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LEVEL SETS OF CONTINUOUS FUNCTIONS INCREASING WITH RESPECT TO EACH VARIABLE

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Abstract

We are going to prove that level sets of continuous functions increasing with respect to each variable are arcwise connected (Theorem 3) and characterize those of them which are arcs (Theorem 2). In [3], we will apply the second result to the classical linear functional equation

$$\varphi \circ f = g\varphi + h$$

(cf., for instance, [1] and [2]) in a case not studied yet, where f is given as a pair of means, that is so-called mean-type mapping.

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If $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ we will write

$$(x_1, y_1) \leq (x_2, y_2) :\Longleftrightarrow x_1 \geq x_2 \quad \text{and} \quad y_1 \leq y_2$$

and

$$(x_1, y_1) < (x_2, y_2) :\Longleftrightarrow (x_1, y_1) \leq (x_2, y_2) \quad \text{and} \quad (x_1, y_1) \neq (x_2, y_2).$$

Of course, $\leq$ is a partial ordering in $\mathbb{R}^2$.

Let I, J be real intervals and let $f : I \times J \longrightarrow \mathbb{R}$ be a function. For every $a \in \mathbb{R}$ define the *level set* $\Gamma_f(a)$ by

$$\Gamma_f(a) = \{(x, y) \in I \times J :\ f(x, y) = a\}.$$

A set $A \subset \Gamma_f(a)$ is called a $\leq$-*interval* if the following conditions hold:

(i) if $(x_1, y_1), (x_2, y_2) \in A$ are $\leq$-comparable, then A has a common point with every straight line of the form $y = x + t$ with t lying between $y_1 - x_1$ and $y_2 - x_2$;

(ii) if $(x_1, y_1), (x_2, y_2) \in A$ are not $\leq$-comparable then A contains a rectangle with the vertices $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$.

Figure 1 shows sets which are $\leq$-intervals, whereas Figures 2 and 3 present sets which are not.

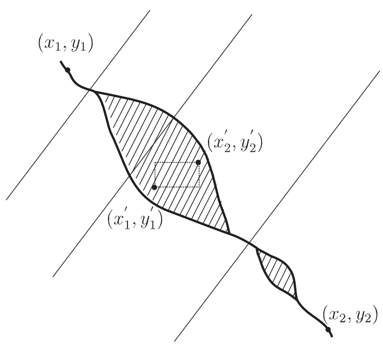

Figure 1

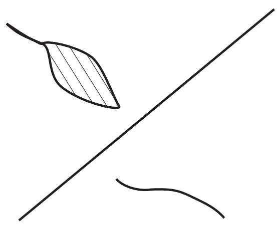

Figure 2

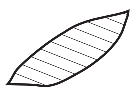

Figure 3

Theorem 1. *Level sets of a continuous function increasing with respect to each variable are $\le$-intervals.*

Proof. Let $f : I \times J \longrightarrow \mathbb{R}$ be continuous and increasing with respect to each variable and take an $a \in \mathbb{R}$. Fix $(x_1, y_1), (x_2, y_2) \in \Gamma_f(a)$. At first assume that the points are $\le$-comparable, say $(x_1, y_1) \le (x_2, y_2)$ and let $t \in \mathbb{R}$ be such that $y_1 - x_1 < t < y_2 - x_2$. There are only two possibilities: either $y_1 - x_1 < t \le y_1 - x_2$, or $y_1 - x_2 < t < y_2 - x_2$. In the first one $y_1 - t \in I$ and $f(y_1 - t, y_1) \le f(x_1, y_1) = a$ and in the other one $x_2 + t \in J$ and $f(x_2, x_2 + t) \le f(x_2, y_2) = a$. In both we can find an $x' \in I$ such that $x' + t \in J$ and $f(x', x' + t) \le a$. Similarly, depending on whether $y_1 - x_1 < t \le y_2 - x_1$ or $y_2 - x_1 < t < y_2 - x_2$, either $f(x_1, x_1 + t) \ge f(x_1, y_1) = a$, or $f(y_2 - t, y_2) \ge f(x_2, y_2) = a$, that is there exists an $x'' \in I$ such that $x'' + t \in J$ and $f(x'', x'' + t) \ge a$. Using the Darboux property we can find an $x \in I$ with $x + t \in J$ and $f(x, x + t) = a$, i.e., $(x, x + t) \in \Gamma_f(a)$.

