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VARIATIONAL INEQUALITIES IN NONCOMPACT NONCONVEX REGIONS

Ching-Yan Lin

Center for General Education
National Taipei University of Technology
Taipei, Taiwan, Republic of China

and

Liang-Ju Chu

Department of Mathematics
National Taiwan Normal University
Taipei, Taiwan, Republic of China

Abstract

In this paper, a general existence theorem on the generalized variational inequality problem $GVI(T, C, \phi)$ is derived by using our new versions of Nikaidô's coincidence theorem, for the case where the region C is noncompact and nonconvex, but merely is a nearly convex set. Equipped with a kind of V_0-Karamardian condition, this general existence theorem contains some existing ones as special cases. Based on a Saigal condition, we also modify the main theorem to obtain another existence theorem on $GVI(T, C, \phi)$, which generalizes a result of Fang and Peterson.

Keywords: Nikaidô's coincidence theorem, variational inequality, nearly convex, V_0-Karamardian condition, Saigal condition, acyclic multifunction, algebraic interior, bounding points.

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1. Introduction and preliminaries

In 1959, Nikaidô established the following remarkable coincidence theorem, by using a result of Begle [1, 2, 3] plus the outline of the Knaster-Kuratowski-Mazurkiewicz [11] proof of Brouwer's fixed point theorem.

Nikaidô's Coincidence Theorem [14, Theorem 3]**.** *Let M be a compact Hausdorff topological space, N a finite-dimensional compact convex set, and σ and τ continuous functions from M to N. If τ is onto and the inverse image $\tau^{-1}(q)$ is acyclic for each $q \in N$ (such a function τ is called a Vietoris map), then there exists some $p \in M$ such that $\sigma(p) = \tau(p)$.*

In [6], we extend the following two new versions of Nikaidô's coincidence theorem using different approaches.

The First Version of Nikaidô's Coincidence Theorem. *Let M be a nonempty compact convex subset of an LC space X, N a nonempty subset of a Hausdorff topological space Y, and σ and τ continuous functions from M to N. If τ is a Vietoris map, then there exists some $p \in M$ such that $\sigma(p) = \tau(p)$.*

The Second Version of Nikaidô's Coincidence Theorem. *Let M be a nonempty compact subset of a Hausdorff topological space X, N a nonempty convex subset of a locally convex topological vector space Y, and σ and τ continuous functions from M to N. If τ is a Vietoris map, then there exists some $p \in M$ such that $\sigma(p) = \tau(p)$.*

The main purpose of the present paper is to deduce several generalized key results of variational inequalities based on the above very powerful results, together with some coercitive property. Indeed, we shall simplify and reformulate the existence theorem of generalized variational inequalities in noncompact regions, which need not be convex, but is merely a nearly convex set. Beyond the realm of monotonicity nor metrizability, the results derived here generalize and unify various earlier ones from the classic optimization theory.

For a subset C of a topological space X, and a multifunction $T : X \longrightarrow 2^Y$ and a single-valued function $\phi : C \times C \times D \longrightarrow R$, we define the *generalized variational inequality problem* to be

$$\mathbf{GVI}(\mathbf{T}, \mathbf{C}, \phi) : \text{ Find } \overline{x} \in C \text{ and } \overline{y} \in T(\overline{x}) \text{ such that } \phi(x, \overline{x}, \overline{y}) \geq 0, \ \forall\, x \in C.$$

In particular, if $\phi(z,x,y) := \langle z-x, y\rangle,\ \forall\ (z,x,y) \in C\times C\times D$, the problem $GVI(T,C,\phi)$ reduces to the usual variational inequality $VI(T,C)$.

