

## ON BOUNDARY VALUE PROBLEMS OF SECOND ORDER DIFFERENTIAL INCLUSIONS

BAPUR CHANDRA DHAGE

*Kasubai, Gurukul Colony*

*Ahmedpur-413 515*

*Dist: Latur, Maharashtra, India*

**e-mail:** bcd20012001@yahoo.co.in

### Abstract

This paper presents sufficient conditions for the existence of solutions to boundary-value problems of second order multi-valued differential inclusions. The existence of extremal solutions is also obtained under certain monotonicity conditions.

**Keywords:** differential inclusion, method of upper and lower solutions, existence theorem.

**2000 Mathematics Subject Classification:** 34A60.

### 1. INTRODUCTION

Let  $\mathbb{R}$  denote the real line and let  $P_f(\mathbb{R})$  denote the class of all non-empty subsets of  $\mathbb{R}$  with a property  $f$ . In particular,  $P_{cl}(\mathbb{R})$ ,  $P_{bd}(\mathbb{R})$ ,  $P_{cv}(\mathbb{R})$ , and  $P_{cp}(\mathbb{R})$  denote respectively the classes of closed, bounded, convex and compact subsets of  $\mathbb{R}$ . Similarly,  $P_{cl,bd}(\mathbb{R})$  and  $P_{cp,cv}(\mathbb{R})$  denote respectively the classes of all closed-bounded and compact-convex subsets of  $\mathbb{R}$ . Let  $J = [t_0, t_1]$  be a closed and bounded interval in  $\mathbb{R}$  for some real numbers  $t_0, t_1 \in \mathbb{R}$  with  $t_0 < t_1$ . Now consider the two point boundary value problem (in short BVP) of second order differential inclusions

$$(1.1) \quad Lx(t) \in F(t, x(t)) \text{ a.e. } t \in J$$

satisfying the boundary conditions

$$(1.2) \quad \left. \begin{aligned} a_0x(t_0) + a_1x'(t_0) &= c_0 \\ b_0x(t_1) - b_1x'(t_1) &= c_1 \end{aligned} \right\}$$

where the functions involved in (1.1) and (1.2) satisfy the following properties:

- (a) the operator  $L : AC^1(J, \mathbb{R}) \rightarrow L^1(J, \mathbb{R})$  has the form  $Lx = -x'' + qx' + rx$ , where  $AC^1(J, \mathbb{R})$  is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on  $J$ , and  $q$  and  $r$  are real functions on  $J$  such that  $q, r \in L^1(J, \mathbb{R})$ ,
- (b)  $F : J \times \mathbb{R} \rightarrow P_f(\mathbb{R})$ ,
- (c)  $a_0, a_1$  and  $b_0, b_1$  are nonnegative real numbers satisfying  $a_0 + a_1 > 0$  and  $b_0 + b_1 > 0$  and
- (d)  $c_0, c_1 \in \mathbb{R}$ .

By the solution of the BVP (1.1)–(1.2) we mean a function  $x \in AC^1(J, \mathbb{R})$  whose 2<sup>nd</sup> derivative exists and is a member of  $L^1(J, \mathbb{R})$  in  $F(t, x)$ , i.e., there exists a  $v \in L^1(J, \mathbb{R})$  such that  $v(t) \in F(t, x(t))$  a.e.  $t \in J$ , and  $Lx(t) = v(t)$  for all  $t \in J$  satisfying (1.2), where  $x \in AC^1(J, \mathbb{R})$  is the space of continuous real-valued functions whose first derivative is absolutely continuous on  $J$ .

The special cases of the BVP (1.1)–(1.2) have been discussed in the literature for the existence of the solution. The special case of the form

$$(1.3) \quad -x''(t) = f(t, x(t)) \text{ a.e. } t \in J$$

satisfying the boundary conditions

$$(1.4) \quad \left. \begin{aligned} a_0x(t_0) + a_1x'(t_0) &= c_0 \\ b_0x(t_1) - b_1x'(t_1) &= c_1 \end{aligned} \right\}$$

where  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $a_0, a_1, b_0, b_1 \in \mathbb{R}_+$ ,  $c_0, c_1 \in \mathbb{R}$  and  $a_0a_1 + a_0b_1 + a_1b_0 > 0$  has been discussed in Heikkilä [19] for the existence of extremal solutions. Again when  $c_0 = c_1, a_1 = 0 = b_1, a_0 = b_0$ , the BVP (1.1)–(1.2) reduces to

$$(1.5) \quad y'' \in F(t, y) \text{ a.e. } t \in J, \quad y(t_0) = y(t_1),$$

where  $y = -x$ . This is a BVP of second order differential inclusions considered in Benchohra and Ntouyas [8]. Similarly, taking  $a_0 = 1, a_1 = 0, b_0 = 0$

and  $b_1 = 1$  in the BVP (1.1)–(1.2) we obtain the following second order differential inclusions, viz.,

$$(1.6) \quad y'' \in F(t, x), \quad \text{a.e. } t \in J$$

$$(1.7) \quad y(t_0) = c_0, \quad y'(t_1) = c_1.$$

Finally, the special case of the BVP (1.1)–(1.2) of the form

$$(1.8) \quad -x''(t) \in F(t, x(t)) \quad \text{a.e. } t \in J$$

satisfying the boundary conditions

$$(1.9) \quad \left. \begin{aligned} a_0x(t_0) + a_1x'(t_0) &= c_0 \\ b_0x(t_1) - b_1x'(t_1) &= c_1, \end{aligned} \right\}$$

has been studied in Halidias and Papageorgiou [18]. Thus the BVP (1.1)–(1.2) is more general and so is its importance in the theory of differential inclusions.

The method of upper and lower solutions has been successfully applied to the problem of nonlinear differential equations and inclusions. For the first problem, we refer to Heikkilä and Lakshmikantham [20] and Bernfield and Lakshmikantham [6] and for the other we refer to Halidias and Papageorgiou [18], Benchohra [7] and Dhage and Kang [13]. In this paper, we apply the multi-valued version of Leray-Schauder fixed point theorem due to Martelli [23] to BVP (1.1)–(1.2) for proving the existence of solutions between the given lower and upper solutions, using the Carathéodory condition of  $F$ . The existence of the extremal solutions is also obtained under certain monotonic conditions of the multi-functions and using the fixed point theorems of Dhage [10, 11] for multi-maps on the ordered spaces.

