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# EQUILIBRIUM OF MAXIMAL MONOTONE OPERATOR IN A GIVEN SET

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**Abstract**

Sufficient conditions for an equilibrium of maximal monotone operator to be in a given set are provided. This partially answers to a question posed in [10].

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## 1 Introduction

Let $C$ be a convex subset of a real Banach space $E$ and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex proper lsc function. It is interesting question under what conditions

$$\inf_E f = \inf_C f \ \ ?$$

One of possibilities is to assume that $C$ is compact and the derivative of $f$ satisfies the following condition:

(1.1) for every $x \in E \setminus C$ there is $c \in C$ such that $f'(x; c - x) \leq 0$,

(see, e.g. [2] for $f$ being locally Lipshitzian with a compact set $C$ [1] for a vector case). If the infimum is attained in $C$ then

(1.2) $$0 \in \partial f(c) \text{ for some } c \in C.$$

Of course, when $C$ is singleton, $C = \{c\}$ then (1.1) imply that

$$\text{(1.3)} \qquad \begin{aligned} &\text{for every } \ x \in E,\ x \neq c, \ \text{and} \ x^* \in \partial f(x) \\ &\text{the inequality } \langle x^*, x - c\rangle \geq 0 \text{ holds true.} \end{aligned}$$

With the *Rockafellar maximal monotonicity theorem* at hand we are able to say that (1.1) (or (1.3)) implies (1.2) at once. Let us assume that $C$ is a weakly compact convex subset of $E$. S. Simons (see [8, 9, 10]) obtained the following Theorem.

**Theorem 11.** (C-c* Theorem) *If $C$ is a nonempty weakly compact convex subset of a real Banach space $E$, $c^* \in E^*$ and for all $(z, z^*) \in$ graph $\partial f$, there exists $c \in C$ such that $\langle z^* - c^*, z - c\rangle \geq 0$ then $(C \times \{c^*\}) \cap \text{graph } \partial f \neq \emptyset$.*

He also posed the question:

If $C$ is a nonempty weakly compact convex subset of a real Banach space $E$, $C^* \subset E^*$ is a nonempty weak$^*$ compact convex subset of $E^*$ and

$$\text{(1.4)} \qquad \begin{aligned} &\text{for all } (z, z^*) \in \text{ graph } \partial f, \text{ there exists } (c, c^*) \in C \times C^* \\ &\text{such that } \langle z^* - c^*, z - c\rangle \geq 0, \end{aligned}$$

does it follow that

$$\text{(1.5)} \qquad (C \times C^*) \cap \text{graph } \partial f \neq \emptyset \ ?$$

In particular, when $E = \mathbb{R}$ the answer is in the affirmative (see [12] or [10] for details). Unfortunately, when $E = \mathbb{R}^2$ we can construct a convex $C^1$ function for which (1.4) holds but (1.5) does not (see [12]). However, if additionally $C \perp C^*$, then (1.4) implies (1.5) (see [12]). It is quite natural to ask (see [8] or Open Question Section in [10], problem 25.9) whether we can take a maximal monotone operator, say $T$, instead of the subdifferential of convex function and to get (1.4) $\Longrightarrow$(1.5). Of course, in view of the above we need an additional assumption to get the implication, but at this moment we do not even know whether the following condition:

$$\text{(1.6)} \quad \text{for all } (z, z^*) \in \text{graph } T, \text{ there exists } c \in C \text{ such that } \langle z^*, z - c\rangle \geq 0,$$

implies

$$\text{(1.7)} \qquad (C \times \{0\}) \cap \text{graph } T \neq \emptyset.$$

Some others questions can be invoked too. Can the compactness assumption be relaxed? Does (1.4) imply (1.5) if $T$ is put instead of the subdifferential and $C \perp C^*$? Can we take others sets than the product $C \times C^*$?