We digress briefly now to introduce the notation and review some definitions. Suppose that C and D are subsets of topological spaces X and Y, respectively. The *interior, relative interior, closure*, and *convex hull* of a set C will be denoted by $intC, riC, clC$, and coC, respectively. In this paper, we shall use *LC spaces* to indicate the class of locally convex Hausdorff topological vector spaces. A *multifunction* T from C to D, written as $T : C \longrightarrow 2^D$, is simply a function which assigns to each point x of C a (possibly empty) subset $T(x)$ of D. The *domain, range, graph* and *inverse* of T are defined, respectively, by

$$\begin{aligned} D(T) &:= \{x \in C;\ T(x) \neq \emptyset\}, \\ R(T) &:= \{y \in D;\ y \in T(x) \text{ for some } x \in D(T)\}, \\ G(T) &:= \{(x,y) \in C \times D;\ y \in T(x)\}, \end{aligned}$$

and

$$T^{-1}(y) := \{x \in C;\ y \in T(x)\}.$$

A multifunction $T : C \longrightarrow 2^D$ is *upper semicontinuous at* x provided for each open set V containing $T(x)$, there exists an open set U containing x such that $T(y)$ is contained in V whenever y is in U. We shall say T is *upper semicontinuous* (u.s.c.) provided T is u.s.c. at each x. T will be called to be *acyclic* provided T is u.s.c. and $T(x)$ is acyclic for each x. Here, an *acyclic* space is a nonempty compact Hausdorff path connected topological space whose n-th homology group is zero for each $n = 1, 2, 3, ...$ Homology, taken over any fixed field of coefficients, is in terms of either Vietoris or Čech cycles, as in Begle [1, 2, 3]. For example, any nonempty compact convex set and any compact contractible space are acyclic.

2. EXISTENCE THEOREMS ON $GVI(T,C,\phi)$

In this section, we shall use our extensions of Nikaidô's coincidence theorem to deal with the existence theorem on $GVI(T,C,\phi)$ for the case where C is noncompact and nonconvex, but merely a nearly convex set. We shall say that a subset C of X is *nearly convex* if for every compact subset K of C

and every neighborhood V of the origin of X, there is a continuous mapping $h : K \longrightarrow C$ such that

$$x - h(x) \in V, \ \forall \ x \in K, \tag{2.1}$$

and $h(K)$ is contained in some convex subset of C. Trivially, all convex sets are nearly convex. Other related nonconvex sets are the following. A subset C of X is said to be *almost convex* (in the sense of Minty [13]), if $ri(coC) \subset C$. The set C is *virtually convex* (in the sense of Rockafellar [15]), if for every compact subset K of coC and every neighborhood V of the origin in X, there exists a continuous mapping $h : K \longrightarrow C$ satisfying (2.1). This condition implies that C is *finitely convex* (in the sense on Halkin [8]); that is, for every finite subset P of C and every neighborhood V of the origin in X, there exists a continuous mapping $h : coP \longrightarrow C$ such that

$$x - h(x) \in V, \ \forall \ x \in coP.$$

It is clear that nearly convex sets may not be convex. However, under some mild conditions, the class of nearly convex sets also contains almost convex sets, virtually convex sets, and finitely convex sets. Furthermore, we have some relations as follows.

Proposition 2.1. *If C is almost convex with $ri(coC) \neq \emptyset$, then C is nearly convex. Conversely, any closed nearly convex set C in an LC space X is almost convex.*

Proof. Let $z \in ri(coC)$. Notice that for any compact subset K of C and any neighborhood V of the origin in X, there is $\lambda \in (0, 1]$ small enough that

$$\lambda(x - z) \in V, \ \forall \ x \in K.$$

Define $h : K \longrightarrow C$ by

$$h(x) := (1 - \lambda)x + \lambda z, \ \forall \ x \in K.$$

Then h is continuous and $x - h(x) = \lambda(x - z) \in V, \ \forall \ x \in K$. Notice that $x \in K \subset coC$ and $z \in ri(coC)$. Hence,

$$h(x) = (1 - \lambda)x + \lambda z \in ri(coC), \ \forall \ x \in K.$$

Since C is almost convex, it follows that

$$h(K) \subset ri(coC) \subset C.$$

This shows that C is nearly convex. Conversely, let C be closed and nearly convex. Notice that for any $z \in coC$, we may write $z = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$, where $x_i \in C$ and $\lambda_i \in [0,1]$, $i = 1, 2, 3, \cdots, n$. Let $K := \{x_1, x_2, \cdots, x_n\}$. For any neighborhood V of the origin in X, we may assume without loss of generality that V is convex, since X is locally convex. Then, by the definition of near convexity of C, we have a continuous mapping $h : K \longrightarrow C$ such that

$$x - h(x) \in V, \ \forall \ x \in K,$$

and $coh(K) \subset C$. Thus.