## 2. PRELIMINARIES

Let  $X$  be a Banach space. A correspondence  $T : X \rightarrow P_f(X)$  is called a multi-valued map or simply a multi-map and  $u \in Tu$  for some  $u \in X$ , then  $u$  is called a fixed point of  $T$ . A multi  $T$  is a closed (resp. convex and compact) if  $Tx$  is a closed (resp. convex and compact) subset of  $X$  for each  $x \in X$ .  $T$  is said to be bounded on bounded sets if  $T(B) = \bigcup_{x \in B} T(x) = \bigcup T(B)$  is a bounded subset of  $X$  for all bounded sets  $B$  in  $X$ .  $T$  is called upper

semi-continuous (u.s.c.) if for every open set  $N \subset X$ , the set  $\{x \in X : Tx \subset N\}$  is open in  $X$ .  $T$  is said to be totally bounded if for any bounded subset  $B$  of  $X$ , the set  $\cup T(B)$  is a totally bounded subset of  $X$ .

Again  $T$  is called completely continuous if it is upper semi-continuous and totally bounded on  $X$ . It is known that if the multi-valued map  $T$  is totally bounded with non empty compact values,  $T$  is upper semi-continuous if and only if  $T$  has a closed graph (that is  $x_n \rightarrow x_*, y_n \rightarrow y_*, y_n \in Tx_n \Rightarrow y_* \in Tx_*$ ).

We apply the following multi-valued version of a fixed point theorem of Leray-Schauder [17] due to Martelli [23] in the sequel.

**Theorem 2.1.** *Let  $T : X \rightarrow P_{cp,cv}(X)$  be a completely continuous multi-valued map. If the set*

$$\mathcal{E} = \{u \in X : \lambda u \in Tu \text{ for some } \lambda > 1\}$$

*is bounded, then  $T$  has a fixed point.*

We need the following definition in the sequel.

**Definition 2.2.** A multi-valued map  $F : J \rightarrow P_{cp,cv}(\mathbb{R})$  is said to be measurable if for every  $y \in \mathbb{R}$ , the function  $t \rightarrow d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$  is measurable.

**Definition 2.3.** A multi-valued map  $F : J \times \mathbb{R} \rightarrow P_f(\mathbb{R})$  is said to be  $L^1$ -Carathéodory if

- (i)  $t \mapsto F(t, x)$  is measurable for each  $x \in \mathbb{R}$ ,
- (ii)  $x \mapsto F(t, x)$  is upper semi-continuous for almost all  $t \in J$ , and
- (iii) for each real number  $k > 0$ , there exists a function  $h_k \in L^1(J, \mathbb{R})$  such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq h_k(t), \quad \text{a.e. } t \in J$$

for all  $x \in \mathbb{R}$  with  $|x| \leq k$ .

Denote

$$S_F^1(x) = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t)) \text{ a.e. } t \in J\}.$$

Then we have the following lemmas due to Lasota and Opial [22].

**Lemma 2.1.** *If  $\dim(X) < \infty$  and  $F : J \times X \rightarrow P_{cp,cv}(X)$ , then  $S_F^1(x) \neq \emptyset$  for each  $x \in X$ .*

**Lemma 2.2.** *Let  $X$  be a Banach space,  $F$  an  $L^1$ -Carathéodory multi-valued map with  $S_F^1 \neq \emptyset$  and  $\mathcal{K} : L^1(J, X) \rightarrow C(J, X)$  be a linear continuous mapping. Then the operator*

$$\mathcal{K} \circ S_F^1 : C(J, X) \longrightarrow P_{cp,cv}(C(J, X))$$

*is a closed graph operator in  $C(J, X) \times C(J, X)$ .*

We define the partial ordering  $\leq$  in  $AC^1(J, \mathbb{R})$  (the Sobolev class of functions  $x : J \rightarrow \mathbb{R}$  for which  $x'$  is absolutely continuous and  $Lx \in L^1(J, \mathbb{R})$ ) as follows. Let  $x, y \in AC^1(J, \mathbb{R})$ . Then we define

$$(2.1) \quad x \leq y \Leftrightarrow x(t) \leq y(t), \quad \forall t \in J.$$

Define a norm  $\|\cdot\|$  in  $AC^1(J, \mathbb{R})$  by

$$(2.2) \quad \|x\| = \sup_{t \in J} |x(t)|.$$

If  $a, b \in AC^1(J, \mathbb{R})$  and  $a \leq b$ , then we define an order interval  $[a, b]$  in  $AC^1(J, \mathbb{R})$  by

$$[a, b] = \{x \in AC^1(J, \mathbb{R}) : a \leq x \leq b\}.$$

The following definition appears in Dhage and Kang [13]. See also Agarwal *et al.* [1].

**Definition 2.4.** A function  $\alpha \in AC^1(J, \mathbb{R})$  is called a lower solution of IVP (1.1) if for all  $v_1 \in L^1(J, \mathbb{R})$  with  $v_1(t) \in F(t, \alpha(t))$  a.e.  $t \in J$  we have that

$$\begin{aligned}
L\alpha(t) &\leq v_1(t) \quad \text{a.e. } t \in J \\
a_0\alpha(t_0) + a_1\alpha'(t_0) &\leq c_0 \\
b_0\alpha(t_1) - b_1\alpha'(t_1) &\leq c_1.
\end{aligned}$$

Similarly, a function  $\beta \in AC^1(J, \mathbb{R})$  is called an upper solution of the BVP (1.1)–(1.2) if for all  $v_2 \in L^1(J, \mathbb{R})$  with  $v_2(t) \in F(t, \beta(t))$  a.e.  $t \in J$  we have that

$$\begin{aligned}
L\beta(t) &\geq v_1(t) \quad \text{a.e. } t \in J \\
a_0\beta(t_0) + a_1\beta'(t_0) &\geq c_0 \\
b_0\beta(t_1) - b_1\beta'(t_1) &\geq c_1.
\end{aligned}$$

Now we are ready to prove in the next section our main existence result for the BVP (1.1)–(1.2).

### 3. EXISTENCE RESULT

Before going to the main existence theorem of this section we give a useful result from the theory of boundary value problems of ordinary differential equations.

