_0}q_{n_0} \in K,$$

it follows that $u_{\nu_0} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$. Hence (A_8) is also satisfied and consequently $(\theta, \beta, g, f, u) \in \mathcal{M}_T$.

Next, we shall prove that there exists at least one (ε, ψ) -approximate solution of (1.1) and (1.2), defined on the whole interval $[\tau - \sigma, T]$. To this end, we shall apply Theorem 2.3 to the set \mathcal{M}_T endowed with the following preorder:

If $(\theta^1, \beta^1, g^1, f^1, u^1)$ and $(\theta^2, \beta^2, g^2, f^2, u^2)$ are two (ε, ψ) -approximate solution on $[\tau - \sigma, \nu^1]$ and respectively on $[\tau - \sigma, \nu^2]$, then we say that

$$(\theta^1, \beta^1, g^1, f^1, u^1) \preceq (\theta^2, \beta^2, g^2, f^2, u^2)$$

if only if $\nu^1 \leq \nu^2$, $\theta^1 = \theta^2|_{[\tau, \nu^1]}$, $\beta^1 = \beta^2|_{[\tau, \nu^1]}$, $g^1 = g^2|_{[\tau, \nu^1]}$, and $f^1 = f^2|_{[\tau, \nu^1]}$.

It is obvious that \preceq is a preorder on \mathcal{M}_T . Moreover, let us notice that $(\theta^1, \beta^1, g^1, f^1, u^1) \preceq (\theta^2, \beta^2, g^2, f^2, u^2)$ implies, by (3.2), that $u^2|_{[\tau-\sigma, \nu_1]} = u^1$.

We define the function $\mathcal{S} : \mathcal{M}_T \rightarrow \mathbb{R}$ by

$$\mathcal{S}((\theta, \beta, g, f, u)) = \nu,$$

if (θ, β, g, f, u) is an (ε, ψ) -approximate solution defined on $[\tau - \sigma, \nu]$. It is clear that \mathcal{S} is increasing on \mathcal{M}_T . Further on, we shall show that each increasing sequence $((\theta^i, \beta^i, g^i, f^i, u^i))_{i \in \mathbb{N}}$ in \mathcal{M}_T is majorated in \mathcal{M}_T . We construct a majorant as follows. We define

$$\nu^* = \lim_i \nu^i$$

and we have $\nu^* \in (\tau, T]$. For each $i \in \mathbb{N}$, we define $\theta^*(t) = \theta^i(t)$ if $t \in [\tau, \nu^i]$ and $\theta^*(\nu^*) = \nu^*$, $\beta^*(t, s) = \beta^i(t, s)$ for every $(t, s) \in \Delta_{\nu^i}$, $g^*(t) = g^i(t)$ and $f^*(t) = f^i(t)$ if $t \in [\tau, \nu^i]$. Since $((\theta^i, \beta^i, g^i, f^i, u^i))_{i \in \mathbb{N}}$ is an increasing sequence in \mathcal{M}_T , the functions θ^*, β^*, g^* , and f^* are well defined. Moreover, for every $i \in \mathbb{N}$ we have that $\|f^i(t)\| \leq \chi(t)$ and $\|g^i(t)\| \leq \varepsilon$ a.e. on $[\tau, \nu^i]$, which yields

$$(3.9) \quad \|f^*(t)\| \leq \chi(t) \text{ and } \|g^*(t)\| \leq \varepsilon \text{ a.e. on } [\tau, \nu^i]$$

and therefore $g^* \in L^\infty([\tau, \nu^*], X)$ and $f^* \in L^1([\tau, \nu^*], X)$. It is obvious that $\theta^* : [\tau, \nu^*] \rightarrow [\tau, \nu^*]$ and thus we can consider the 5-tuple $(\theta^*, \beta^*, g^*, f^*, u^*)$ with the function u^* defined by (3.2). Since $t \rightarrow \beta^*(t, s)$ is nonexpansive on (s, ν^*) , in view of Remark 3.3 we infer that u^* is continuous on $[t - \tau, \nu^*]$.