$$z = \sum_{i=1}^{n} \lambda_i x_i \in \sum_{i=1}^{n} \lambda_i \left(h(x_i) + V\right) = \sum_{i=1}^{n} \lambda_i h(x_i) + V \subset coh(K) + V \subset C + V.$$

It follows that

$$coC \subset C + V \ \text{ for all neighborhood } V \text{ of the origin in } X.$$

This yields that

$$ri\ coC \subset coC \subset \bigcap_{V} (C + V) = clC = C,$$

and hence C is almost convex. ∎

Proposition 2.2. *If X is finite-dimensional and C is nonempty virtually convex, then C is nearly convex. Conversely, any closed nearly convex set C, with $ri(coC) \neq \emptyset$, in an LC space X is virtually convex.*

Proof. Since C is nonempty in a finite-dimensional topological vector space, we have $ri\ coC \neq \emptyset$. So by Proposition 1, it is sufficient to show that C is almost convex. Without loss of generality, we may assume that the affine hull of C is the whole space, so that $ri(coC) = int(coC)$. For any $z \in int(coC)$, we can choose $r > 0$ so small that

$$z \in B(z, r) \subset coC,$$

where $B(z,r)$ denotes the closed ball of radius r around z in X. Let V be a neighborhood of the origin in X such that $z+V \subset B(z,r)$. Since $B(z,r)$ is a compact subset of coC and C is virtually convex, there is a continuous mapping $h : B(z,r) \longrightarrow C$ such that

$$x - h(x) \in V, \ \forall\, x \in B(z,r).$$

Define $f : B(z,r) \longrightarrow X$ by

$$f(x) = z + x - h(x), \ \forall\, x \in B(z,r).$$

Then f is continuous from $B(z,r)$ to $B(z,r)$, since

$$f(x) = z + x - h(x) \in z + V \subset B(z,r), \ \forall\, x \in B(z,r).$$

Since $B(z,r)$ is a compact convex subset of X, by Brouwer's fixed point theorem, there exists some $x \in B(z,r)$ such that $f(x) = x$. It follows that

$$z = f(x) - x + h(x) = h(x) \in C\,.$$

This yields that $int(coC) \subset C$, and hence C is almost convex. Thus, the assertion follows from Proposition 2.1. Conversely, let C be closed and nearly convex, with $ri(coC) \neq \emptyset$. By Proposition 2.1, C is almost convex. Thus, we have $ri(coC) \subset C$. Let $z \in ri(coC)$. Then for any compact subset K of coC and any neighborhood V of the origin in X, there is $\lambda \in (0,1]$ small enough that

$$\lambda(x - z) \in V, \ \forall\, x \in K.$$

Notice that

$$(1-\lambda)x + \lambda z \in ri(coC) \subset C, \ \forall\, x \in K.$$

Thus, the mapping $h : K \longrightarrow C$, defined by

$$h(x) := (1-\lambda)x + \lambda z, \ \forall\, x \in K,$$

is continuous, and $x - h(x) = \lambda(x-z) \in V, \ \forall\, x \in K$. This shows that C is virtually convex. ■

Before proceeding with our main result, let us first define a variant Karamardian condition [9, 10, 16]. Suppose that $T : X \longrightarrow 2^Y$ is a multifunction and $\phi : X \times X \times Y \longrightarrow R$ is a single-valued function. For a neighborhood V_0 of the origin of X, we shall say that T and ϕ satisfy the V_0*−Karamardian condition* on C, if there exists a compact convex subset K of C, such that for each $x \in (C + V_0)\backslash K$ and $y \in T(x)$, there exists some $s \in K$ such that $\phi(s, x, y) < 0$.

Remark. When C is convex, we may take $V_0 = \{0\}$, so that the V_0-Karamardian condition reduces to the usual Karamardian condition. In fact, we say that T and ϕ satisfy the *Karamardian condition* on C, if there exists a compact convex subset K of C, such that for each $x \in C\backslash K$ and $y \in T(x)$, there is some $s \in K$ satisfying $\phi(s, x, y) < 0$.