Now consider the case of noncomparable $(x_1, y_1), (x_2, y_2)$. Then either $x_1 < x_2$ and $y_1 < y_2$, or $x_2 < x_1$ and $y_2 < y_1$. By the monotonicity assumption on f we deduce that f takes the value a at any point of the rectangle with the vertices $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$. ■

In what follows by an arc we mean any homeomorphic image of a real interval.

Theorem 2. *Assume that $f : I \times J \longrightarrow \mathbb{R}$ is continuous and increasing with respect to each variable. Let A be a $\le$-interval contained in a level set of f. Then the following conditions are pairwise equivalent:*

(i) *the ordering $\le$ is linear on A,*

(ii) *A is an arc,*

(iii) $\mathrm{int} A = \emptyset$.

Proof. (i)$\Longrightarrow$(ii). Assume that the ordering $\le$ is linear on A. Since the function $v : \mathbb{R}^2 \to \mathbb{R}^2$ given by $v(x, y) = (x + y, -x + y)$ is a homeomorphism of the plane it is enough to prove that the set

$$A^* := v^{-1}(A)$$

is an arc. Let $(x, y_1), (x, y_2) \in A^*$. Then $(x + y_1, -x + y_1), (x + y_2, -x + y_2) \in A$. Assume, for instance, that

$$(x + y_1, -x + y_1) \le (x + y_2, -x + y_2).$$

We have

$$x + y_1 \geq x + y_2 \quad \text{and} \quad -x + y_1 \leq -x + y_2,$$

whence $y_1 = y_2$. This means that A^* is a graph of some function. We will show that this function is Lipschitz with the constant 1. To this aim fix $(x_1, y_1), (x_2, y_2) \in A^*$ and assume, for instance, that

$$(x_1 + y_1, -x_1 + y_1) \leq (x_2 + y_2, -x_2 + y_2).$$

Then

$$x_1 + y_1 \geq x_2 + y_2 \quad \text{and} \quad -x_1 + y_1 \leq -x_2 + y_2,$$

whence

$$-(x_1 - x_2) \leq y_1 - y_2 \quad \text{and} \quad y_1 - y_2 \leq x_1 - x_2,$$

that is $\mid y_1 - y_2 \mid \leq \mid x_1 - x_2 \mid$. Thus A^* is the graph of a continuous function. To complete the argument it is enough to show that the domain of this function, i.e. the projection

$$\pi(A^*) = \{x \in \mathbb{R} : \exists_{y \in \mathbb{R}} (x, y) \in A^*\},$$

is an interval. To this aim, fix $x_1, x_2 \in \pi(A^*)$, $x_1 < x_2$ and let $x \in (x_1, x_2)$. There are y_1, y_2 such that $(x_1, y_1), (x_2, y_2) \in A^*$, that is $(x_1 + y_1, -x_1 + y_1)$, $(x_2 + y_2, -x_2 + y_2) \in A$. Since $\leq$ is linear on A they are $\leq$-comparable. Hence, by the fact that A is a $\leq$-interval, for every $t \in \mathbb{R}$ satisfying

$$-x_2 + y_2 - (x_2 + y_2) < t < -x_1 + y_1 - (x_1 + y_1)$$

there exists a $u \in I$ such that $(u, u + t) \in A$. In particular, we can find a $u \in I$ with $(u, u - 2x) \in A$. Put $y := u - x$. Then $(x + y, -x + y) \in A$, that is $(x, y) \in A^*$ and, consequently, $x \in \pi(A^*)$.

The implication (ii)$\Longrightarrow$(iii) is obvious.

(iii)$\Longrightarrow$(i). Suppose that there exist noncomparable points (x_1, y_1), $(x_2, y_2) \in A$. Then, since A is a $\leq$-interval it contains the rectangle with the vertices (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), which means that $\mathrm{int}A \neq \emptyset$. ■

Remark 1. If $f : I \times J \longrightarrow \mathbb{R}$ is one-to-one with respect to at least one variable, then $\mathrm{int}\Gamma_f(a) = \emptyset$ for every $a \in \mathbb{R}$.