Now, we show that $(\theta^*, \beta^*, g^*, f^*, u^*) \in \mathcal{M}_T$. To this end, we fix an arbitrary $i \in \mathbb{N}$ and we observe that for every $t \in [\tau - \sigma, \tau]$ we have $u^*(t) = \varphi(t - \tau) = u^i(t)$ and for every $t \in [\tau, \nu^i]$ we have

$$\begin{aligned} u^*(t) &= S(t - \tau)\varphi(0) + \int_\tau^t S(t - s)f^*(s)ds + \int_\tau^t S(\beta^*(t, s))g^*(s)ds \\ &= S(t - \tau)\varphi(0) + \int_\tau^t S(t - s)f^i(s)ds + \int_\tau^t S(\beta^i(t, s))g^i(s)ds = u^i(t). \end{aligned}$$

Consequently, $u^*(t) = u^i(t)$ for every $t \in [\tau - \sigma, \nu^i]$. Moreover, for every $t \in [\tau, \nu^i]$ and every $s \in [-\sigma, 0]$ we have

$$\tau - \sigma \leq \theta^*(t) + s = \theta^i(t) + s \leq t + s \leq t \leq \nu^i,$$

and

$$u_{\theta^*(t)}^*(s) = u^*(\theta^*(t) + s) = u^*(\theta^i(t) + s) = u^i(\theta^i(t) + s) = u_{\theta^i(t)}^i(s).$$

Therefore

$$(3.10) \quad u_{\theta^*(t)}^* = u_{\theta^i(t)}^i,$$

for every $t \in [\tau, \nu^i]$. Taking into account the above relations, it readily follows that $(\theta^*, \beta^*, g^*, f^*, u^*)$ satisfies $(A_2)-(A_7)$.

Let us verify the conditions (A_1) and (A_8) . For any $t \in [\tau, \nu^*]$ there exists $i \in \mathbb{N}$ such that $t \in [\tau, \nu^i]$ so, by (3.10) we get $u_{\theta^*(t)}^* = u_{\theta^i(t)}^i \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$. For $t = \nu^*$ we have $\theta^*(\nu^*) = \nu^*$ and $u_{\theta^*(\nu^*)}^* = u_{\nu^*}^*$. Then, by (3.9), we can use the relation (3.4) which, together with (3.6), yields $\|u_{\nu^*}^* - \varphi\|_\sigma \leq r$ and thus $u_{\theta^*(\nu^*)}^* = u_{\nu^*}^* \in B_\sigma(\varphi, r)$. By the continuity of u^* we have

$$u_{\nu^*}^*(0) = u^*(\nu^*) = \lim_i u^*(\nu^i) = \lim_i u^i(\nu^i).$$

Since $u_{\nu^i}^i \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ we have that $u^i(\nu^i) = u_{\nu^i}^i(0) \in K \cap B(\varphi(0), r)$, for every $i \in \mathbb{N}$. The set $K \cap B(\varphi(0), r)$ is closed, hence $u_{\nu^*}^*(0) \in K \cap B(\varphi(0), r)$ and therefore $u_{\theta^*(\nu^*)}^* = u_{\nu^*}^* \in \mathcal{K}_0$. It follows that $(\theta^*, \beta^*, g^*, f^*, u^*) \in \mathcal{M}_T$. In addition, $(\theta^i, \beta^i, g^i, f^i, u^i) \preceq (\theta^*, \beta^*, g^*, f^*, u^*)$ for each $i \in \mathbb{N}$ and thus the sequence $((\theta^i, \beta^i, g^i, f^i, u^i))_{i \in \mathbb{N}}$ is majorated in \mathcal{M}_T . Therefore, the set \mathcal{M}_T , endowed with the preorder \preceq and the function \mathcal{S} , satisfies the hypotheses of Theorem 2.3.

Before using the conclusion of Theorem 2.3, we shall show that any element $(\theta, \beta, g, f, u) \in \mathcal{M}_T$ with $\mathcal{S}((\theta, \beta, g, f, u)) < T$ is majorated in \mathcal{M}_T by an element $(\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u}) \in \mathcal{M}_T$ with $\mathcal{S}((\theta, \beta, g, f, u)) < \mathcal{S}((\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u}))$.