**Theorem 3.1.** *If  $f \in L^1(J, \mathbb{R})$ , then the BVP*

$$(3.1) \quad Lx(t) = f(t) \quad \text{a.e. } t \in J$$

*satisfying the boundary conditions*

$$(3.2) \quad \left. \begin{aligned} a_0x(t_0) + a_1x'(t_0) &= c_0 \\ b_0x(t_1) - b_1x'(t_1) &= c_1 \end{aligned} \right\}$$

*has a unique solution  $x$  given by*

$$(3.3) \quad x(t) = z(t) + \int_{t_0}^{t_1} G(t, s)f(s) ds, \quad t \in J,$$

*where  $z$  is a unique solution of the homogeneous differential equation*

$$(3.4) \quad Lx(t) = 0$$

satisfying the non homogeneous boundary conditions

$$(3.5) \quad \left. \begin{aligned} a_0x(t_0) + a_1x'(t_0) &= c_0 \\ b_0x(t_1) - b_1x'(t_1) &= c_1 \end{aligned} \right\}$$

and  $G(t, s)$  is the Green's function associated to the differential equation

$$(3.6) \quad Lx(t) = 0$$

satisfying the homogeneous boundary conditions

$$(3.7) \quad \left. \begin{aligned} a_0x(t_0) + a_1x'(t_0) &= 0 \\ b_0x(t_1) - b_1x'(t_1) &= 0 \end{aligned} \right\}.$$

**Remark 3.2.** It is known that the function  $z$  belongs to the class  $C^1(J, \mathbb{R})$ . Therefore it is bounded on  $J$  and there is a constant  $K_1 > 0$  such that  $\|z\| \leq K_1$ . The explicit form of the function  $z$  of (3.3) is given in Heikkilä *et al.* [21]. Similarly, the Green's function  $G(t, s)$  involved in (3.3) is a continuous real-valued function on  $J \times J$  and so there is a constant  $K_2 > 0$  such that  $\sup_{t,s \in J} |G(t, s)| \leq K_2$ .

We consider the following assumptions:

- (H<sub>1</sub>) The multi  $F(t, x)$  has compact and convex values for each  $(t, x) \in J \times \mathbb{R}$ .
- (H<sub>2</sub>)  $F(t, x)$  is  $L^1$ -Carathéodory.
- (H<sub>3</sub>) The BVP (1.1)–(1.2) has a lower solution  $\alpha$  and an upper solution  $\beta$  with  $\alpha \leq \beta$ .

**Theorem 3.3.** Assume that (H<sub>1</sub>)–(H<sub>3</sub>) hold. Then the BVP (1.1)–(1.2) has at least one solution  $x$  such that

$$\alpha(t) \leq x(t) \leq \beta(t), \text{ for all } t \in J.$$

**Proof.** First we transform the BVP (1.1)–(1.2) into a fixed point inclusion in a suitable Banach space. Consider the following BVP

$$(3.8) \quad \left. \begin{aligned} Lx(t) &\in F(t, \tau x(t)) \quad \text{a.e. } t \in J, \\ a_0x(t_0) + a_1x'(t_0) &= c_0 \\ b_0x(t_1) - b_1x'(t_1) &= c_1 \end{aligned} \right\}$$

for all  $x \in AC^1(J, \mathbb{R})$ , where  $\tau : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$  is the truncation operator defined by

$$(3.9) \quad (\tau x)(t) = \begin{cases} \alpha(t), & \text{if } x(t) < \alpha(t) \\ x(t), & \text{if } \alpha(t) \leq x(t) \leq \beta(t) \\ \beta(t), & \text{if } \beta(t) < x(t). \end{cases}$$

The problem of existence of a solution of the BVP (1.1)–(1.2) reduces to finding the solution of the integral inclusion

$$(3.10) \quad x(t) \in z(t) + \int_{t_0}^{t_1} G(t, s)F(s, \tau x(s)) ds, \quad t \in J.$$

We study the integral inclusion (3.10) in the space  $C(J, \mathbb{R})$  of all continuous real-valued functions on  $J$  with a supremum norm  $\|\cdot\|$ . Define a multi-valued map  $T : C(J, \mathbb{R}) \rightarrow P_f(C(J, \mathbb{R}))$  by

$$(3.11) \quad Tx = \left\{ u \in C(J, \mathbb{R}) : u(t) = z(t) + \int_{t_0}^{t_1} G(t, s)v(s)ds, \quad v \in \overline{S_F^1}(\tau x) \right\}$$

where

$$\begin{aligned} \overline{S_F^1}(\tau x) &= \{v \in S_F^1(\tau x) : v(t) \geq \alpha(t) \quad \text{a.e. } t \in A_1 \text{ and} \\ &\quad v(t) \leq \beta(t), \quad \text{a.e. } t \in A_2\} \end{aligned}$$

and

$$\begin{aligned} A_1 &= \{t \in J : x(t) < \alpha(t) \leq \beta(t)\}, \\ A_2 &= \{t \in J : \alpha(t) \leq \beta(t) < x(t)\}, \\ A_3 &= \{t \in J : \alpha(t) \leq x(t) \leq \beta(t)\}. \end{aligned}$$



By Lemma 2.1,  $S_F^1(\tau x) \neq \emptyset$  for each  $x \in C(J, \mathbb{R})$  which further yields that  $\overline{S_F^1}(\tau x) \neq \emptyset$  for each  $x \in C(J, \mathbb{R})$ . Indeed, if  $v \in S_F^1(x)$ , then the function  $w \in L^1(J, \mathbb{R})$  defined by

$$w = \alpha \chi_{A_1} + \beta \chi_{A_2} + v \chi_{A_3},$$

is in  $\overline{S_F^1}(\tau x)$  by virtue of decomposability of  $w$ .

We shall show that the multi  $T$  satisfies all the conditions of Theorem 3.3.

**Step I.** First we prove that  $T(x)$  is a convex subset of  $C(J, \mathbb{R})$  for each  $x \in C(J, \mathbb{R})$ . Let  $u_1, u_2 \in T(x)$ . Then there exists  $v_1$  and  $v_2$  in  $\overline{S_F^1}(\tau x)$  such that

$$u_j(t) = z(t) + \int_{t_0}^{t_1} G(t, s) v_j(s) ds, \quad j = 1, 2.$$

Since  $F(t, x)$  has convex values, one has for  $0 \leq k \leq 1$

$$[kv_1 + (1 - k)v_2](t) \in S_F^1(\tau x)(t), \quad \forall t \in J.$$

As a result we have

$$[ku_1 + (1 - k)u_2](t) = z(t) + \int_{t_0}^{t_1} G(t, s) [kv_1(s) + (1 - k)v_2(s)] ds.$$

Therefore  $[ku_1 + (1 - k)u_2] \in Tx$  and consequently  $T$  has convex values in  $C(J, \mathbb{R})$ .

**Step II.**  $T$  maps bounded sets into bounded sets in  $C(J, \mathbb{R})$ . To see this, let  $B$  be a bounded set in  $C(J, \mathbb{R})$ . Then there exists a real number  $r > 0$  such that  $\|x\| \leq r, \forall x \in B$ .