We now establish our main result, which is a general version of Chan and Pang [5, Theorem 3.1]. Also, the result derived here generalizes Lin [12] and some results in [7, 10].

Theorem 2.3. *Let C be a nonempty nearly convex subset of an LC space X, and Y a topological space. Suppose that $T : X \longrightarrow 2^Y$ is an acyclic multifunction in $C + V$ for some neighborhood V of the origin in X, and $\phi : X \times X \times Y \longrightarrow R$ is a continuous function satisfying*

(i) $\phi(x, x, y) \geq 0, \ \forall \ (x, y) \in G(T)$,

(ii) *for each $(x, y) \in G(T)$, the map $z \mapsto \phi(z, x, y)$ is quasiconvex.*

If T and ϕ satisfy the V_0-Karamardian condition on C for some neighborhood V_0 of the origin in X, then there is a solution to $GVI(T, C, \phi)$.

Proof. Observe that for any convex neighborhood U of the origin in X with $U \subset V \cap V_0$, the maps T and ϕ also satisfy the U-Karamardian condition on C. Hence, without loss of generality, we may assume V_0 is convex and T is acyclic in $C + V_0$. For each $z \in C$, we define

$$A(z) := \{(x, y) \in G(T); \ \phi(z, x, y) \geq 0\}.$$

To complete the proof, we need to show that

$$\bigcap\{A(z); \ z \in C\} \neq \emptyset. \tag{2.2}$$

We first consider the case where $G(T)$ is compact. By (i), each $A(z)$ is nonempty and compact. Thus, to prove (2.2), it will suffice to show just that

$$\bigcap\{A(z_i);\ i \in I\} \neq \emptyset, \tag{2.3}$$

for any finite subset $\{z_i;\ i \in I\}$ of C. Assume now that there is a finite subset $\{z_i;\ i \in I\}$ of C such that

$$\bigcap\{A(z_i);\ i \in I\} = \emptyset. \tag{2.4}$$

Since C is nearly convex, for the compact subset $H := \{z_i;\ i \in I\}$ of C and every neighborhood V of the origin in X, there is a continuous mapping $h_V : H \longrightarrow C$ such that

$$z_i - h_V(z_i) \in V,\ \forall\, i \in I,$$

and $coh_V(H) \subset C$. We now show that there exists a neighborhood U of the origin such that

$$\bigcap\{A(h_U(z_i));\ i \in I\} = \emptyset.$$

Assume that for any neighborhood V of the origin,

$$\bigcap\{A(h_V(z_i));\ i \in I\} \neq \emptyset.$$

Then there is some $(x_V, y_V) \in A(h_V(z_i))$ for each $i \in I$. This implies that $(x_V, y_V) \in G(T)$ and $\phi(h_V(z_i), x_V, y_V) \geq 0$ for each $i \in I$. Since $G(T)$ is compact, there exists a subnet of (x_V, y_V) converging to some $(\bar{x}, \bar{y})$ as V tends toward $\{0\}$. Without loss of generality, we may let (x_V, y_V) converge to $(\bar{x}, \bar{y}) \in G(T)$, since $G(T)$ is closed. It follows from the upper semicontinuity of ϕ that

$$\phi(s, \bar{x}, \bar{y}) \geq \overline{\lim_{V}}\ \phi(h_V(s), x_V, y_V) \geq 0,\ \forall\, s \in H.$$