Herein we deal with two of them. Namely, in Section 2 we get a partial answer to the question, when the condition (1.6) implies (1.7). In Section 3 we relax the compactness assumption having

$$\text{graph } \partial f \cap \Big((C + B(0,\varepsilon)) \times B(0,\varepsilon)\Big) \neq \emptyset \text{ for every } \varepsilon > 0, \tag{1.8}$$

where $B(0,\varepsilon)$ is the ball at the origin with the radius $\varepsilon$.

## 2 Maximal monotone operators

The main tool of our reasoning is the notion of subdifferential of convex function (we refer to [3] for details, see also [6]). One of advantages of subdifferential calculus is that it allows us to treat nondifferentiable objects in differentiable manner, so let us recall the notion. If $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex and finite at $x \in E$ then

$$\partial f(x) := \{x^* \in E^* \mid f(x+h) \geq f(x) + \langle x^*, h\rangle, \forall h \in E\}.$$

For any subset $C$ of $E$, $d_C(x)$ stands for the distance of $x$ from $C$ as usual. If $C$ is convex, then $d_C^2$ is a convex continuous function on $E$. Below several of properties of $d_C^2$ are gathered.

**Lemma 21.** *Let $E$ be a real normed space and $C$ be a convex subset of $E$, $C \neq \emptyset$. Then*

(1) $\forall c \in \text{cl}\, C, \quad \partial d_C^2(c) = \{0\}$, *where* cl *is the topological closure of* $C$.

(2) $\forall x \in E \setminus C$, $\forall c \in C$, $\forall x^* \in \partial d_C^2(x)$, $\langle x^*, x - c\rangle \geq d_C^2(x)$.

(3) $(\, d_C^2(x_n) \to 0$ *and* $x_n^* \in \partial d_C^2(x_n)) \implies \| x_n^* \| \to 0$.

(4) $\forall x \in E, \forall c \in C,\ d_C^2(x) \geq (\max\{0, \| x - c \| - \text{diam } C\})^2$.

***Proof.*** 1. Let $c \in \text{cl}\, C$ and $c^* \in \partial d_C^2(c)$. By the definition of the subdifferential we get

$$\forall t > 0,\ \forall h \in E, \quad \langle c^*, th\rangle \leq d_C^2(c + th) - d_C^2(c) \leq t^2 \| h \|^2,$$

thus

$$\forall h \in E, \quad \langle c^*, h\rangle \leq 0.$$

2. Let $x \in E \setminus C$, $x^* \in \partial d_C^2(x)$ and $c \in C$. For every $1 \geq t > 0$ we have

$$\begin{aligned}\langle x^*, t(c-x)\rangle &\leq d_C^2(x + t(c-x)) - d_C^2(x)\\ &\leq (1-t)d_C^2(x) + td_C^2(c) - d_C^2(x) = -td_C^2(x),\end{aligned}$$

thus $\langle x^*, (x-c)\rangle \geq d_C^2(x)$.

3. We have for all $h \in E$ and for all $t > 0$

$$\begin{aligned}\langle x_n^*, th\rangle &\leq d_C^2(x_n + th) - d_C^2(x_n)\\ &\leq (d_C(x_n+th) - d_C(x_n))(d_C(x_n+th) + d_C(x_n))\\ &\leq t \parallel h \parallel (d_C(x_n+th) + d_C(x_n)),\end{aligned}$$

hence $\langle x_n^*, h\rangle \leq 2 \parallel h \parallel d_C(x_n)$, which implies $\parallel x_n^* \parallel \leq 2d_C(x_n)$ and $\parallel x_n^* \parallel \to 0$, whenever $n \to 0$.