Now for each  $u \in Tx$ , there exists a  $v \in \overline{S_F^1}(\tau x)$  such that

$$u(t) = z(t) + \int_{t_0}^{t_1} G(t, s) v(s) ds.$$

Then for each  $t \in J$ ,

$$\begin{aligned} |u(t)| &\leq |z(t)| + \int_{t_0}^{t_1} |G(t, s)| |v(s)| ds \\ &\leq |z(t)| + \int_{t_0}^{t_1} |G(t, s)| h_r(s) ds \\ &= K_1 + K_2 \|h_r\|_{L^1}. \end{aligned}$$

This further implies that

$$\|u\| \leq K_1 + K_2 \|h_r\|_{L^1}.$$

for all  $u \in Tx \subset \bigcup T(B)$ . Hence  $\bigcup T(B)$  is bounded.

**Step III.** Next we show that  $T$  maps bounded sets into equi-continuous sets. Let  $B$  be a bounded set as in Step II, and  $u \in Tx$  for some  $x \in B$ . Then there exists  $v \in \overline{S_F^1}(\tau x)$  such that

$$u(t) = z(t) + \int_{t_0}^{t_1} G(t, s) v(s) ds.$$

Then for any  $t, \tau \in J$  we have

$$\begin{aligned} |u(t) - u(\tau)| &\leq |z(t) - z(\tau)| + \left| \int_{t_1}^{t_2} G(t, s) v(s) ds - \int_{t_0}^{t_2} G(\tau, s) v(s) ds \right| \\ &\leq |z(t) - z(\tau)| + \int_{t_0}^{t_1} |G(t, s) - G(\tau, s)| |v(s)| ds \\ &\leq |z(t) - z(\tau)| + \int_{t_0}^{t_1} |G(t, s) - G(\tau, s)| h_r(s) ds \\ &\leq |z(t) - z(\tau)| + |p(t) - p(\tau)| \end{aligned}$$

where  $p(t) = \int_{t_0}^{t_1} G(t, s) h_r(s) ds$ .

Now the function  $p$  is continuous on the compact interval  $J$ , hence it is uniformly continuous on  $J$ . Hence we have

$$|u(t) - u(t')| \rightarrow 0 \text{ as } t \rightarrow t'.$$

As a result  $\bigcup T(B)$  is an equi-continuous set in  $C(J, \mathbb{R})$ . Now an application of Arzelà-Ascoli theorem yields that the multi  $T$  is totally bounded on  $C(J, \mathbb{R})$ .

**Step IV.** Next we prove that  $T$  has a closed graph. Let  $\{x_n\} \subset C(J, \mathbb{R})$  be a sequence such that  $x_n \rightarrow x_*$  and let  $\{y_n\}$  be a sequence defined by  $y_n \in Tx_n$  for each  $n \in \mathbb{N}$  such that  $y_n \rightarrow y_*$ . We just show that  $y_* \in Tx_*$ .

Since  $y_n \in Tx_n$ , there exists a  $v_n \in \overline{S_F^1}(\tau x_n)$  such that

$$y_n(t) = z(t) + \int_{t_0}^{t_1} G(t, s) v_n(s) ds.$$

Consider the linear and continuous operator  $\mathcal{K} : L^1(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$  defined by

$$\mathcal{K}v(t) = z(t) + \int_{t_0}^{t_1} G(t, s) v(s) ds.$$

Now

$$\begin{aligned} |y_n(t) - z(t) - (y_*(t) - z(t))| &\leq |y_n(t) - y_*(t)| \\ &\leq \|y_n - y_*\|_C \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

From Lemma 2.2 it follows that  $(\mathcal{K} \circ \overline{S_F^1})$  is a closed graph operator and from the definition of  $\mathcal{K}$  one has

$$y_n(t) \in (\mathcal{K} \circ \overline{S_F^1})(\tau x_n).$$

As  $x_n \rightarrow x_*$  and  $y_n \rightarrow y_*$ , there is a  $v_* \in \overline{S_F^1}(\tau x_*)$  such that

$$y_* = z(t) + \int_{t_0}^{t_1} G(t, s) v_*(s) ds.$$

Hence the multi  $T$  is an upper semi-continuous operator on  $C(J, \mathbb{R})$ .

**Step V.** Finally we show that the set

$$\mathcal{E} = \{x \in C(J, \mathbb{R}) : \lambda x \in Tx \text{ for some } \lambda > 1\}$$

is bounded.

Let  $u \in \mathcal{E}$ . Then there exists  $v \in \overline{S_F^1}(\tau x)$  such that

$$u(t) = \lambda^{-1}z(t) + \lambda^{-1} \int_{t_0}^{t_1} G(t, s)v(s)ds.$$

Then

$$|u(t)| \leq |z(t)| + \int_{t_0}^{t_1} |G(t, s)| |v(s)| ds.$$

Since  $\tau x \in [\alpha, \beta]$ ,  $\forall x \in C(J, \mathbb{R})$ , we have

$$\|\tau x\| \leq \|\alpha\| + \|\beta\| := l.$$

By  $(H_2)$  there is a function  $h_l \in L^1(J, \mathbb{R})$  such that

$$\|F(t, \tau x)\| = \sup\{|u| : u \in F(t, \tau x)\} \leq h_l(t) \text{ a.e. } t \in J$$

for all  $x \in C(J, \mathbb{R})$ . Therefore

$$\begin{aligned} |u(t)| &\leq |z(t)| + \int_{t_0}^{t_1} |G(t, s)| h_l ds \\ &= K_1 + K_2 \|h_l\|_{L^1} \end{aligned}$$

for all  $t \in J$  and so, the set  $\mathcal{E}$  is bounded in  $C(J, \mathbb{R})$ .

Thus  $T$  satisfies all the conditions of Theorem 2.1 and so an application of it yields that the multi-map  $T$  has a fixed point. Consequently, the BVP (1.1)–(1.2) has a solution  $u$  on  $J$ .