Proof. Assume, for instance, that f is one-to-one with respect to the first variable. Fix an $a \in \mathbb{R}$ and suppose that $\mathrm{int}\Gamma_f(a) \neq \emptyset$. Let $(x_0, y_0) \in \mathrm{int}\Gamma_f(a)$ and choose an $\varepsilon \in (0, \infty)$ such that $(x_0-\varepsilon, x_0+\varepsilon) \times \{y_0\} \subset \mathrm{int}\Gamma_f(a)$. Then

$$f(x, y_0) = a \qquad \text{for } x \in (x_0 - \varepsilon, x_0 + \varepsilon),$$

which is impossible. ∎

Corollary 1. *Assume that $f : I \times J \longrightarrow \mathbb{R}$ is continuous and increasing with respect to each variable, strictly increasing with respect to at least one of them. If A is a $\leq$-interval contained in a level set of f, then A is an arc and the ordering $\leq$ is linear on A. In particular, every level set of f is an arc.*

In what follows, a function $f : I \times J \longrightarrow \mathbb{R}$ will be called *homogeneous* if

$$f(tx, ty) = tf(x, y)$$

for every $(x, y) \in I \times J$ and $t \in \mathbb{R}$ with $(tx, ty) \in I \times J$.

Remark 2. If $f : I \times J \longrightarrow \mathbb{R}$ is homogeneous, then $\mathrm{int}\Gamma_f(a) = \emptyset$ for every $a \in \mathbb{R} \setminus \{0\}$.

Proof. Let $a \in \mathbb{R} \setminus \{0\}$. Suppose that $\mathrm{int}\, \Gamma_f(a) \neq \emptyset$. Let $(x_0, y_0) \in \mathrm{int}\Gamma_f(a)$ and choose a $t \neq 1$ such that $(tx_0, ty_0) \in \mathrm{int}\Gamma_f(a)$. Then, since f is homogeneous,

$$a = f(tx_0, ty_0) = tf(x_0, y_0) = ta,$$

whence $a = 0$, which contradicts our assumption. ∎

Corollary 2. *Assume that $f : I \times J \longrightarrow \mathbb{R}$ is continuous, increasing with respect to each variable and homogeneous. If A is a $\leq$-interval contained in a non-zero level set of f, then A is an arc and the ordering $\leq$ is linear on A. In particular, every non-zero level set of f is an arc.*

Proposition 1. *Assume that $f : I \times J \longrightarrow \mathbb{R}$ is continuous and increasing with respect to each variable. If $a \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2) \in \Gamma_f(a)$, then there exists an arc which is a $\leq$-interval, joins $(x_1, y_1), (x_2, y_2)$ and is contained in $\Gamma_f(a)$ as well as in the rectangle with the vertices (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2).*

Proof. Assume, for instance, that $x_1 \le x_2$. We may assume that

$$x_1 < x_2 \quad \text{and} \quad y_1 > y_2$$

since otherwise, due to the monotonicity assumption on f, the rectangle with the vertices $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$ is contained in $\Gamma_f(a)$. For every $x \in [x_1, x_2]$, using the inequality

$$f(x, y_2) \le f(x_2, y_2) = a = f(x_1, y_1) \le f(x, y_1)$$

and the Darboux property, we infer that the set

$$A(x) := \{y \in [y_2, y_1] : \ (x, y) \in \Gamma_f(a)\}$$

is nonvoid. By the assumed property of f it is a closed interval. Define $\beta : [x_1, x_2] \longrightarrow [y_2, y_1]$ by $\beta(x) := \sup A(x)$. To see that it is decreasing fix x', x'' satisfying $x_1 \le x' < x'' \le x_2$ and observe that

$$a = f(x', \beta(x')) < f(x', y)$$

for every $y \in (\beta(x'), y_1]$ so if $\beta(x'') > \beta(x')$, then

$$a < f(x', \beta(x'')) \le f(x'', \beta(x'')) = a,$$

which is impossible. Therefore $\beta(x') \ge \beta(x'')$.