To this aim let us consider that (θ, β, g, f, u) is an (ε, ψ) -approximate solution defined $[\tau - \sigma, \nu]$ with $\nu \in (\tau, T)$. Since $u_\nu \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ we can apply Theorem 2.2 on $[\nu, T]$ for $G(\cdot) = F(\cdot, u_\nu)$. It follows that there exists a measurable function $\bar{f} : [\nu, T] \rightarrow X$ such that $\bar{f}(t) \in F(t, u_\nu)$ a.e. on $[\nu, T]$ and

$$\|\bar{f}(t) - \psi(t)\| \leq d(\psi(t), F(t, u_\nu)) + \varepsilon\mu(t) \text{ a.e. on } [\nu, T].$$

By (H_1) we have $\|\bar{f}(t)\| \leq \chi(t)$ a.e. on $[\nu, T]$ and hence $\bar{f} \in L^1([\nu, T], X)$. Since $(\nu, u_\nu) \in \mathcal{J} \times \mathcal{K}_0$ we can apply the tangency condition (H_4) at (ν, u_ν) .

Therefore, there exist $(h_n)_n$ in \mathbb{R}_+ with $h_n \downarrow 0$ and $(q_n)_n$ in X with $q_n \rightarrow 0$ such that

$$(3.11) \quad S(h_{n_0})u_\nu(0) + \int_\nu^{\nu+h_n} S(\nu+h_n-s)\bar{f}(s)ds + h_n q_n \in K,$$

for every $n \in \mathbb{N}$.

We define

$$\hat{f}(t) := \begin{cases} f(t) & \text{if } t \in [\tau, \nu], \\ \bar{f}(t) & \text{if } t \in (\nu, T]. \end{cases}$$

Since

$$\lim_{\delta \downarrow 0} \omega(\varphi, [-\sigma, 0], \delta) = 0 \text{ and } \lim_{\delta \downarrow 0} \omega(Q\hat{f}, [\tau, T], \delta) = 0$$

we can fix $\tilde{n} \in \mathbb{N}$ such that $h_{\tilde{n}} \in (0, T - \nu]$, $\|q_{\tilde{n}}\| \leq \varepsilon$ and

$$\omega(\varphi, [-\sigma, 0], h_{\tilde{n}}) + \omega(Q\hat{f}, [\tau, T], h_{\tilde{n}}) + 2M(T - \tau)\|q_{\tilde{n}}\| \leq \varepsilon.$$

Further on, we define $\tilde{\nu} := \nu + h_{\tilde{n}}$, $\tilde{f}(t) := \hat{f}(t)$ for $t \in [\tau, \tilde{\nu}]$ and

$$\tilde{\theta}(t) := \begin{cases} \theta(t) & \text{if } t \in [\tau, \nu], \\ \nu & \text{if } t \in (\nu, \tilde{\nu}]; \end{cases}$$

$$\tilde{g}(t) := \begin{cases} g(t) & \text{if } t \in [\tau, \nu] \\ q_{n_0} & \text{if } t \in (\nu, \tilde{\nu}] \end{cases}$$

$$\tilde{\beta}(t, s) := \begin{cases} \beta(t, s) & \text{if } \tau \leq s < t \leq \nu, \\ \tau - \nu + \beta(\nu, s) & \text{if } \tau \leq s < \nu < t \leq \tilde{\nu}, \\ 0 & \text{if } \nu \leq s < t \leq \tilde{\nu}. \end{cases}$$