Next we show that  $u$  is also a solution of the BVP (1.1)–(1.2) on  $J$ . First we show that  $u \in [\alpha, \beta]$ . Suppose not. Then either  $\alpha \not\leq u$  or  $u \not\leq \beta$  on some

subinterval  $J'$  of  $J$ . If  $u \not\geq \alpha$ , then there exist  $t_0, t_1 \in J, t_0 < t_1$  such that

$$\begin{aligned} a_0 u(t_0) + a_1 u'(t_0) &= a_0 \alpha(t_0) + a_1 \alpha'(t_0) = c_1 \\ b_0 u(t_1) - b_1 u'(t_1) &= a_0 \alpha(t_1) + a_1 \alpha'(t_1) = c_2, \end{aligned}$$

and  $\alpha(t) > u(t)$  for all  $t \in (t_0, t_1) \subset J$ . From the definition of the operator  $\tau$  it follows that

$$Lx(t) \in F(t, \alpha(t)) \quad \text{a.e. } t \in J.$$

Then there exists a  $v(t) \in F(t, \alpha(t))$  such that  $v(t) \geq \alpha(t), \forall t \in J$  with

$$Lu(t) = v(t) \quad \text{a.e. } t \in J.$$

Integrating from  $t_0$  to  $t_1$  yields

$$u(t) - z(t) = \int_{t_0}^{t_1} G(t, s) v(s) ds.$$

Since  $\alpha$  is a lower solution of the BVP (1.1)–(1.2), we have

$$\begin{aligned} u(t) &= z(t) + \int_{t_0}^{t_1} G(t, s) v(s) ds \\ &\geq z(t) + \int_{t_0}^{t_1} G(t, s) \alpha(s) ds \\ &= \alpha(t) \end{aligned}$$

for all  $t \in (t_0, t_1)$ . This is a contradiction. Similarly, if  $u \not\leq \beta$  on some subinterval of  $J$ , then also we get a contradiction. Hence  $\alpha \leq u \leq \beta$  on  $J$ . As a result the BVP (1.1)–(1.2) has a solution  $u$  in  $[\alpha, \beta]$ . Finally, since  $\tau x = x, \forall x \in [\alpha, \beta]$ ,  $u$  is a required solution of the BVP (1.1)–(1.2) on  $J$ . This completes the proof.  $\blacksquare$

#### 4. EXISTENCE OF EXTREMAL SOLUTIONS

##### 4.1. Carathéodory case

In this section, we establish the existence of extremal solutions to the BVP (1.1)–(1.2) when the multi-function  $F(t, x)$  is Carathéodory and isotone increasing in  $x$ . Here our technique involves combining the method of upper and lower solutions with an algebraic fixed point theorem of Dhage [11] on ordered Banach spaces.

Define a cone  $K$  in  $C(J, \mathbb{R})$  by

$$(4.1) \quad K = \{x \in C(J, \mathbb{R}) : x(t) \geq 0, \forall t \in J\}.$$

Then the cone  $K$  defines an order relation  $\leq$  in  $C(J, \mathbb{R})$  by

$$(4.2) \quad x \leq y \text{ iff } x(t) \leq y(t), \forall t \in J.$$

It is known that the cone  $K$  is normal in  $C(J, \mathbb{R})$ . See Heikkilä and Lakshmikantham [20] and the references therein. For any  $A, B \in P_{cl, bd}(\mathbb{R})$  we define the order relation  $\leq$  in  $P_{cl, bd}(\mathbb{R})$  by

$$(4.3) \quad A \leq B \text{ iff } a \leq b, \forall a \in A \text{ and } \forall b \in B.$$

In particular,  $a \leq B$  implies that  $a \leq b, \forall b \in B$  and if  $A \leq A$ , then it follows that  $A$  is a singleton set.

**Definition 4.1.** A multi-map  $T : C(J, \mathbb{R}) \rightarrow P_{cl, bd}(\mathbb{R})$  is said to be isotone increasing if for any  $x, y \in C(J, \mathbb{R})$  with  $x < y$  we have that  $Tx \leq Ty$ .

We need the following fixed point theorem of Dhage [10] in the sequel.

**Theorem 4.2.** Let  $[\alpha, \beta]$  be an order interval in a Banach space  $X$  and let  $T : [\alpha, \beta] \rightarrow P_{cl}([\alpha, \beta])$  be a completely continuous and isotone increasing multi-map. Further if the cone  $K$  in  $X$  is normal, then  $T$  has a least  $x_*$  and a greatest fixed point  $y^*$  in  $[\alpha, \beta]$ . Moreover, the sequences  $\{x_n\}$  and  $\{y_n\}$  defined by  $x_{n+1} \in Tx_n, x_0 = \alpha$  and  $y_{n+1} \in Ty_n, y_0 = \beta$ , converge to  $x_*$  and  $y^*$  respectively.

We consider the following assumptions in the sequel.

- (B<sub>1</sub>) The BVP (1.1)–(1.2) has a lower solution  $\alpha$  and an upper solution  $\beta$  with  $\alpha \leq \beta$ .
- (B<sub>2</sub>) The multi-map  $F(t, x)$  is  $L^1$ -Carathéodory.
- (B<sub>3</sub>)  $F(t, x)$  is nondecreasing in  $x$  almost everywhere for  $t \in J$ , i.e., if  $x < y$ , then  $F(t, x) \leq F(t, y)$  almost everywhere for  $t \in J$ .

**Remark 4.3.** Suppose that hypotheses (B<sub>1</sub>)–(B<sub>3</sub>) hold. Then the function  $h : J \rightarrow \mathbb{R}$  defined by

$$h(t) = \|F(t, \alpha(t))\| + \|F(t, \beta(t))\|, \quad \text{for } t \in J,$$

is Lebesgue integrable and such that

$$|F(t, x)| \leq h(t), \quad \forall t \in J, \quad \forall x \in [\alpha, \beta].$$

We need the following definition in the sequel.

**Definition 4.4.** A solution  $x_M$  of the BVP (1.1)–(1.2) is called maximal if for any other solution of the BVP (1.1)–(1.2) we have that  $x(t) \leq x_M(t)$ ,  $\forall t \in J$ . Similarly, a minimal solution  $x_m$  of the BVP (1.1)–(1.2) is defined.

**Theorem 4.5.** Assume that hypotheses (H<sub>1</sub>), (B<sub>1</sub>), (B<sub>2</sub>) and (B<sub>3</sub>) hold. Then the BVP (1.1)–(1.2) has a minimal and a maximal solution on  $J$ .

**Proof.** Clearly, the BVP (1.1)–(1.2) is equivalent to the operator inclusion

$$(4.4) \quad x(t) \in Tx(t), \quad t \in J$$

where the multi-map  $T : C(J, \mathbb{R}) \rightarrow P_{cl, bd}(\mathbb{R})$  is defined by

$$Tx = \left\{ u \in C(J, \mathbb{R}) : u(t) = z(t) + \int_{t_0}^{t_1} G(t, s)v(s)ds, \quad v \in S_F^1(x) \right\}.$$

Now the multi-map  $T$  exists in view of hypothesis (B<sub>2</sub>). We show that the multi-map  $T$  satisfies all the conditions of Theorem 3.1. First we show that  $T$  is isotone increasing on  $C(J, \mathbb{R})$ . Let  $x, y \in C(J, \mathbb{R})$  be such that  $x < y$ .