Now we will show that β is left continuous. Let $x_0 \in (x_1, x_2]$ and suppose that β is not left continuous at x_0. Then

$$\lim_{x \to x_0-} \beta(x) > \beta(x_0).$$

Thus there exists a $c \in \mathbb{R}$ satisfying

$$\lim_{x \to x_0-} \beta(x) > c > \beta(x_0).$$

We can find a sequence $(u_n : n \in \mathbb{N})$ of points of $[x_1, x_2]$ convergent to x_0 from the left and such that

$$\lim_{n \to \infty} \beta(u_n) > c.$$

Since $c > \beta(x_0)$ we have

$$f(x_0, c) > a.$$

Therefore, because of the continuity of f,

$$a < f(u_n, c) \leq f(u_n, v_n) = a$$

for n large enough, which is impossible.

Define a set A by

$$A = \{(x, y) \in [x_1, x_2] \times [y_2, y_1] : \ \beta(x+) \leq y \leq \beta(x)\},$$

where $\beta(x_2+) := y_2$. By the monotonicity assumption and the definition of β the set A is contained in $\Gamma_f(a)$. We will prove that $\leq$ is linear on A. To this aim fix $(u_1, v_1), (u_2, v_2) \in A$. If $u_1 = u_2$, then, of course, (u_1, v_1) and (u_2, v_2) are comparable. So assume, for instance, that $u_1 < u_2$. Suppose that $v_1 < v_2$. Then

$$v_1 \in [\beta(u_1+), \beta(u_1)], \ v_2 \in [\beta(u_2+), \beta(u_2)]$$

whence

$$\beta(u_1+) \leq v_1 < v_2 \leq \beta(u_2) \leq \beta(u_1+),$$

which is impossible. Consequently, $(u_2, v_2) \leq (u_1, v_1)$.

According to Theorem 2 it is enough to show that A is a $\leq$-interval. Fix points $(u_1, v_1), (u_2, v_2) \in A$. Since $\leq$ is linear on A they are $\leq$-comparable. Assume, for instance, that $(u_1, v_1) \leq (u_2, v_2)$. Take an arbitrary $t \in \mathbb{R}$ such that

$$v_1 - u_1 < t < v_2 - u_2.$$

We must distinguish two cases.

(i) $\beta(u_1) - u_1 < t$. Observe that $\beta(u_2) - u_2 \geq v_2 - u_2 > t$. Define

$$\overline{u} := \sup\{u \in I : \ \beta(u) - u \geq t\}.$$

Then $u_2 \leq \overline{u} < u_1$,

$$\beta(u) - u < t \qquad \text{for } u \in I \cap (\overline{u}, \infty)$$

and

$$\beta(\overline{u}+) - \overline{u} \leq t \leq \beta(\overline{u}) - \overline{u}.$$

If $\beta(\overline{u}) - \overline{u} = t$ then

$$(\overline{u}, \overline{u} + t) = (\overline{u}, \beta(\overline{u})) \in A.$$

If $\beta(\overline{u}) - \overline{u} > t$ then

$$\beta(\overline{u}) - \overline{u} > t \geq \beta(\overline{u}+) - \overline{u}.$$

Therefore there exists a $v \in [\beta(\overline{u}+), \beta(\overline{u}))$ such that $v - \overline{u} = t$. Consequently, $(\overline{u}, \overline{u} + t) \in A$.

(ii) $\beta(u_1) - u_1 \geq t$. Then

$$\beta(u_1) - u_1 \geq t > v_1 - u_1 \geq \beta(u_1+) - u_1.$$

Therefore there exists a $v \in (\beta(u_1+), \beta(u_1)]$ such that $t = v - u_1$. Consequently, $(u_1, u_1 + t) \in A$. ■

As an immediate consequence of Proposition 1 we get what follows.

Theorem 3. *Assume that $f : I \times J \longrightarrow \mathbb{R}$ is continuous and increasing with respect to each variable. Then every level set of f is arcwise connected.*

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