4. Let $x \in E$ and $c \in C$.
We have

$$d_C(x) = \inf_{c' \in C} \parallel x - c' \parallel \geq \inf_{c' \in C}(\parallel x - c \parallel - \parallel c - c' \parallel) \geq \parallel x - c \parallel - \operatorname{diam} C,$$

hence

$$d_C^2(x) \geq \{\max\{0, \parallel x - c \parallel - \operatorname{diam} C\}\}^2. \qquad \blacksquare$$

It follows from the proof of (1) that $d_C^2$ is Frechet differentiable at each $c \in \operatorname{cl} C$ with the derivative equal to 0. We know also that the subdifferential mapping is upper semicontinuous (norm-to-norm, see Lemma 2.6 of [3]) at such points, so (3) is a consequence of (1). However, herein for the sake of the reader convenience the proof is done directly. Let us also observe that by (2) and (4), the subdifferential $\partial d_C^2$ is a coercive operator, thus $T + \partial d_C^2$ is coercive as well. If $E$ were a reflexive Banach space then 0 would be in the range of $T + \partial d_C^2$ (see [11, Section 32.14, Corollary 32.35] for example). However, if $E$ is not reflexive then we neither know if $0 \in \operatorname{cl} R(T + \partial d_C^2)$ nor even whether $T + \partial d_C^2$ is maximal monotone operator, where $R(T + \partial d_C^2)$ stands for the range of the operator $T + \partial d_C^2$.

Below under the assumption that (1.6) holds true we provide a necessary and sufficient conditon for $0 \in T(C)$, whenever $T$ is a maximal monotone operator on a Banach space and $C$ is weakly compact convex and nonempty. This result has been obtained together with M. Przeworski (we refer also to [5] for the nonreflexive case and to [4] for the reflexive one).

**Theorem 22.** *Let $E$ be a real Banach space, $C \subset E$ be a weakly compact convex nonempty subset of $E$, and $T : E \rightrightarrows E^*$ be a maximal monotone operator with dom $T \neq \emptyset$. Then*

$$(1.6) \text{ holds and } 0 \in \operatorname{cl} R(T + \partial d_C^2) \iff 0 \in T(C)$$

***Proof.*** Of course, if $0 \in T(C)$ for some $c \in C$ then by (1) of Lemma 21 we get $\{0\} = \partial d_C^2(c)$ and

$$0 \in T(c) + \partial d_C^2(c) \subset \operatorname{cl} R(T + \partial d_C^2).$$

Condition (1.6) is a consequence of the monotonicity (keep in mind $(c, 0) \in$ graph $T$).

Let us consider the case $0 \in \operatorname{cl} R(T + \partial d_C^2)$. Then there are sequences $(x_n) \subset$ dom $T$ and $(t_n^*)$, $(y_n^*) \subset E^*$ such that

$$t_n^* \in T(x_n),\ y_n^* \in \partial d_C^2(x_n) \text{ for every } n \in \mathbb{N}$$

and $x_n^* = t_n^* + y_n^*$ tends to 0, whenever $n$ tends to $\infty$. By (1.6) there are $c_n \in C$ such that $\langle t_n^*, x_n - c_n \rangle \geq 0$ for every $n \in \mathbb{N}$, so by (2) of Lemma 21 we arrive at

$$(2.1) \quad \| x_n^* \| \| x_n - c_n \| \geq \langle t_n^* + y_n^*, x_n - c_n \rangle \geq d_C^2(x_n) \text{ for every } n \in \mathbb{N}.$$

Using (4) of Lemma 21 we get

$$\| x_n^* \| \| x_n - c_n \| \geq \Big\{ \max\{0, \| x_n - c_n \| - \operatorname{diam} C\} \Big\}^2 \text{ for every } n \in \mathbb{N}.$$