Let  $a \in Tx$  be arbitrary. Then there is a  $v_1 \in S_F^1(x)$  such that

$$a(t) = z(t) + \int_{t_0}^{t_1} G(t, s)v_1(s) ds.$$

Since  $F(t, x)$  is nondecreasing in  $x$  we have that  $S_F^1(x) \leq S_F^1(y)$ . As a result for any  $v_2 \in S_F^1(y)$  one has

$$\begin{aligned} \alpha(t) &\leq z(t) + \int_{t_0}^{t_1} G(t, s)v_2(s) ds \\ &= b(t) \end{aligned}$$

for all  $t \in J$  and any  $b \in Ty$ . This shows that the multi-map  $T$  is isotone increasing on  $C(J, \mathbb{R})$  and in particular on  $[\alpha, \beta]$ . Since  $\alpha$  and  $\beta$  are lower and upper solutions of the BVP (1.1)–(1.2) on  $J$ , we have

$$\alpha(t) \leq z(t) + \int_{t_0}^{t_1} G(t, s)v(s)ds, \quad t \in J$$

for all  $v \in S_F^1(\alpha)$ , and so  $\alpha \leq T\alpha$ . Similarly  $T\beta \leq \beta$ . Now let  $x \in [\alpha, \beta]$  be arbitrary. Then by the isotonicity of  $T$ ,

$$\alpha \leq T\alpha \leq Tx \leq T\beta \leq \beta.$$

Therefore,  $T$  defines a multi-map  $T : [\alpha, \beta] \rightarrow P_f([\alpha, \beta])$ . Finally proceeding as in Theorem 3.1, it is proved that  $T$  is a completely continuous multi-valued operator on  $[\alpha, \beta]$ . Since  $T$  satisfies all the conditions of Theorem 3.1 and the cone  $K$  in  $C(J, \mathbb{R})$  is normal, an application of Theorem 3.1 yields that  $T$  has a least and a greatest fixed point in  $[\alpha, \beta]$ . This further implies that the BVP (1.1)–(1.2) has a minimal and a maximal solution on  $J$ . This completes the proof.  $\blacksquare$

#### 4.2. Discontinuous case

In this section, we obtain the existence of the extremal solutions of the BVP (1.1)–(1.2) under the weaker continuity and monotonic conditions of the multi-function  $F$ . We use the following notations in the sequel.

Let  $BM(J, \mathbb{R})$  denote the space of all bounded and measurable real-valued functions on  $J$ . Define a norm  $\|\cdot\|$  and an order relation  $\leq$  in  $BM(J, \mathbb{R})$  by (2.2) and (2.1) respectively. It is known that  $BM(J, \mathbb{R})$  is a complete lattice with respect to this order relation  $\leq$ . See Birkhoff [5] for details.



We define the order relation “ $\leq$ ” in  $P_{cl}(X)$  as follows. Let  $A, B \in P_{cl}(X)$ . Then we have

$$(4.5) \quad A \leq B \Leftrightarrow \begin{cases} \text{for each } a \in A \exists a \in B \text{ such that } a \leq b, \text{ and} \\ \text{for each } b' \in B \exists a' \in A \text{ such that } a' \leq b'. \end{cases}$$

The above order relation in  $P_f(X)$  has been used in Dhage [10, 11] in the study of extremal solutions for differential and integral inclusions and it is an improvement upon the order relation defined in Dhage and Regan [16] and Agarwal *et al.* [1].

We need the following definition in the sequel.

**Definition 4.6.** A multi-map  $T : X \rightarrow P_{cl}(X)$  is said to be nondecreasing on  $X$  if  $x \leq y$  implies that  $Tx \leq Ty$  for all  $x, y \in X$ .

The following key fixed point theorem for multi-maps in complete lattices will be used in proving the main existence results. For details see Dhage [11] and the references therein.

**Theorem 4.7.** *Let  $X$  be an ordered Banach space and let  $T : X \rightarrow P_{cl}(X)$  be a multi-map such that*

- (a)  $(X, \leq)$  is a complete lattice,
- (b)  $T$  is nondecreasing and
- (c)  $\mathcal{F} = \{u \in X : u \in Tu\}$ .

*Then  $\mathcal{F}$  is a non-empty and complete lattice.*

We consider the following hypotheses in the sequel.

- (C<sub>1</sub>) The BVP (1.1)–(1.2) has a lower solution  $a$  and an upper solution  $b$  with  $a \leq b$ .
- (C<sub>2</sub>)  $F(t, x)$  is closed for each  $(t, x) \in J \times \mathbb{R}$ .
- (C<sub>3</sub>)  $F(t, x)$  is isotone increasing in  $x$  almost everywhere for  $t \in J$ .
- (C<sub>4</sub>)  $S_F^1(x) \neq \emptyset$  for all  $x \in BM(J, \mathbb{R})$ .
- (C<sub>5</sub>) The function

$$h(t) = \|F(t, a(t))\| + \|F(t, b(t))\|, \quad t \in J$$

is Lebesgue integrable.

**Remark 4.8.** Hypothesis (C<sub>5</sub>) is considered for making (C<sub>4</sub>) sense. To see this, let  $x \in [a, b]$  be any element. Then by (H<sub>3</sub>),

$$\begin{aligned} |F(t, x(t))| &\leq |F(t, a(t))| + |F(t, b(t))| \\ &\leq \|F(t, a(t))\| + \|F(t, b(t))\| \\ &= h(t) \end{aligned}$$

for all  $t \in J$ . So if  $F(\cdot, a(\cdot))$  has a measurable selection, then it is integrable. As a result  $S_F^1(x) \neq \emptyset$  for all  $x \in BM(J, \mathbb{R})$  with  $a \leq x \leq b$ .

We remark that hypotheses (C<sub>2</sub>), (C<sub>4</sub>)–(C<sub>5</sub>) have extensively been used in the literature on differential inclusions. See references [7, 8, 9] and [18]. The hypothesis (C<sub>3</sub>) is not much routine, but has been used in [1, 13] and [10].

