Discussiones Mathematicae
Differential Inclusions, Control and Optimization 23 (2003) 21–29

SOME APPLICATIONS OF GIRSANOV'S THEOREM TO THE THEORY OF STOCHASTIC DIFFERENTIAL INCLUSIONS

Michał Kisielewicz

Institute of Mathematics
University of Zielona Góra
Podgórna 50, 65–246 Zielona Góra, Poland

Abstract

The Girsanov's theorem is useful as well in the general theory of stochastic analysis as well in its applications. We show here that it can be also applied to the theory of stochastic differential inclusions. In particular, we obtain some special properties of sets of weak solutions to some type of these inclusions.

Keywords: stochastic process, Girsanov's theorem, stochastic differential inclusion, weak solution, Brownian motion.

2000 Mathematics Subject Classification: 93E03, 93C30.

1. Introduction

Stochastic differential inclusions, introduced independently by F. Hiai [1] and M. Kisielewicz [2], are defined as the relations of the from:

$$x_t - x_s \in cl_{L^2}\left(\int_s^t F(\tau, x_\tau)d\tau + \int_s^t G(\tau, x_\tau)dB_\tau\right) \tag{1}$$

that have to be satisfied for $0 \le s \le t \le T$ by a continuous $(\mathcal{F}_t)$-adapted stochastic process $x = (x_t)_{0\le t\le T}$ on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ satisfying usual hypotheses ([5]). We assume that set-valued mappings $F : [0,T] \times \mathbb{R}^n \to Cl(\mathbb{R}^n)$ and $G : [0,T] \times \mathbb{R}^n \to Cl(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ are given. We assume that they are measurable on $[0,T] \times \mathbb{R}^n$. Let $Cl(\mathbb{R}^n)$ and $Cl(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ denote the families of all nonempty closed subsets of the n-dimensional Euclidean space $\mathbb{R}^n$ and on the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of all $(n \times m)$ – matrices, respectively. As usual, for a given $g = (g_{ij})_{n\times m} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ we define $\|g\| = \sum_{i=1}^n \sum_{j=1}^m |g_{ij}|$. By $B = (B_t)_{0\le t\le T}$ we denote an m-dimensional $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, P)$ such that $B_t(\omega) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ for $t \in [0,T]$, $\omega \in \Omega$ and $P(B_0 = 0) = 1$.

Having given the probability space $(\Omega, \mathcal{F}, P)$ mentioned above with a filtration $(\mathcal{F}_t)_{0\le t\le T}$, $\mathcal{F}_t$-Brownian motion B and set-valued mappings F and G, we can look for a continuous $\mathcal{F}_t$-adapted stochastic process $x = (x_t)_{0\le t\le T}$ on $(\Omega, \mathcal{F}, P)$ satisfying (1) for $0 \le s \le t \le T$. Such a process x is said to be a strong solution to (1).

If we have given only F and G we can look for a system $\{(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P), (x_t)_{0\le t\le T}, (B_t)_{0\le t\le T}\}$ satisfying conditions mentioned above and such that (1) is satisfied for $0 \le s \le t \le T$. Such a system is said to be a weak solution to (1).

It is clear that weak solutions can be identified with pairs (x, B) of processes $x = (x_t)_{0\le t\le T}$ and $B = (B_t)_{0\le t\le T}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$.

In what follows the Banach space of all n-dimensional ($n \times m$ – type matrices) $\mathcal{F}_t$-adapted processes $f = (f_t)_{0\le t\le T}$ $(\sigma_t = (\sigma_{ij})_{n\times m}(t);\ 0 \le t \le T)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ such that $E\int_0^T |f_t|^p dt < \infty$ $(E\int_0^T \|\sigma_t\|^p dt < \infty)$; $p \ge 1$ is denoted by $\mathcal{L}_p^n(\mathcal{F}_t)(\mathcal{L}_p^{n\times m}(\mathcal{F}_t))$. We also assume that $B = (B_t)_{0\le t\le T}$, being an m-dimensional $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, P)$, is such that $P(B_0 = 0) = 1$.

Finally, by $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{Conv}(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ we denote the space of all nonempty compact and convex subsets of $\mathbb{R}^n$ and $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, respectively.