We show that $(\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u})$, with \tilde{u} given by (3.2), is an (ε, ψ) -approximate solution defined on $[\tau - \sigma, \tilde{\nu}] \subset [\tau - \sigma, T]$. First, we notice that $\tilde{\beta}$ satisfies (A_3) , $\tilde{g} \in L^\infty([\tau, \tilde{\nu}], X)$ and $\tilde{f} \in L^1([\tau, \tilde{\nu}], X)$ so, in view of Remark 3.3, it follows that $\tilde{u} \in C([\tau - \sigma, \tilde{\nu}], X)$. Moreover,

$$\tilde{u}(t) = u(t) \text{ for every } t \in [\tau - \sigma, \nu]$$

and

$$\begin{aligned}
\tilde{u}(t) &= S(t - \tau)\varphi(0) + \int_{\tau}^t S(t - s)\tilde{f}(s)ds + \int_{\tau}^t S(\tilde{\beta}(t, s))\tilde{g}(s)ds \\
&= S(t - \nu)\tilde{u}(\nu) + \int_{\nu}^t S(t - s)\tilde{f}(s)ds + \int_{\nu}^t S(\tilde{\beta}(t, s))\tilde{g}(s)ds \\
&= S(t - \nu)u_{\nu}(0) + \int_{\nu}^t S(t - s)\hat{f}(s)ds + (t - \nu)q_{\tilde{n}},
\end{aligned}$$

for every $t \in [\nu, \tilde{\nu}]$. Also, it is obvious that $(\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u})$ satisfies (A_2) – (A_6) .

Since for every $t \in [\tau, \nu]$ we have $\tilde{\theta}(t) = \theta(t)$ and

$$\tilde{u}_{\tilde{\theta}(t)} = \tilde{u}_{\theta(t)} = u_{\theta(t)} \in \mathcal{K}_0 \cap B_{\sigma}(\varphi, r)$$

and for every $t \in (\nu, \tilde{\nu}]$ we have

$$\tilde{u}_{\tilde{\theta}(t)} = \tilde{u}_{\nu} = u_{\nu} \in \mathcal{K}_0 \cap B_{\sigma}(\varphi, r),$$

we deduce that (A_1) is fulfilled.

Let us verify the conditions (A_7) and (A_8) . For every $t \in [\tau, \nu]$ we have

$$\|\tilde{u}_t - u_{\tilde{\theta}(t)}\|_{\sigma} = \|\tilde{u}_t - \tilde{u}_{\theta(t)}\|_{\sigma} = \|u_t - u_{\theta(t)}\|_{\sigma} \leq \varepsilon$$

and for $t \in (\nu, \tilde{\nu}]$, using (3.5), Remark 3.5 and our choice of $h_{\tilde{n}}$, we obtain

$$\begin{aligned}
\|\tilde{u}_t - \tilde{u}_{\tilde{\theta}(t)}\|_{\sigma} &= \|\tilde{u}_t - \tilde{u}_{\nu}\|_{\sigma} \leq \omega(\varphi, [-\sigma, 0], t - \nu) \\
&+ \omega(Q\tilde{f}, [\tau, \tilde{\nu}], t - \nu) + 2M(\tilde{\nu} - \tau)\|g\|_{\infty} \leq \omega(\varphi, [-\sigma, 0], h_{\tilde{n}}) \\
&+ \omega(Q\hat{f}, [\tau, T], h_{\tilde{n}}) + 2M(T - \tau)\|q_{\tilde{n}}\| \leq \varepsilon
\end{aligned}$$

and so (A_7) is fulfilled.

By (A_1) , (A_4) and (A_5) we have that $\|f(t)\| \leq \chi(t)$ and $\|g(t)\| \leq \varepsilon \leq 1$ a.e. on $[\tau, \tilde{\nu}]$ and therefore we can use (3.4) which, together with (3.6), yields

$$\|\tilde{u}_{\tilde{\nu}} - \varphi\|_{\sigma} = \|\tilde{u}_{\tilde{\nu}} - \tilde{u}_{\tau}\|_{\sigma} \leq r$$

and thus $\tilde{u}_{\tilde{\nu}} \in B_{\sigma}(\varphi, r)$. Since, by (3.2) and (3.11), we have

$$\tilde{u}_{\tilde{\nu}}(0) = S(h_{\tilde{n}})u_{\nu}(0) + \int_{\nu}^{\nu+h_{\tilde{n}}} S(\nu + h_{\tilde{n}} - s)\widehat{f}(s)ds + h_{\tilde{n}}q_{\tilde{n}} \in K,$$