**Theorem 4.9.** *Assume that hypotheses (C<sub>1</sub>)–(C<sub>5</sub>) hold. Then the BVP (1.1)–(1.2) has a minimal and a maximal solution on  $[a, b]$ .*

**Proof.** Let  $X = BM(J, \mathbb{R})$  and consider the lattice interval  $[a, b]$  in  $X$  which does exist in view of hypothesis (C<sub>1</sub>). Obviously  $[a, b]$  is a complete lattice. The details of complete lattices appear in Birkhoff [5]. Define a multi-map  $T : X \rightarrow P_f(X)$  by

$$\begin{aligned} Tx &= \left\{ u \in X : u(t) = z(t) + \int_0^t G(t, s)v(s) ds, v \in S_F^1(x) \right\} \\ &= (\mathcal{K} \circ S_F^1)(x) \end{aligned}$$

where the operator  $\mathcal{K} : L^1(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$  is defined by

$$\mathcal{K}y(t) = z(t) + \int_0^t G(t, s)y(s) ds.$$

We shall show that the multi-map  $T$  satisfies all the conditions of Theorem 4.7 on  $[a, b]$ .

**Step I.** First we show that  $T$  is isotone increasing on  $X$ . Let  $x, y \in X$  be such that  $x \leq y$ . Since (C<sub>3</sub>) holds, it follows that  $S_F^1(x) \leq S_F^1(y)$ . Therefore from the definition of  $T$ , we have

$$Tx = (\mathcal{K} \circ S_F^1)(x) = \mathcal{K}(S_F^1(x)) \leq \mathcal{K}(S_F^1(y)) = (\mathcal{K} \circ S_F^1)(y) = Ty.$$

As a result the multi-map  $T$  is isotone increasing on  $X$  and in particular on  $[a, b]$ .

**Step II.** Next we show that  $T : [a, b] \rightarrow P_{cl, bd}([a, b])$ . From (C<sub>4</sub>) it follows that

$$a(t) \leq z(t) + \int_0^t G(t, s)v(s) ds, \quad t \in J$$

for all  $v \in S_F^1(a)$ , and so

$$\begin{aligned} a(t) &\leq \left\{ u(t) : u(t) = z(t) + \int_0^t G(t, s)v(s) ds \text{ for all } v \in S_F^1(a) \right\} \\ &= Ta(t) \end{aligned}$$

for all  $t \in J$ . Hence  $a \leq Ta$ . Similarly, it is proved that  $Tb \leq b$ . To conclude, it is enough to prove that if  $x \in [a, b]$  is any element, then  $a \leq Tx$ . Now  $a \leq x$ , then by the isotonicity of  $T$ , one has

$$a \leq Ta \leq Tx \leq Tb \leq b.$$

As a result we have that  $T : [a, b] \rightarrow P_{cl, bd}([a, b])$ .

**Step III.** Finally hypothesis (C<sub>2</sub>) implies that  $Tx$  is a closed subset of  $[a, b]$  for each  $x \in [a, b]$ . This follows very easily if we show  $S_F^1(x)$  have closed values in  $L^1(J, \mathbb{R})$ . The last property is clear because of assumption (H<sub>2</sub>). Then for each  $x \in [a, b]$  we have that  $Tx$  is a closed subset of  $[a, b]$ .

Thus the multi-map  $T$  satisfies all the conditions of Theorem 4.7 and so an application of it yields that the fixed point set  $\mathcal{F}$  for  $T$  is non-empty and it has maximal and minimal elements. This further implies that BVP (1.1)–(1.2) has a minimal and a maximal solution in  $[a, b]$ . This completes the proof. ■

Note that hypothesis (C<sub>1</sub>) could also be replaced with

(C<sub>6</sub>) There exists a  $k \in L^1(J, \mathbb{R})$  such that

$$|F(t, x)| \leq k(t), \quad \text{a.e. } t \in J$$

for all  $x \in \mathbb{R}$ .

**Theorem 4.10.** *Assume that hypotheses (C<sub>2</sub>)–(C<sub>5</sub>) and (C<sub>6</sub>) hold. Then the BVP (1.1)–(1.2) has a minimal and a maximal solution on  $J$ .*

**Proof.** Define the functions  $a$  and  $b$  on  $J$  by

$$\alpha(t) = z(t) - \int_0^t G(t, s)k(s) ds$$

and

$$\beta(t) = z(t) + \int_0^t G(t, s)k(s) ds.$$

We shall prove that  $a$  and  $b$  serve respectively as the lower and upper solutions of BVP (1.1)–(1.2) on  $J$ . Obviously  $\alpha, \beta \in AC^1(J, \mathbb{R})$ . Since (C<sub>6</sub>) holds,  $|S_F^1(x)(t)| \leq k(t)$  a.e.  $t \in J$  for all  $x \in C(J, \mathbb{R})$ . Then we have

$$L\alpha(t) = -k(t) \leq v(t), \quad t \in J,$$

for all  $v \in S_F^1(x)$ . As a result

$$L\alpha(t) \leq v(t), \quad t \in J \quad \text{and} \quad \begin{cases} a_0\alpha(t_0) + a_1\alpha'(t_0) = c_1 \\ b_0\alpha(t_1) - b_1\alpha'(t_1) = c_2, \end{cases}$$

for all  $v \in S_F^1(\alpha)$ . Similarly,

$$L\beta(t) \geq v(t), \quad t \in J \quad \text{and} \quad \begin{cases} a_0\beta(t_0) + a_1\beta'(t_0) = c_1 \\ b_0\beta(t_1) - b_1\beta'(t_1) = c_2, \end{cases}$$

for all  $v \in S_F^1(\beta)$ . Again note that

$$\alpha(t) = z(t) - \int_0^t G(t, s)k(s) ds \leq z(t) + \int_0^t G(t, s)k(s) ds = \beta(t)$$

for all  $t \in J$ , so that  $\alpha \leq \beta$ . Thus hypothesis (C<sub>1</sub>) holds with  $a = \alpha$  and  $b = \beta$ . Now the desired conclusion follows by an application of Theorem 3.1. The proof is complete.  $\blacksquare$

Next we consider the following hypothesis:

(C<sub>7</sub>) There exists a nondecreasing multi-function  $H : J \times \mathbb{R}_+ \rightarrow P_{cl, bd}(\mathbb{R}_+)$  such that

$$|F(t, x)| \leq H(t, |x|), \quad \text{a.e. } t \in J$$

for all  $(t, x) \in J \times \mathbb{R}$  and that the BVP

$$Lx(t) \in H(t, x(t)), \quad \text{a.e. } t \in J$$

$$a_0x(t_0) + a_1x'(t_0) = |c_0|$$

$$b_0x(t_1) - b_1x'(t_1) = |c_1|$$

has an upper solution  $w \in AC_+^1(J, \mathbb{R})$ .