Since $\| x_n^* \| \to 0$, so the above inequality ensures the existence of $M \geq 0$ such that $\| x_n - c_n \| \leq M$ for every $n \in \mathbb{N}$, thus (2.1) implies $d_C^2(x_n) \to 0$, whenever $n \to \infty$. The set $C$ is weakly compact and $d_C^2$ is weakly lower semicontinuous (see Corollary 3.19 of [3]), so the Weierstrass theorem ensures the existence of $(\tilde{c_n}) \subset C$ such that

$$d_C(x_n) = \| x_n - \tilde{c_n} \|$$

for every $n \in \mathbb{N}$. By the Eberlein-Smulian therorem (see e.g. [7]) we are able to choose a subsequence of $(\tilde{c_n}) \subset C$ weakly converging to $\bar{c} \in C$. Without loss of generality we assume that $\tilde{c_n} \overset{\text{weakly}}{\rightharpoonup} \bar{c}$. Of course, $x_n \overset{\text{weakly}}{\rightharpoonup} \bar{c}$, since $\| x_n - \tilde{c_n} \| \to 0$. It follows from (3) of Lemma 21 that $y_n^* \longrightarrow 0$, so $t_n^* \to 0$. By the monotonicity of $T$ we have

$$\langle t_n^* - t^*, x_n - t \rangle \geq 0$$

for every $(t, t^*) \in$ graph $T$. Since $t_n^* \longrightarrow 0$, $x_n \overset{\text{weakly}}{\rightharpoonup} \bar{c}$, so

$$\langle t^*, t - \bar{c} \rangle \geq 0$$

for every $(t, t^*) \in$ graph $T$, which by the maximal monotonicity of $T$ implies $0 \in T(\bar{c})$. ∎

# 3 Convex function

In this section we shall show that (1.6) entails the existence of $(c_n) \subset C$ and $c_n^* \in T(c_n)$ such that $\| c_n^* \| \to 0$, whenever $C$ is convex closed and bounded, $T = \partial f$ and some mild additional assumptions on $f$ are imposed. This suggests that for some class of maximal monotone operators we are able to get positive answer to the question concerning the compactness assumption. In the proof of the result we need the following Corollary.

**Corollary 31.** *Let $g : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function and $C$, $S$ be subsets of a real Banach space $E$, $\varepsilon > 0$ be fixed. If for every sequence $(v_n) \subset S$ such that*

$$g(v_n) > g(v_{n+1}) + \varepsilon \min\{d_C^2(v_{n+1}), d_C^2(v_n)\} \| v_n - v_{n+1} \|$$

*for every $n \in \mathbb{N}$, there is $v \in S$ such that for some subsequence $(v_{n_k}) \subset (v_n)$*

$$g(v_{n_k}) > g(v) + \varepsilon \min\{d_C^2(v), d_C^2(v_{n_k})\} \| v_{n_k} - v \|$$

*for every $k \in \mathbb{N}$, then there is $\bar{v} \in S$ such that*

$$g(\bar{v}) \leq g(z) + \varepsilon \min\{d_C^2(\bar{v}), d_C^2(z)\} \| \bar{v} - z \|$$

*for every $z \in S$.*

***Proof.*** This is a straightforward consequence of Example 2.3 and Theorem 3.7 from [2]. ∎

Let us notice that if we introduce a preference relation $\succ \subseteq S \times S$ as follows

$$u \succ v \quad \Longleftrightarrow \quad g(u) > g(v) + \varepsilon \min\{d_C^2(v),\ d_C^2(u)\} \| v - u \|$$

then $\bar{v}$, which existence is ensured by the above Corollary, is $(\succ, \succ^*)$-maximal element of $S = S(g, \alpha) := \{x \in E \mid g(x) \leq \alpha\}$ for any $(w, \alpha) \in \operatorname{epi} g$ (see [2]).

The below result corresponds to Theorem 6.1 of [9] (see also [8] and [1]). The main difference between these two results is that the weak compactness of $C$ is not assummed.

**Theorem 32.** *Let $E$ be a real Banach space and $C \subset E$ be a convex bounded and nonempty subset of $E$. Assume that $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous function such that*

(1) *if $d_C(x_n) \to 0$ and $\sum_{n=1}^{\infty} d_C^2(x_{n+1}) \parallel x_n - x_{n+1} \parallel < \infty$ then* $\limsup_{n\to\infty} f(x_n) \geq \inf_C f$,

(2) $\forall (x, x^*) \in \operatorname{graph} \partial f, \quad \sup_{c \in C} \langle x^*, x - c \rangle \geq 0$.