Let us recall ([2], Theorem 4) that for given measurable and square integrable bounded set-valued mappings $F : [0,T] \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ and $G : [0,T] \times \mathbb{R}^n \to \mathrm{Conv}(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ and the $\mathcal{F}_t$-Brownian motion $B = (B_t)_{0\le t\le T}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ given above, a continuous $\mathcal{F}_t$-adapted stochastic process $x = (x_t)_{0\le t\le T}$ on $\Omega, \mathcal{F}, P)$ satisfies (1) for every $0 \le s \le t \le T$ if and only if there are $f \in S(F \circ x)$ and $g \in S(F \circ x)$ such that

$$x_t - x_s = \int_s^t f_\tau d\tau + \int_s^t g_\tau dB_\tau; \qquad (P.1) \tag{2}$$

for $0 \le s \le t \le T$, where $S(F \circ x)$ and $S(G \circ x)$ denote the families of all $\mathcal{F}_t$-adapted selectors for $(F \circ x)_\tau(\omega) = F(t, x_t(\omega))$ and $(G \circ x)_t(\omega) = G(t, x_t(\omega))$, respectively.

Let Φ and Ψ be $\mathcal{F}_t$-adapted ([2]) set-valued mappings $\Phi : [0,T] \times \Omega \to Cl(\mathbb{R}^n)$ and $\Psi : [0,T] \times \Omega \to Cl(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ such that $E\int_0^T \|\Phi_t\| dt < \infty$ and $E\int_0^T \|\Psi_t\|^2 dt < \infty$, where $\|\Phi_t\| = \sup\{|a| : a \in \Phi_t\}$ and $\|\Psi_t\| = \sup\{|b| : b \in \Psi_t\}$. We define stochastic set-valued integrals for Φ and Ψ on $[s,t] \subset [0,T]$ by setting

$$\int_s^t \Phi_\tau d\tau = \left\{ \int_s^t \varphi_\tau d\tau : \varphi \in S(\Phi) \right\} \tag{3}$$

and

$$\int_s^t \Psi_\tau d\tau = \left\{ \int_s^t \psi_\tau dB\tau : \psi \in S(\Psi) \right\}, \tag{4}$$

where $S(\Phi)$ and $S(\Psi)$ denote again the families of all $\mathcal{F}_t$-adapted selectors for Φ and Ψ, respectively. For Φ and Ψ given above a family

$$\left(\int_0^t \Phi_\tau d\tau + \int_0^t \Psi_\tau dB_\tau \right)_{0 \le t \le T} \tag{5}$$

will be denoted by $\mathcal{Z}(\Phi, \Psi, B)$ and called a set-valued Itô process. In what follows for a fixed $[s,t] \subset [0,T]$, by $\mathcal{Z}(\Phi, \Psi, B)$ $([s,t])$ we shall denote the sum

$$\mathcal{Z}(\Phi, \Psi, B)([s,t]) = \int_s^t \Phi_\tau d\tau + \int_s^t \Psi_\tau dB\tau. \tag{6}$$

It follows immediately from the properties of the set-valued stochastic integrals ([2], Theorem 4) that for the set-valued Itô process $\mathcal{Z}(\Phi, \Psi, B)$ with Φ and Ψ taking convex values and a continuous n-dimensional stochastic process $x = (x_t)_{0 \le t \le T}$ on $(\Omega, \mathcal{F}, P)$ the relation

$$x_t - x_s \in \mathcal{Z}(\Phi, \Psi, B)\ ([s,t]) \tag{7}$$

is satisfied for every $0 \le s \le t \le T$ if and only if there exist $\varphi \in S(\Phi)$ and $\psi \in S(\Psi)$ such that

$$x_t - x_s = \int_s^t \varphi_\tau d\tau + \int_s^t \psi_\tau dB_\tau; \qquad (P.1)$$

for every $0 \le s \le t \le T$.

2. Continuous selectors of set-valued Itô processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a complete probability space satisfying usual hypotheses.