**Theorem 4.11.** *Assume that hypotheses (C<sub>2</sub>)–(C<sub>5</sub>) and (C<sub>7</sub>) hold. Then the BVP (1.1)–(1.2) has a minimal and a maximal solution on  $J$ .*

**Proof.** To finish, we just show that hypothesis (C<sub>7</sub>) implies hypothesis (C<sub>1</sub>). Notice that (C<sub>7</sub>) implies

$$|S_F^1(x)| \leq S_H^1(|x|)$$

for all  $x \in AC^1(J, \mathbb{R})$ . Therefore for any  $x \in AC^1(J, \mathbb{R})$ ,

$$\sup \{|S_F^1(x)| : |x| \leq w\} \leq \sup \{|S_H^1(|x|)| : |x| \leq w\} \leq S_H^1(w).$$

Therefore

$$|S_F^1(w)| \leq S_H^1(w) \leq Lw,$$

which yields that

$$L(-w(t)) \leq F(t, -w(t)) \quad \text{and} \quad L(w(t)) \geq F(t, w(t)) \quad \text{for all } t \in J.$$

Because

$$a_0w(t_0) + a_1w'(t_0) \geq |c_1|$$

$$b_0(w)(t_1) - b_1(w')(t_1) \geq |c_2|,$$

it follows that

$$a_0w(t_0) + a_1w'(t_0) \geq c_1$$

$$b_0w(t_1) - b_1w'(t_1) \geq c_2,$$

and

$$a_0(-w(t_0)) + a_1(-w'(t_0)) \leq c_1$$

$$b_0(-w(t_1)) - b_1(-w'(t_1)) \leq c_2.$$

Thus hypothesis  $(C_1)$  holds with  $a = -w$  and  $b = w$ . Hence the desired conclusion follows by an application of Theorem 3.1. The proof is complete. ■

## 5. CONCLUSION

The existence theorems for differential inclusions involving the convex right hand side have been studied in the literature for a long time. The method of upper and lower solutions has been introduced by Halidias and Papa-georgiou [18]. Since our BVP (1.1)–(1.2) is more general, the existence results proved in this paper using the convexness and the upper and lower solution method include several existence results in the literature including those of Benchohra [7], Benchohra and Ntouyas [8, 9] and Halidias and Papa-georgiou [18] etc. as special cases. The study of differential inclusions involving the nonconvex and discontinuous right hand side is rather new to and has been initiated recently by Dhage [11], Dhage and Regan [16] and Agarwal *et al.* [1]. The existence results of this paper follow the line of arguments of Dhage [11] and so they constitute an important contribution to the theory of differential inclusions under less restrictive monotonic conditions. The monotonicity assumed in the discontinuous differential inclusions of the present paper is of simple nature, however, more generalized monotonicity of the discontinuous multi-function like in Dhage [12] can also be considered to achieve the desired goal. Some of the results for the BVP (1.1)–(1.2) along this direction with nonconvex right hand side and using the fixed point theorem of Nadler [17] will be reported elsewhere.

## REFERENCES

- [1] R. Agarwal, B.C. Dhage and D. O'Regan, *The upper and lower solution method for differential inclusions via a lattice fixed point theorem*, Dynamic Systems Appl. **12** (2003), 1–7.
- [2] J. Appell, H.T. Nguven and P. Zabreiko, *Multi-valued superposition operators in ideal spaces of vector functions*, Indag. Math. **3** (1992), 1–8.
- [3] J. Aubin and A. Cellina, *Differential Inclusions*, Springer Verlag 1984.

- [4] P.B. Bailey, L.F. Shampine and P.E. Waltman, *Nonlinear Two Point Boundary Value Problems*, Academic Press, New York 1968.
- [5] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Coll. Publ. vol. 25, New York 1967.
- [6] S. Bernfield and V. Lakshmikantham, *An Introduction to Boundary Value Problems*, Academic Press, New York 1974.
- [7] M. Benchohra, *Upper and lower solutions method for second order differential inclusions*, Dynam. Systems Appl. **11** (2002), 13–20.
- [8] M. Benchohra and S.K. Ntouyas, *On second order differential inclusions with periodic boundary conditions*, Acta Math. Univ. Comenianea **LXIX** (2000), 173–181.
- [9] M. Benchohra and S.K. Ntouyas, *The lower and upper solutions method for first order differential inclusions with nonlinear boundary conditions*, J. Ineq. Pure Appl. Math. **3** (1) (2002), Art. 14.
- [10] B.C. Dhage, *A functional integral inclusion involving discontinuities*, Fixed Point Theory **5** (2004), 53–64.
- [11] B.C. Dhage, *A fixed point theorem for multi-valued mappings in Banach spaces with applications*, Nonlinear Anal. (to appear).
- [12] B.C. Dhage, *Monotone method for discontinuous differential inclusions*, Math. Sci. Res. J. **8** (3) (2004), 104–113.
- [13] B.C. Dhage and S.M. Kang, *Upper and lower solutions method for first order discontinuous differential inclusions*, Math. Sci. Res. J. **6** (2002), 527–533.
- [14] B.C. Dhage and S. Heikkilä, *On nonlinear boundary value problems with deviating arguments and discontinuous right hand side*, J. Appl. Math. Stoch. Anal. **6** (1993), 83–92.
- [15] B.C. Dhage, T.L. Holambe and S.K. Ntouyas, *Upper and lower solutions method for second order discontinuous differential inclusions*, Math. Sci. Res. J. **7** (2003), 206–212.
- [16] B.C. Dhage and D.O. Regan, *A lattice fixed point theorem and multi-valued differential equations*, Functional Diff. Equations **9** (2002), 109–115.
- [17] J. Dugundji and A. Granas, *Fixed point theory*, Springer Verlag 2003.
- [18] N. Halidias and N. Papageorgiou, *Second order multi-valued boundary value problems*, Arch. Math. (Brno) **34** (1998), 267–284.
- [19] S. Heikkilä, *On second order discontinuous scalar boundary value problem*, Nonlinear Studies **3** (2) (1996), 153–162.

- [20] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker Inc., New York 1994.
- [21] S. Heikkilä, J.W. Mooney and S. Seikkilä, *Existence, uniqueness and comparison results for nonlinear boundary value problems involving deviating arguments*, J. Diff. Eqn **41** (3) (1981), 320–232.
- [22] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.
- [23] M. Martelli, *A Rothe's type theorem for non compact acyclic-valued maps*, Boll. Un. Mat. Ital. **4** (Suppl. Fasc.) (1975), 70–76.

Received 4 May 2004