*Then*

$$\inf_E f = \inf_C f.$$

***Proof.*** Let us suppose that $\inf_E f < \inf_C f$. Let us fix any $\bar{x} \in E$ such that $f(\bar{x}) < \inf_C f$. We are able to find $\varepsilon, \varepsilon' > 0$ such that

$$f(\bar{x}) + \varepsilon d_C^2(\bar{x}) < \inf_C \left( f + \varepsilon d_C^2 \right) \tag{3.1}$$

and for all $v \in E$,

$$\left( f(v) + \varepsilon d_C^2(v) \leq f(\bar{x}) + \varepsilon d_C^2(\bar{x}) \right) \Longrightarrow \varepsilon'(d_C(v) + \operatorname{diam} C) < 2^{-1}\varepsilon. \tag{3.2}$$

Put $g(y) := f(y) + \varepsilon d_C^2(y)$ for every $y \in E$ and define the following relation

$$u \succ v \Longleftrightarrow g(u) > g(v) + \varepsilon' \min\{d_C^2(v), d_C^2(u)\} \parallel v - u \parallel$$

for every $u, v \in S(g, g(\bar{x})) := \{z \in E \mid g(\bar{x}) \geq g(z)\}$. We shall show that there is $(\succ, \succ^*)$-maximal element in $S(g, g(\bar{x}))$, namely for some $\bar{v} \in S(g, g(\bar{x}))$ and every $z \in S(g, g(\bar{x}))$ the following inequality holds true

$$g(\bar{v}) \leq g(z) + \varepsilon' \min\{d_C^2(\bar{v}), d_C^2(z)\} \parallel \bar{v} - z \parallel .$$

For this reason let us take any sequence $(v_n) \subset S(g, g(\bar{x}))$ such that $v_n \succ v_{n+1}$ for every $n \in \mathbb{N}$. Of course, the function $g$ is bounded from below on $S(g, g(\bar{x}))$ since it is the sum of a convex function and $\varepsilon d_C^2$. Thus we get

$$\infty > g(v_n) - g(v_{n+1}) > \varepsilon' \min\{d_C^2(v_n), d_C^2(v_{n+1})\} \parallel v_n - v_{n+1} \parallel \tag{3.3}$$

for every $n \in \mathbb{N}$, and for some $m \in \mathbb{R}$

$$g(v_1) \geq m + \varepsilon' \sum_{i=1}^{k} \min\{d_C^2(v_i), d_C^2(v_{i+1})\} \parallel v_i - v_{i+1} \parallel \text{ for every } k \in \mathbb{N}.$$

If there is $\delta > 0$ such that $d_C^2(v_i) \geq \delta$ for every $i \in \mathbb{N}$, then $\infty > \sum_{i=1}^{\infty} \| v_i - v_{i+1} \|$, so $v_i \to v_0$ for some $v_0 \in E$. Let us put $a_k := \inf_{i \geq k} d_C^2(v_i)$ and consider the following two cases.

*Case* I. There is $k \in \mathbb{N}$ such that $a_k = \lim_{i\to\infty} d_C^2(v_i)$.
Of course, we get

$$g(v_i) > g(v_{i+1}) + \varepsilon' a_k \| v_i - v_{i+1} \|$$

for every $i \geq k$, which implies

$$\begin{aligned} g(v_i) &> g(v_0) + \varepsilon' a_k \| v_i - v_0 \| \\ &= g(v_0) + \varepsilon' \min\{d_C^2(v_i), d_C^2(v_0)\} \| v_i - v_0 \| \end{aligned}$$

for every $i \geq k$. So the assumption of Corollary 31 is satisfied.