It follows that

$$(\bar{x}, \bar{y}) \in \bigcap\{A(z_i);\ i \in I\}.$$

This contradicts (2.4), and hence there exists a neighborhood U of the origin such that

$$\bigcap\{A(h_U(z_i));\ i \in I\} = \emptyset. \tag{2.5}$$

For the sake of convenience, we denote h_U by h, and for each $p := (x, y) \in G(T)$, we define

$$f_I(p) := \min\{\phi(h(z_i), x, y);\ i \in I\}.$$

Then $f_I(p) < 0$ and the term

$$\epsilon := \inf\{-f_I(p);\ p \in G(T)\} > 0,$$

since f_I is u.s.c. on the compact set $G(T)$. Therefore, for each $p \in G(T)$ there exists some $i \in I$ such that

$$\phi(h(z_i), p) = f_I(p) < \epsilon + f_I(p) \leq 0.$$

Let

$$\theta_i(p) := \max\{0, \epsilon + f_I(p) - \phi(h(z_i), p)\}.$$

Thus, the set $J_p := \{i \in I;\ \theta_i(p) > 0\}$ is nonempty and the formula

$$\theta(p) := \sum_{i \in I} \frac{\theta_i(p)}{\sum_{j \in I} \theta_j(p)} h(z_i)$$

specifies a well-defined continuous function from $G(T)$ to the convex hull coH of the finite set H. On the other hand, let α be the natural projection of the graph $G(T)$ onto C. That is, for any $p := (x, y) \in G(T)$ we have $\alpha(p) := x$. Notice that the projection α maps $M := G(T) \cap (coh(H) \times T(coh(H))$ into $coh(H)$ continuously, with $\alpha^{-1}(x) = \{x\} \times T(x)$, an acyclic subset of M for each $x \in coh(H) \subset C$. Therefore, Nikaidô's coincidence theorem yields some $\bar{p} := (\bar{x}, \bar{y}) \in M$ such that $\theta(\bar{p}) = \alpha(\bar{p})$. Since $J_{\bar{p}}$ is nonempty, for any $i \in J_{\bar{p}}$, we have $\theta_i(\bar{p}) > 0$ and

$$-f_I(\bar{p}) \geq \epsilon > \phi(h(z_i), \bar{p}) - f_I(\bar{p}).$$

From this we have

$$\phi(h(z_i), \bar{p}) < 0, \ \forall\, i \in J_{\bar{p}}. \tag{2.6}$$

Notice that the summation in $\theta(\bar{p})$ can be taken just over $J_{\bar{p}}$. It follows that

$$\bar{x} = \alpha(\bar{p}) = \theta(\bar{p}) \in co\{h(z_i);\ i \in J_{\bar{p}}\} \subset C.$$

Since the function $\phi(\cdot, \bar{y})$ is quasiconvex on C, by (2.6) we then have

$$\phi(\bar{x}, \bar{x}, \bar{y}) \leq \max\{\phi(h(z_i), \bar{x}, \bar{y});\ i \in J_{\bar{p}}\} < 0.$$

This contradicts with the condition (i), and hence (2.3) holds.

Next, we consider the general case where $G(T)$ is not compact. Let $\{z_i;\ i \in I\}$ be any finite subset of C, and define $H := K \bigcup \{z_i;\ i \in I\}$, where K is a compact convex subset of C, defined in the V_0-Karamardian condition. Since C is nearly convex, for the compact subset H of C, there is a continuous mapping $h : H \longrightarrow C$ such that

$$x - h(x) \in V_0,\ \forall\, x \in H,$$

and $coh(H) \subset C$. Then the set $L := coH$ is a compact convex subset of $C + V_0$. In fact, we have

$$H \subset h(H) + V_0 \subset coh(H) + V_0.$$

Since $coh(H) + V_0$ is convex, we obtain

$$L = coH \subset coh(H) + V_0 \subset C + V_0.$$

For each $z \in C$, we define

$$A_L(z) := \{(x, y) \in G(T_L);\ \phi(z, x, y) \geq 0\},$$

where $T_L : L \longrightarrow 2^{T(L)}$ is the restriction of T to the nonempty compact convex subset L of X. Then T_L is an acyclic multifunction on L, and its graph $G(T_L) = G(T)\bigcap(L \times T(L))$ is compact. By a similar argument (after (2.5)), and applying the second version of Nikaidô's coincidence theorem to $M := G(T_L)$ and $N := L$, we have

$$\bigcap\{A_L(x_j);\ j \in J\} \neq \emptyset, \tag{2.7}$$

for any finite subset $\{x_j;\ j \in J\}$ of L. In view of compactness of each set $A_L(z)$ for $z \in L$, we conclude that