it follows that $\tilde{u}_{\tilde{\nu}} \in \mathcal{K}_0 \cap B_{\sigma}(\varphi, r)$. Therefore, $(\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u})$ is an (ε, ψ) -approximate solution defined on $[\tau - \sigma, \tilde{\nu}]$. Moreover, by construction, we have $(\theta, \beta, g, f, u) \preceq (\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u})$ and $\mathcal{S}((\theta, \beta, g, f, u)) = \nu < \tilde{\nu} = \mathcal{S}((\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u}))$.

Now, from Theorem 2.3 we infer that there exists $(\theta, \beta, g, f, u) \in \mathcal{M}_T$ such that $\mathcal{S}((\theta, \beta, g, f, u)) = \mathcal{S}((\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u}))$, for each $(\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u}) \in \mathcal{M}_T$ with $(\theta, \beta, g, f, u) \preceq (\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u})$. If $\mathcal{S}((\theta, \beta, g, f, u)) < T$ then, by the last step, there exists $(\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u}) \in \mathcal{M}_T$ with $(\theta, \beta, g, f, u) \preceq (\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u})$ and such that $\mathcal{S}((\theta, \beta, g, f, u)) < \mathcal{S}((\tilde{\theta}, \tilde{\beta}, \tilde{g}, \tilde{f}, \tilde{u}))$. We conclude that $\mathcal{S}((\theta, \beta, g, f, u)) = T$ and this completes the proof of Lemma 3.1. ■

We are now prepared to complete the proof of Theorem 2.1.

Proof. Let $T \in (\tau, \tau + \rho]$ be given by Lemma 3.1 and let $(\varepsilon_n)_n$ be a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$ and $\varepsilon_n \in (0, 1)$ for every $n \in \mathbb{N}$.

Starting with one measurable selection $f_0(\cdot) \in F(\cdot, \varphi)$, in view of Lemma 3.1 we can define inductively the sequence $((\theta^n, \beta^n, g^n, f^n, u^n))_{n \in \mathbb{N}}$ such that $(\theta^n, \beta^n, g^n, f^n, u^n)$ is an (ε_n, f^n) -approximate solution on $[\tau - \sigma, T]$ for every $n \in \mathbb{N}$.

Thus, for every $n \in \mathbb{N}$ we have

$$(3.12) \quad u^n(t) = (Qf^n)(t) + \begin{cases} 0, & \text{for } t \in [\tau - \sigma, \tau], \\ \int_{\tau}^t S(\beta^n(t, s))g^n(s)ds, & \text{for } t \in (\tau, T] \end{cases}$$

and

$$(B_1) \quad u_{\theta^n(t)}^n \in \mathcal{K}_0 \cap B_{\sigma}(\varphi, r) \text{ for every } t \in [\tau, T];$$

$$(B_2) \quad 0 \leq t - \theta^n(t) \text{ for every } t \in [\tau, T];$$

$$(B_3) \quad 0 \leq \beta^n(t, s) \leq t - \tau \text{ for } \tau \leq s < t \leq T;$$

$$(B_4) \quad \|g^n(t)\| \leq \varepsilon_n \text{ a.e. on } [\tau, T];$$

$$(B_5) \quad f^n(t) \in F(t, u_{\theta^n(t)}^n) \text{ a.e. on } [\tau, T];$$

$$(B_6) \quad \|f^n(t) - f^{n-1}(t)\| \leq d(f^{n-1}(t), F(t, u_{\theta^n(t)}^n)) + \varepsilon_n \mu(t) \text{ a.e. on } [\tau, T];$$

$$(B_7) \quad \|u_t^n - u_{\theta^n(t)}^n\|_\sigma \leq \varepsilon_n \text{ for every } t \in [\tau, T];$$

$$(B_8) \quad u_T^n \in \mathcal{K}_0 \cap B_\sigma(\varphi, r).$$

We shall prove that $(u^n)_n$ converges uniformly to a function $u : [\tau - \sigma, T] \rightarrow X$ that is a solution of (1.1) and (1.2).