*Case* II. There is a subsequence $n_k$ such that $d_C^2(v_{n_k}) \leq d_C^2(v_{n_{(k+1)}}) \leq \dots$ and $d_C^2(v_{n_{(k+j)}}) \leq d_C^2(v_i)$ for every $j = 1, 2, \dots$ and $i \geq n_{(k+j)}$.
In this case we have

$$g(v_i) > g(v_j) + \varepsilon' d_C^2(v_{n_k}) \| v_i - v_j \|$$

for every $k$ and $j > i \geq n_k$, thus

$$g(v_{n_k}) > g(v_0) + \varepsilon' \min\{d_C^2(v_{n_k}),\, d_C^2(v_0)\} \| v_{n_k} - v_0 \|$$

for every $k$. Again the assumption of Corollary 31 is satisfied.

It follows from Corollary 31 that for some $v \in S(g, g(\bar{x}))$ and every $y \in S(g, g(\bar{x}))$ we have

$$g(v) \leq g(y) + \varepsilon' \min\{d_C^2(v),\, d_C^2(y)\} \| v - y \| .$$

If $g(z) \geq g(\bar{x})$ then, of course, $g(v) \leq g(z) + \varepsilon' \min\{d_C^2(v), d_C^2(z)\} \| v - z \|$, so

$$g(v) \leq g(y) + \varepsilon' \min\{d_C^2(v), d_C^2(y)\} \| v - y \|$$

for every $y \in E$. Hence, we infer the existence $v^* \in \partial f(v)$, $y^* \in \varepsilon \partial d_C^2(v)$ and $z^* \in \varepsilon' d_C^2(v) B(0,1)$ ( $B(0,1)$ stands for the unit ball in the dual space) such that $v^* + y^* + z^* = 0$. On the other hand, by assumption (2) of the Theorem, property 2 of Lemma 21 and (3.2) we get

$$0 = \sup_{c \in C} \langle v^* + y^* + z^*, v - c \rangle \geq \varepsilon d_C^2(v) - \varepsilon' d_C^2(v) \{ d_C(v) + \text{diam } C \} > 0,$$

a contradiction.

In order to finish the proof let us assume that $\liminf_{n\to\infty} d_C^2(v_n) = 0$. Then, choose a subsequence $(v_{n_k}) \subset (v_n)$ such that $\lim_{k\to\infty} d_C^2(v_{n_k}) = 0$ and $d_C^2(v_i) \geq d_C^2(v_{n_k})$ for every $k \in \mathbb{N}$, $i \leq n_k$ and $d_C^2(v_{n_k})$ is decreasing. By (3.3) we get

$$g(\bar{x}) > g(v_1) > g(v_{n_k}) \tag{3.4}$$

for every $k \in \mathbb{N}$, and by the triangle inequality

$$\infty > \sum_{i=1}^{\infty} \min\{d_C^2(v_i), d_C^2(v_{i+1}\} \parallel v_i - v_{i+1} \parallel \geq \sum_{k=1}^{\infty} d_C^2(v_{n_{(k+1)}}) \parallel v_{n_k} - v_{n_{(k+1)}} \parallel .$$

Hence, by (3.1) and (3.4) we get

$$\inf_{c \in C} f(c) > \lim_{k\to\infty} g(v_{n_k}) + g(\bar{x}) - g(v_1), \; d_C^2(v_{n_k}) \longrightarrow 0$$

and

$$\infty > \sum_{k=1}^{\infty} d_C^2(v_{n_{(k+1)}}) \parallel v_{n_k} - v_{n_{(k+1)}} \parallel,$$

which contradicts assumption 1. ∎

Assumption (1) of the above Theorem needs some comments. It is obvious that if $C$ is weakly compact then $d_C(x_n) \to 0$ implies a weak convergence of some subsequence of $(x_n)$ to a point $c$ of $C$, so the weak lower semicontinuity of $f$ ($f$ is assumed to be convex) forces

$$\liminf_{k\to\infty} f(x_{n_k}) \geq f(c) \geq \inf_C f,$$

which, of course, ensures (1). The assumption is also satisfied if $f$ is continuous on $E$. Then $f$ is Lipschitzian on some neighbourhood of $C$, say with $M > 0$ and

$$f(x_n) \geq f(c_n) - 2M d_C(x_n)$$

for some $c_n \in C$, which implies

$$d_C(x_n) \to 0 \quad \Longrightarrow \quad \liminf_{n\to\infty} f(x_n) \geq \inf_C f.$$