$$\bigcap\{A_L(z);\ z \in L\} \neq \emptyset.$$

That is, there exists some $(\bar{x}, \bar{y}) \in G(T_L)$ such that

$$\phi(z, \bar{x}, \bar{y}) \geq 0,\ \forall\ z \in L.$$

In particular, we have

$$\phi(s, \bar{x}, \bar{y}) \geq 0,\ \forall\ s \in K, \tag{2.8}$$

and

$$\phi(z_i, \bar{x}, \bar{y}) \geq 0,\ \forall\ i \in I. \tag{2.9}$$

Now, we claim that $\bar{x} \in K$. Suppose this were not true. Since $(\bar{x}, \bar{y}) \in G(T_L)$, $\bar{x} \in C + V_0$. Therefore, $\bar{x} \in (C + V_0)\backslash K$. By applying the V_0-Karamardian condition, we have some $s \in K$ such that $\phi(s, \bar{x}, \bar{y}) < 0$, which contradicts (2.8). The contradiction shows that $\bar{x} \in K$. Consequently, by (2.9) we obtain

$$(\bar{x}, \bar{y}) \in \bigcap\{A_K(z_i);\ i \in I\},$$

for any finite subset $\{z_i;\ i \in I\}$ of C. Since each $A_K(z)$ is compact for each $z \in C$, we conclude

$$\bigcap\{A_K(z);\ z \in C\} \neq \emptyset.$$

Hence (2.2) holds by the fact that $A_K(z) \subset A(z)$. Thus, we complete the proof. ■

Following the remark before Theorem 2.3, when C is convex, we can relax Theorem 2.3 to the case where T is acyclic on C and ϕ is continuous on $C \times C \times Y$. Indeed, we obtain the following result, which is a version of Saigal theorem [16].

Theorem 2.4. *Let C be a nonempty convex subset of an LC space X, and Y a topological space. Suppose that $T : C \longrightarrow 2^Y$ is acyclic and $\phi : C \times C \times Y \longrightarrow R$ is a continuous function satisfying*

(i) $\phi(x, x, y) \geq 0, \ \forall \ (x, y) \in G(T)$,

(ii) *for each $(x, y) \in G(T)$, the map $z \mapsto \phi(z, x, y)$ is quasiconvex.*

If T and ϕ satisfy the Karamardian condition on C, then there is a solution to $GVI(T, C, \phi)$.

Notice that the V_0-Karamardian condition requires a compact convex set K satisfying $\phi(z, x, y) < 0$. We next develop a condition on K such that the strict inequality can be replaced by $\phi(z, x, y) \leq 0$. For this, we introduce the concept of core and bounding points of K. Recall that for a subset A of X, the *algebraic interior* of A, denoted by *core* A, consists of all points a in A such that for all $b \in X \backslash \{a\}$ there exists some $x \in A \cap (a, b)$ such that $[a, x] \subset A$, where

$$(a, b) := \{(1 - \lambda)a + \lambda b; \ 0 < \lambda < 1\},$$

and

$$[a, x] := \{(1 - \lambda)a + \lambda x; \ 0 \leq \lambda \leq 1\}.$$

The points neither in *core* A nor in *core*$(X \backslash A)$ are called *bounding points* of A. We shall denote by ∂A the set consisting of all the bounding points of A. We remark that for any convex set C, the algebraic interior *core* C is again convex, and in general the following inclusion relations hold:

$$\text{int}\, C \subset \ core\ C, \quad A \subset \ core\ A \cup \partial A.$$

Lemma 2.5. *Let A and B be nonempty subsets of X such that $A \subset B$, and let $\phi : X \times X \times Y \longrightarrow R$ be a single-value function satisfying*

(i) $\phi(x, x, y) = 0, \ \forall \ (x, y) \in G(T)$,

(ii) *for each $(x, y) \in G(T)$, the map $z \mapsto \phi(z, x, y)$ is convex.*

If $\bar{x} \in coreA$, $\bar{y} \in T(\bar{x})$, and $\phi(z, \bar{x}, \bar{y}) \geq 0$, $\forall\ z \in A$, then $\phi(z, \bar{x}, \bar{y}) \geq 0$, $\forall\ z \in B$.