For this, we first show that for every $n \in \mathbb{N}$ we have

$$(C_1) \quad \|f^n(t)\| \leq \chi(t) \text{ a.e. on } [\tau, T];$$

$$(C_2) \quad \|u^n(t) - (Qf^n)(t)\| \leq M(T - \tau)\varepsilon_n \text{ for every } t \in [\tau, T];$$

$$(C_3) \quad \|u_{\theta^{n+1}(t)}^{n+1} - u_{\theta^n(t)}^n\|_\sigma \leq 2\varepsilon_n + \|u^{n+1} - u^n\|_T, \text{ for every } t \in [\tau, T],$$

where $\|\cdot\|_T$ is the norm in $C([\tau - \sigma, T]; X)$;

$$(C_4) \quad \|f^{n+1}(t) - f^n(t)\| \leq \mu(t)(\|u^{n+1} - u^n\|_T + 3\varepsilon_n) \text{ a.e. on } [\tau, T].$$

Indeed, (C_1) follows from (H_1) and (B_5) , (C_2) follows from (3.3), (3.12) and (B_4) . In order to prove (C_3) we observe that

$$\begin{aligned} \|u_t^{n+1} - u_t^n\|_\sigma &= \sup_{-\sigma \leq s \leq 0} \|u^{n+1}(t+s) - u^n(t+s)\| \\ &\leq \sup_{\tau-\sigma \leq \nu \leq T} \|u^{n+1}(\nu) - u^n(\nu)\| = \|u^{n+1} - u^n\|_T \end{aligned}$$

and thus by (B_7) we obtain that

$$\begin{aligned} \|u_{\theta^{n+1}(t)}^{n+1} - u_{\theta^n(t)}^n\|_\sigma &\leq \|u_{\theta^{n+1}(t)}^{n+1} - u_t^{n+1}\|_\sigma + \|u_t^{n+1} - u_t^n\|_\sigma + \|u_t^n - u_{\theta^n(t)}^n\|_\sigma \\ &\leq \varepsilon_{n+1} + \|u_t^{n+1} - u_t^n\|_\sigma + \varepsilon_n \leq 2\varepsilon_n + \|u^{n+1} - u^n\|_T \end{aligned}$$

for every $t \in [\tau, T]$. Finally, by (H_2) , (B_5) and (C_3) we have

$$\begin{aligned}
\|f^{n+1}(t) - f^n(t)\| &\leq d\left(f^n(t), F\left(t, u_{\theta^{n+1}(t)}^{n+1}\right)\right) + \varepsilon_{n+1}\mu(t) \\
&\leq d_{HP}\left(F\left(t, u_{\theta^n(t)}^n\right), F\left(t, u_{\theta^{n+1}(t)}^{n+1}\right)\right) + \varepsilon_{n+1}\mu(t) \\
&\leq \mu(t)\left(\|u_{\theta^n(t)}^n - u_{\theta^{n+1}(t)}^{n+1}\|_\sigma + \varepsilon_{n+1}\right) \\
&\leq \mu(t)\left(\|u^{n+1} - u^n\|_T + 3\varepsilon_n\right)
\end{aligned}$$

a.e. on $[\tau, T]$ and hence (C_4) is also checked. Further on, for every $t \in [\tau, T]$, by (3.6), (3.11), (C_3) and (C_5) we have

$$\begin{aligned}
\|u^{n+1}(t) - u^n(t)\| &\leq \|u^{n+1}(t) - (Qf^{n+1})(t)\| + \|(Qf^{n+1})(t) - (Qf^n)(t)\| \\
&+ \|(Qf^n)(t) - u^n(t)\| \leq M(T - \tau)(\varepsilon_{n+1} + \varepsilon_n) + \int_\tau^T \|f^{n+1}(s) - f^n(s)\| ds \\
&\leq 2M(T - \tau)\varepsilon_n + M(3\varepsilon_n + \|u^{n+1} - u^n\|_T) \int_\tau^T \mu(s) ds \\
&\leq M\left(2(T - \tau) + \frac{3}{2}\right)\varepsilon_n + \frac{1}{2}\|u^{n+1} - u^n\|_T.
\end{aligned}$$