Similar reasoning can be done if there is $c \in C$ at which $f$ is continuous or $\mathrm{dom} f \cap \mathrm{int}\, C \neq \emptyset$. Then

$$0 \in \mathrm{cl}\, \partial(f + \psi_C)(C) = \mathrm{cl}\,(\partial f(\cdot) + \partial\psi(\cdot))(C),$$

where $\psi_C$ is 0 on $C$ and $+\infty$ outside the set. Let $d_C(x_n) \longrightarrow 0$, $\| x_n - c_n \| \to 0$ and for some $c' \in C$

$$p^* \in \partial f(c') + N_C(c'),\ p^* = c^* + y^*,\ c^* \in \partial f(c'), y^* \in N_C(c').$$

For every $n \in \mathbb{N}$ we have

$$\begin{aligned} f(x_n) \geq f(c') + \langle c^*, x_n - c'\rangle &= f(c') + \langle c^*, x_n - c_n\rangle + \langle c^*, c_n - c'\rangle \\ &\geq \inf_C f - \| c^* \| \| x_n - c_n \| + \langle p^*, c_n - c'\rangle, \end{aligned}$$

hence $\limsup_{n\to\infty} f(x_n) \geq \inf_C f - \| p^* \| \,\mathrm{diam}\, C$, since $0 \in \mathrm{cl}\,(\partial f(\cdot) + \partial\psi(\cdot))(C)$, so

$$d_C(x_n) \to 0 \Longrightarrow \limsup_{n\to\infty} f(x_n) \geq \inf_C f,$$

thus (1) of Theorem is satisfied.

Let us assume that $\inf_C f \in \mathbb{R}$ and $\inf_E f = \inf_C f$. Of course

$$\mathrm{graph}\, \partial f \cap ((C + B(0,\varepsilon)) \times B(0,\varepsilon)) \neq \emptyset$$

for every $\varepsilon > 0$. Thus (1.8) holds true. Hence the assumptions of Theorem 3.2 are sufficient to get (1.8). In order to get $0 \in \partial f(C)$ we have to preserve that $\mathrm{argmin} f \neq \emptyset$.

**Example 33.** Let $C$ be a bounded convex subset of a real Banach space $E$ and $C^*$ be convex weak* compact subset of the dual $E^*$. Let us assume that $0 \in C^*$ and

$$\sup_{c^*\in C^*} \inf_{c\in C} \langle c^*, c\rangle = 0,$$

then

$$\inf_{c\in C} \sup_{c^*\in C^*} \langle c^*, c\rangle = 0.$$

Indeed, let us define $f(x) := \max_{c^*\in C^*} \langle c^*, x\rangle$ for $x \in E$. Of course, $f(x) \geq 0$ for every $x \in E$, and $f(0) = 0$. Let $x \in E \setminus C$ and $x^* \in \partial f(x)$.

The function $f$ is convex Lipschitz continuous and positively homogenous, so $\langle x^*, x\rangle = f(x)$ (see e.g. Lemma 5.10 of [3]). Thus

$$\sup_{c\in C}\langle x^*, x-c\rangle = f(x) - \inf_{c\in C}\langle x^*, c\rangle \geq f(x) \geq 0.$$

It follows from the above Theorem that

$$\inf_{c\in C} f = \inf_{x\in E} f = 0,$$

which guarantees the desired equality.

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