Theorem 1. *Let $\mathcal{Z}(\Phi, \mathcal{E}, B)$ be a set-valued Itô process corresponding to an n-dimensional $\mathcal{F}_t$-Brownian motion B on $(\Omega, \mathcal{F}, P)$ and convex valued $\mathcal{F}_t$-adapted set-valued process Φ, where $\mathcal{E}$ denotes $n\times n$-unit matrix, i.e., $\mathcal{E} = (\sigma_{ij})_{n\times n}$ with $\sigma_{ij} = 1$ for $i = j$ and $\sigma_{ij} = 0$ for $i \neq j$. Assume Φ is such that*

$$E \exp\left(\frac{1}{2}\int_0^T \|\Phi(t,\cdot)\|^2 dt\right) < \infty.$$

Then for every continuous $\mathcal{F}_t$-adapted stochastic process $x = (x_t)_{0\leq t\leq T}$ on $(\Omega, \mathcal{F}, P)$ such that $x_t - x_s \in \mathcal{Z}(\Phi, \mathcal{E}, B)([s,t])$ for $0 \leq s \leq t \leq T$ there is a probability measure Q^x on $\mathcal{F}$ such that

(i) *Q^x is equivalent to P*

(ii) *x is an n-dimensional $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, Q^x)$.*

Proof. As mentioned in the previous section, there exists $f^x \in S(\Phi)$ such that

$$dx_t = f_t^x dt + dB_t; \qquad t \in [0,T] \tag{8}$$

and such that $E(\exp\frac{1}{2}\int_0^T |f_t^x|^2 dt) < \infty$. Now, similarly as in the proof of Girsanov's theorem ([4], Theorem 8.6.3) we can put

$$M_t = \exp\left(-\int_0^t f_\tau^x dB_\tau - \frac{1}{2}\int_0^t |f_\tau^x|^2 d\tau\right)$$

for $t \in [0,T]$ and define the measure Q^x on $\mathcal{F}_T$ such that $dQ^x = M_T dP$. It follows immediately from formula (8) and Theorem 8.6.3 of [4] that x is an n-dimensional $\mathcal{F}_t$-Brownian motion with respect to the probability measure Q^x. ■

Theorem 2. *Let* $\mathcal{Z}(\Phi,\Psi,B)$ *be a set-valued Itô process corresponding to an* m*-dimensional* $\mathcal{F}_t$*-Brownian motion* B *and* $\mathcal{F}_t$*-adapted set-valued stochastic processes* $\Phi : [0,T]\times\Omega \to \mathrm{Conv}(\mathbb{R}^n)$ *and* $\Psi : [0,T]\times\mathbb{R}^n \to \mathrm{Conv}(\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m))$. *Suppose* Φ *and* Ψ *are such that there are* $u \in \mathcal{L}_2^m(\mathcal{F}_t)$ *and* $\alpha \in \mathcal{L}_1^n(\mathcal{F}_t)$ *such that* $\Psi(t,\omega)\cdot u(t,\omega) = \Phi(t,\omega) - \alpha(t,\omega)$ *for* $(t,\omega) \in [0,T]\times\Omega$ *and* $E[\exp(\frac{1}{2}\int_0^T |u(t,\cdot)|^2 dt) < \infty$. *Then for every continuous* $\mathcal{F}_t$*-adapted stochastic process* $x = (x_t)_{0\le t\le T}$ *on* $(\Omega,\mathcal{F},P)$ *such that*

$$x_t - x_s \in \mathcal{Z}(\Psi\cdot u + \alpha, \Psi, B)([s,t])$$

for every $0 \le s \le t \le T$, *there is a probability measure* Q^x *on* $\mathcal{F}$ *such that*

(i) Q^x *is equivalent to* P,

(ii) $\widetilde{B}_t(\omega) = \int_0^t u(\tau,\omega)d\tau + B(t)$; $t \in [0,T]$, *is an* $\mathcal{F}_t$*-Brownian motion on* $(\Omega,\mathcal{F},Q^x)$,

(iii) $M_t =: x_t - x_0 - \int_0^t u(\tau,\cdot)d\tau$; $t \in [0,T]$ *is an* $\mathcal{F}_t$*-martingale on* $(\Omega,\mathcal{F},Q^x)$,

(iv) $\mathcal{Z}(\Phi,\Psi,B)([s,t]) = \int_s^t u(\tau,\cdot)d\tau + \int_s^t \Psi(\tau,\cdot)d\widetilde{B}_\tau$ *for* $0 \le s \le t \le T$.