Therefore, since $\|u^{n+1}(t) - u^n(t)\| = 0$ for every $t \in [\tau - \sigma, \tau]$, we obtain

$$\|u^{n+1}(t) - u^n(t)\| \leq M\left(2(T - \tau) + \frac{3}{2}\right)\varepsilon_n + \frac{1}{2}\|u^{n+1} - u^n\|_T,$$

for every $t \in [\tau - \sigma, T]$, and thus

$$\|u^{n+1} - u^n\|_T \leq M\left(2(T - \tau) + \frac{3}{2}\right)\varepsilon_n + \frac{1}{2}\|u^{n+1} - u^n\|_T,$$

for every $n \in \mathbb{N}$. It follows that

$$(3.13) \quad \|u^{n+1} - u^n\|_T \leq M(4(T - \tau) + 3)\varepsilon_n$$

for every $n \in \mathbb{N}^*$ with $\sum_{n=1}^\infty \varepsilon_n < +\infty$ and so we deduce that $(u^n)_n$ converge uniformly to a function $u : [\tau - \sigma, T] \rightarrow X$.

From (C_4) and (3.13) we deduce that, for almost all $t \in [\tau, T]$, we have

$$\begin{aligned} \|f^{n+1}(t) - f^n(t)\| &\leq \mu(t)(\|u^{n+1} - u^n\|_T + 3\varepsilon_n) \\ &\leq \mu(t)(4M(T - \tau) + 3M + 3)\varepsilon_n \end{aligned}$$

for every $n \in \mathbb{N}^*$. This implies that $(f^n)_n$ converge pointwise almost everywhere to a measurable function f . For any fixed $t \in [\tau - \sigma, T]$, by (C_1) and Lebesgue's Theorem, we obtain that $\lim_{n \rightarrow \infty} (Qf^n)(t) = (Qf)(t)$. Consequently, by (C_2) , we conclude that $u(t) = (Qf)(t)$ for every $t \in [\tau - \sigma, T]$.

For every $t \in [\tau, T]$ and every $n \in \mathbb{N}^*$, by (B_7) and (C_3) , we have

$$\|u_{\theta^n(t)}^n - u_t\|_\sigma \leq \|u_{\theta^n(t)}^n - u_t^n\|_\sigma + \|u_t^n - u_t\|_\sigma \leq \varepsilon_n + \|u_t^n - u_t\|_T$$

and thus $u_{\theta^n(t)}^n \rightarrow u_t$ in $\|\cdot\|_\sigma$ as $n \rightarrow \infty$. In view of (B_1) and Remark 3.1 it follows that $u_t \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ and hence $u(t) \in K \cap B(\varphi(0), r)$, for every $t \in [\tau, T]$.

Now, let us observe that, a.e. on $[\tau, T]$, we have

$$\begin{aligned} d(f(t), F(t, u_t)) &\leq \|f(t) - f^n(t)\| + d\left(F\left(t, u_{\theta^n(t)}^n\right), F(t, u_t)\right) \\ &\leq \|f(t) - f^n(t)\| + \mu(t)\|u_{\theta^n(t)}^n - u_t\|_\sigma, \end{aligned}$$

for every $n \in \mathbb{N}^*$. It follows, by letting $n \rightarrow \infty$, that $d(f(t), F(t, u_t)) = 0$ and thus, since F has closed values, $f(t) \in F(t, u_t)$ a.e on $[\tau, T]$.

We have proved that $u : [\tau - \sigma, T] \rightarrow X$ is a solution of (1.1) and (1.2), with $u(t) \in K$ for every $t \in [\tau, T]$, and so, (τ, φ) being arbitrarily fixed in $\mathcal{J} \times \mathcal{K}_0$, we have shown that K is a viable domain for (1.1). ■

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