Proof. Let $x \in B$. If $x = \bar{x}$, then $\phi(z, \bar{x}, \bar{y}) = 0$. If $x \in B \backslash \{\bar{x}\}$, it follows from $\bar{x} \in coreA$ that there exists some $\lambda > 0$ sufficiently small so that $z := (1-\lambda)\bar{x} + \lambda x \in A$. Taking this z, we obtain $\phi((1-\lambda)\bar{x} + \lambda x), \bar{x}, \bar{y})$ ≥ 0. By conditions (i) and (ii), it follows that

$$\lambda\phi(x, \bar{x}, \bar{y}) = (1-\lambda)\phi(\bar{x}, \bar{x}, \bar{y}) + \lambda\phi(x, \bar{x}, \bar{y}) \geq \phi((1-\lambda)\bar{x} + \lambda x), \bar{x}, \bar{y}) \geq 0.$$

Thus, $\phi(z, \bar{x}, \bar{y}) \geq 0$ for all $x \in B$. ■

The following existence theorem for $GVI(T, C, \phi)$ generalizes a result of Fang and Peterson [7] which is based on Saigal condition [16]. Indeed, the mapping T they deal with is a u.s.c. multifunction from R^n to R^n whose images are nonempty compact contractible subsets of R^n.

Theorem 2.6. *Let C be a nonempty subset of an LC space X, and Y a topological space. Suppose that $T : C \longrightarrow 2^Y$ is an acyclic multifunction and $\phi : X \times X \times Y \longrightarrow R$ is a continuous function satisfying*

(i) $\phi(x, x, y) = 0$, $\forall\ (x, y) \in G(T)$,

(ii) *for each $(x, y) \in G(T)$, the map $z \mapsto \phi(z, x, y)$ is convex.*

It is assumed that there exists a nonempty compact convex subset K of C such that for each $x \in \partial K$, and $y \in T(x)$, there exists some $w \in coreK$ such that

$$\phi(w, x, y) \leq 0. \tag{2.10}$$

Then there exists a solution to $GVI(T, C, \phi)$.

Proof. Since K is a nonempty compact convex subset of C, it follows from Theorem 2.4 that the problem $GVI(T, K, \phi)$ has a solution. That is, there exist $\bar{x} \in K$ and $\bar{y} \in T(\bar{x})$ such that

$$\phi(z, \bar{x}, \bar{y}) \geq 0, \quad \forall\ z \in K. \tag{2.11}$$

Since $\bar{x} \in K \subset coreK \cup \partial K$, we conclude the proof by considering two possible cases.

Case 1. $\bar{x} \in coreK$. Applying Lemma 2.5 with $A := K$ and $B := C$, together with (2.11), yields a solution to $GVI(T, C, \phi)$.

Case 2. $\bar{x} \in \partial K$. It follows from (2.10) that there exists some $w \in coreK$ such that

$$\phi(w, \bar{x}, \bar{y}) \leq 0. \tag{2.12}$$

Combining (2.11) with (2.12) yields

$$\phi(w, \bar{x}, \bar{y}) = 0. \tag{2.13}$$

For any $x \in C\backslash\{\bar{x}\}$, it follows from the fact $w \in coreK$ that there exists some $\lambda > 0$ sufficiently small so that $z := (1-\lambda)w + \lambda x \in K$. Taking this z in (2.11), we obtain $\phi((1-\lambda)w + \lambda x), \bar{x}, \bar{y}) \geq 0$. By the condition (ii) and (2.13), it follows that

$$\lambda\phi(x, \bar{x}, \bar{y}) = (1-\lambda)\phi(w, \bar{x}, \bar{y}) + \lambda\phi(x, \bar{x}, \bar{y}) \geq \phi((1-\lambda)w + \lambda x), \bar{x}, \bar{y}) \geq 0.$$

Thus, $\phi(x, \bar{x}, \bar{y}) \geq 0, \ \forall \ x \in C$, and hence $\bar{x}$ is a solution of $GVI(T, C, \phi)$. ■

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