Proof. Similarly as in the proof of Theorem 4 from [2] we can verify that $x_t - x_s \in \mathcal{Z}(\Psi\cdot u + \alpha, \Psi, B)([s,t])$ for every $0 \le s \le t \le T$, implies the existence of $g^x \in S(\Psi)$ such that $x_t - x_s = \int_s^t (g^x\cdot u + \alpha)_\tau d\tau + \int_s^t g_\tau^x dB_\tau$ for $0 \le s \le t \le T$. Hence it follows that

$$dx_t = f_t^x dt + g_t^x dB_t \tag{9}$$

with $f_t^x - \alpha_t = g_t^x \cdot u_t$ on Ω for $t \in [0,T]$. Therefore, by virtue of Theorem 8.6.4 from [4] there is a probability measure Q^x on $\mathcal{F}$ such that:

$$dQ^x = M_T dP,$$

where

$$M_T = \exp\left(-\int_0^t u_\tau dB_\tau - \frac{1}{2}\int_0^t |u_\tau|^2 d\tau\right).$$

Hence, in particular it follows that Q^x is equivalent to P. Furthermore, by Theorem 8.6.4 from [4] it follows that $\widetilde{B}_t = \int_0^t u_\tau d\tau + B_t$ for $t \in [0,T]$ is an $\mathcal{F}_t$-Brownian motion on $(\Omega,\mathcal{F},Q^x)$. Therefore, in particular, we have

$d\widetilde{B}_t = u_t dt + dB_t$. Hence and by (9) it follows that

$$dx_t = g_t^x u_t dt + \alpha_t dt + g_t^x dB_t = \alpha_t dt + g_t^x (u_t dt + dB_t) = \alpha_t dt + g_t^x d\widetilde{B}_t$$

for $t \in [0, T]$. Then

$$x_t - x_0 - \int_0^t \alpha_\tau d\tau = \int_0^t g_\tau^x d\widetilde{B}_\tau; \qquad 0 \le t \le T.$$

Thus $x_t - x_0 - \int_0^t \alpha_\tau d\tau$ is an $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, Q^x)$. But $\mathcal{Z}(\Phi, \Psi, B)\ ([s,t]) = \{\int_s^t (g^x \cdot u + \alpha)_\tau d\tau + \int_s^t g_\tau^x dB_\tau : g^x \in S(\Psi)\}$ for fixed $0 \le s \le t \le T$. Denoting

$$y_r^g = y_0^g + \int_0^r (g^x \cdot u + \alpha)_\tau d\tau + \int_0^r g_\tau^x dB_\tau$$

for $r \in [s, t]$ we get

$$\begin{aligned} dy_r^g &= g_r^x u_r dr + \alpha_r dr + g_r^x dB_r = \alpha_r dt + g_r^x (u_r dr + dB_r) \\ &= \alpha_r dr + g_r^x d\widetilde{B}_r; \quad r \in [s, t]. \end{aligned}$$

Therefore,

$$y_t^g - y_s^g = \int_s^t \alpha_\tau d\tau + \int_s^t g_r d\widetilde{B}_\tau \in \int_s^t \alpha_\tau d\tau + \int_s^t \Psi_\tau d\widetilde{B}_\tau$$

for fixed $0 \le s \le t \le T$ and $g^x \in S(\Psi)$.
Therefore

$$\mathcal{Z}(\Phi, \Psi, B)([s,t]) \subset \int_s^t \alpha_\tau d\tau + \int_s^t \Psi(\tau, \cdot) d\widetilde{B}_\tau.$$

for $0 \le s \le t \le T$. It is easy to see that we also have

$$\int_s^t \alpha_\tau + \int_s^t \Psi_\tau(\tau, \cdot) d\widetilde{B}_\tau \subset \mathcal{Z}(\Phi, \Psi, B)([s,t])$$

for $0 \le s \le t \le T$. ∎

3. Stochastic differential inclusion

Let us consider the stochastic differential inclusion of the from:

$$x_t - x_s \in \int_s^t F(\tau, x_\tau) d\tau + \int \mathcal{E} \cdot dB_\tau \tag{10}$$

for $0 \le s \le t \le T$, where $\mathcal{E}$ is the $n \times n$ – unit matrix.

Theorem 3. *Let $B = (B_t)_{0 \le t \le T}$ be an n-dimensional $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, P)$ and assume $F : [0,T] \times \mathrm{I\!R}^n \to \mathrm{Conv}(\mathrm{I\!R}^n)$ is measurable and square intergrably bounded. Then*

(i) *for every solution $x = (x_t)_{0 \le t \le T}$ to (10) there is a probability measure Q^x on $\mathcal{F}$, equivalent to P and such that x is an n-dimensional $\mathcal{F}_t$-Brownian montion on $(\Omega, \mathcal{F}, Q^x)$,*

(ii) *for every $\mathcal{F}_0$-measurable random variable $\eta : \Omega \to \mathrm{I\!R}^n$ and every $f \in S(F \circ (\eta + B))$ the system $\{(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \widetilde{P}), x, \widetilde{B}\}$ with $x_t = \eta + B_t, d\widetilde{P} = M_T dP$, where $M_t = \exp[\int_s^t f_\tau dB_\tau - \frac{1}{2}\int_s^t |f_\tau|^2 d\tau]$ and $\widetilde{B}_t = B_t - \int_0^t f_\tau d\tau$ is a weak solution to (10) with an initial distribution equal to the distribution P^η of η.*

Proof. Let us observe that every solution x to (10) is $\mathcal{F}_t$-adapted. Then $(F \circ x)$ is $\mathcal{F}_t$-adapted on $[0,T] \times \Omega$ with convex values. Furthermore, there is $m \in L^2([0,T], \mathrm{I\!R})$ such that $\|F(t,x)\| \le m(t)$ for $t \in [0,T]$ and $x \in \mathrm{I\!R}^n$. Therefore $\int_0^\tau \|F(t,x_t)\|^2 dt \le \int_0^T m^2(t) dt < +\infty$ with $(P.1)$. This implies that

$$E \exp\left(\frac{1}{2}\int_0^T \|F(t,x_t)\|^2 dt\right) \le \exp\left(\frac{1}{2}\int_0^T m^2(t) dt\right).$$

Therefore, by virtue of Theorem 1 there is a probability measure Q^x on $\mathcal{F}$ such that condition (i) is satisfied.

If $\eta : \Omega \to \mathrm{I\!R}^n$ is $\mathcal{F}_0$-measurable, then $x_t = \eta + B_t$ is $\mathcal{F}_t$-measurable for $t \in [0,T]$. Then $x = (x_t)_{0 \le t \le T}$ is continuous with an initial distribution $P^x = P^\eta$. Taking now $f \in S(F \circ x)$ we also obtain $E \exp\left(\frac{1}{2}\int_0^T |f_\tau|^2 d\tau\right) < +\infty$.

Let $M = (M_t)_{0\le t\le T}$ be defined by

$$M_t = \exp\left[\int_0^t f_\tau dB_\tau - \frac{1}{2}\int_0^t |f_\tau|^2 d\tau\right].$$

By Girsanov's theorem it follows that $d\widetilde{P} = M_T dP$ is a probability measure on $\mathcal{F}$ equivalent to P and such that $\widetilde{B}_t = B_t - \int_0^t f_\tau d\tau$ is an $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, \widetilde{P})$. Hence it follows that $x_t = \eta + B_t = \eta + \int_0^t f_\tau d\tau + \widetilde{B}_t = \eta + \int_0^t f_\tau d\tau + \int_0^t \mathcal{E}\cdot d\widetilde{B}_\tau$ for $t \in [0,T]$ is such that

$$x_t - x_s = \int_s^t f_\tau d\tau + \int_s^t \mathcal{E}\cdot d\widetilde{B}_\tau^f \in \int_s^t F(\tau, x_\tau)d\tau + \int_s^t \mathcal{E} d\widetilde{B}_\tau.$$

Then the system $\{(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \widetilde{P}), x, \widetilde{B}\}$ is a weak solution to (10) with an initial distribution P^η. ■

Corollary 1. *If F satisfies the asumptions of Theorem 3, then for every $\mathcal{F}_0$-measurable random variable $\eta : \Omega \to \mathbb{R}^n$ the set of all weak solutions to stochastic differential inclusion (5) with an initial distribution P^η corresponding to a fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ and an n-dimensional $\mathcal{F}_t$-Brownian motion $B = (B_t)_{0\le t\le T}$ starting with zero is defined by*

$$\mathcal{X}_\eta = \{(\eta + B_t, \widetilde{B}_t^f)_{0\le t\le T} : f \in S(F\circ(\eta + B))\}$$

with

$$\widetilde{B}_t^f = B_t - \int_0^t f_\tau d\tau;\quad 0 \le t \le T$$

or

$$\mathcal{X}_\eta = (\eta + B_t)_{0\le t\le T} \times \{(\widetilde{B}_t^f)_{0\le t\le T} : f \in S(F\circ(\eta + B))\}.$$

Moreover, it is a convex weakly compact subset of the space $C([0,T], L^2((\Omega, \mathcal{F}, \widetilde{P}), \mathbb{R}^n))$.

Theorem 4. *Let $\Phi : [0,T]\times\mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ and $G : [0,T]\times\mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^{n\times m})$ be measurable and such that*

(i) *G is square integrably bounded*

(ii) *there are* $u \in \mathcal{L}^2_m(\mathcal{F}_t)$ *and* $\alpha \in \mathcal{L}^1_n(\mathcal{F}_t)$ *measurable on* $[0,T] \times \Omega \times \mathbb{R}^n$ *and such that*

(a) $G(t,x) \cdot u(t,\omega,x) = F(t,x) - \alpha(t,\omega,x)$ *for* $(t,x) \in [0,T] \times \mathbb{R}^n$ *and* $\omega \in \Omega$,

(b) $E[\exp(\frac{1}{2}\sup_{x\in\mathbb{R}^n}\int_0^T |u(t,\cdot,x)|^2 dt)] < +\infty$.

Let $B = (B_t)_{0\le t\le T}$ *be an* m*-dimensional Brownian motion. Then for every solution* $x = (x_t)_{0\le t\le T}$ *to the stochastic differential inclusion*

$$x_t - x_s \in \int_s^t F(\tau, x_\tau)d\tau + \int_s^t G(\tau, x_\tau)dB\tau \tag{11}$$

for $0 \le s \le t \le T$ *there is a probability measure* Q^x *on* $\mathcal{F}$ *equivalent to* P *such that* $M^x = (M^x_t)_{0\le t\le T}$ *with* $M^x_t = x_t - x_0 - \int_0^t u(\tau,\cdot,x_t)d\tau$ *is an* $\mathcal{F}_t$*-martingale on* $(\Omega,\mathcal{F},Q^x)$*. Furthermore, if* $\alpha \in \mathcal{L}^2_n(\mathcal{F}_t), \mathcal{A} = \{M^x : x \in \mathcal{X}(F,G,B)\}$ *where* $\mathcal{X}(F,G,B)$ *denotes the set of all solution to* (11) *and* $\mathcal{M}^2(\mathcal{A}) = \{Q^x : x \in \mathcal{X}(F,G)\}$*, then* $\mathcal{M}^2(\mathcal{A})$ *is a convex set.*

Proof. The existence of a probability measure Q^x corresponding to a solution $x \in \mathcal{X}(F,G,B)$ such that M^x is an $\mathcal{F}_t$-martintgale on $(\Omega,\mathcal{F},Q^x)$ follows from Theorem 2.

If $\alpha \in \mathcal{L}^2_n(\mathcal{F}_t)$, then M^x is a square integrable martingale. Then for every $x \in \mathcal{X}(F,G,B)$ one has $Q^x \sim P, Q^x = P$ on $\mathcal{F}_0$ and M^x is square intergrable $\mathcal{F}_t$-martingale on $(\Omega,\mathcal{F},Q^x)$. Therefore, by virtue of ([5], p. 151), the set $\mathcal{M}^2(\mathcal{A})$ is convex. ∎

REFERENCES

[1] F. Hiai, *Multivalued stochastic integrals and stochastic inclusions*, not published.

[2] M. Kisielewicz, *Set-valued stochastic integral and stochastic inclusions*, Stoch. Anal. Appl. **15** (5) (1997), 783–800.

[3] M. Kisielewicz, Differential Inclusions and Optimal Control, Kluwer Acad. Publ. Dordrecht, Boston, London 1991.

[4] B. Øksendal, Stochastic Differential Equations, Springer Verlag, Berlin, Heildelberg 1998.

[5] Ph. Proter, Stochastic Integration and Differential Equations, Springer Verlag, Berlin, Heidelberg 1990.

Received 5 February 2003