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A CONSTRUCTIVE METHOD FOR SOLVING STABILIZATION PROBLEMS

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Abstract

The problem of asymptotic stabilization for a class of differential inclusions is considered. The problem of choosing the Lyapunov functions from the parametric class of polynomials for differential inclusions is reduced to that of searching saddle points of a suitable function. A numerical algorithm is used for this purpose. All the results thus obtained can be extended to cover the discrete systems described by difference inclusions.

Keywords: differential inclusions, difference inclusions, Lyapunov function, asymptotic stability.

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1 Introduction

The conditions under which control systems are stable are in essence the real conditions of normal operation of systems. For this reason, in designing control systems it is important to know how to solve the problem of stabilization constructively. It is well known [1, 2], that the method of Lyapunov functions is one of the most efficient methods for analyzing the stability of nonlinear dynamic systems.

We consider dynamic systems described by the differential inclusion [3]

$$\begin{aligned} \dot{x} &\in F_q(x), \\ x(t_0) &= x_0, \\ F_q(x) &= \{y : \ y = Ax, \ A \in \mathcal{B}_q\}, \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$ and $\mathcal{B}_q$ is the convex hule of real $(n \times n)$-matrices $A_1, ..., A_q$, i.e.

$$\mathcal{B}_q = \mathrm{co}(A_1, ..., A_q) \equiv \Big\{ A: \ A = \sum_{\nu=1}^{q} \lambda_\nu A_\nu, \ \lambda_\nu \geq 0, \ \sum_{\nu}^{q} \lambda_\nu = 1 \Big\}$$

In parallel with (1) we examine a more general type of differential inclusion

$$\begin{aligned} \dot{x} &\in F(x), \\ x(t_0) &= x_0, \\ F(x) &= \{y: \ y = Ax, \ A \in \mathcal{B}\}, \end{aligned} \tag{2}$$

where $\mathcal{B}$ is a compactum (in general, nonconvex) in the n^2-dimensional space of real $(n \times n)$-matrices A. The stabilization problem for differential inclusions of type (1) and (2), to which many practically important control systems can be reduced, consists of choosing a Lyapunov function $V(x)$.

For example, we consider the controlled object with varying control region [4]

$$\begin{aligned} \dot{x} = f(x, u), \ x \in \mathbf{R}^n, \ u \in U(x), \\ x(t_0) = x_0. \end{aligned}$$

This gives us a differential inclusion of type

$$\begin{aligned} \dot{x} &\in Q(x), \\ x(t_0) &= x_0, \\ Q(x) &= \{f(x, u), \ u \in U(x)\}: \end{aligned}$$

denoting by $Q(x)$ the set of all vektors $f(x, u)$ obtained as u runs over the control region $U(x)$. The Lyapunov functions $V(x)$ used below are chosen from the class of convex $\mathbf{R}^n$-valued functions.

Definition 1. An absolutely continuous vector function $x(t)$ satisfying the condition $\dot{x}(t) \in F_q(x(t))$ ($\dot{x}(t) \in F(x(t))$) almost everywhere on a considered interval of time $[t_0, t]$ is called a solution of the inclusion (1) (2).

Note that any solution $x(t)$ of inclusion (1) ((2)) can be continued on the whole semi-infinite axis $[t_0, \infty)$.

Definition 2. The zero solution $x = 0$ of the differential inclusion (1) ((2)) is called asymptotically stable if:

a) for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for each solution $x(t)$ of the inclusion (1) ((2)) the inequality $||x(t)|| < \epsilon$ holds for all $t \geq t_0$, if only $||x(t_0)|| < \delta(\epsilon)$

b) there exists $\Delta > 0$ such that for any solution of the inclusion (1) ((2)) with $||x_0|| < \Delta$, the limiting relation $\lim_{t\to\infty} x(t) = 0$ holds.

The rest of the paper is organized as follows. In Section 2, we consider some basic results about the asymptotic stability of differential inclusions. In Section 3, we discuss a constructive algorithm for solving stabilization problem. In Section 4, we give a constructive solution of a stabilization problem.

2 Mathematical preliminaries

First let us discuss some basic results about the asymptotic stability of systems (1) and (2) (Theorems 1, 2 and 3).

Theorem 1. *For the zero solution $x = 0$ of the differential inclusion* (2) *to be asymptotically stable it is necessary and sufficient that there exists a strictly convex, homogeneous* (*of second order*) *Lyapunov function $V(x)$ of a quasiquadratic form, namely*

$$\begin{gathered} V(x) = x^T \mathcal{L} x, \quad \mathcal{L}(x) = (l_{i,j}(x))_{i,j=1}^n, \\ \mathcal{L}^T(x) = \mathcal{L}(x) = \mathcal{L}(\tau x),\ x \neq 0,\ \tau \neq 0, \\ V(0) = 0, \end{gathered}$$

whose derivative along solutions of the inclusion (2) *satisfies the inequality*

$$W(x) = \max_{y \in F(x)} \frac{\partial V(x)}{\partial y} \ \leq \ -\gamma ||x||^2,\ \gamma > 0. \tag{3}$$

Note that for inclusion (2) the role of the usual derivative is played by the function $W(x) = \max_{y\in F(x)} \frac{\partial V(x)}{\partial y}$, where

$$\frac{\partial V(x)}{\partial y} = \lim_{h\to+0} h^{-1}(V(x+hy) - V(x))$$

is a derivative of the convex function $V(x)$ in the point $x \in \mathbf{R}^n$ in the direction $y \in F(x)$ [5, 6].

Theorem 2. *The zero solution $x = 0$ of the differential inclusion* (2) *is asymptotically stable if and only if there exists a Lyapunov function in the class of homogeneous forms of order $2p$:*

$$V_{m,p}(l,x) = \sum_{i=1}^{m} (l_i, x)^{2p},$$

where $l_i \in \mathbf{R}^n,\ i = 1, ..., m$ are constant vectors

$$\text{rank}\ \mathcal{L} = n \leq m,\ \mathcal{L} = (l_1, ..., l_m) \tag{4}$$

such that for its derivative

$$W_{m,p}(l,x) = 2p \max_{y \in F(x)} \{(l_i, x)^{2p-1}(l_i, y)\}$$

along solutions of inclusion (2) *for some integer $p \geq 1$ the inequality*

$$W_{m,p}(l,x) \leq -\nu ||x||^{2p},\ \nu > 0$$

is satisfied.

Here the bracket $(\cdot, \cdot)$ denotes the Euclidean scalar product.

Theorem 3. *The zero solution $x = 0$ of the differential inclusion* (2) *is asymptotically stable if and only if there exists a piecewise-quadratic Lyapunov function*

$$V_m(l,x) = \max_{1 \leq i \leq m} (l_i, x)^2,\ l_i \in \mathbf{R}^n,\ i = 1, ..., m,$$

whose derivative

$$W_m(l,x) = \max_{y \in F(x)} \frac{\partial V_m(x)}{\partial y}$$

along solutions of inclusion (2) *satisfies inequality* (3)*, and the vectors l_i satisfy condition* (4)*.*

The class of convex functions $V_m(l,x)$ is obtained by approximating the level surfaces of the strictly convex function $V(x)$ by centrally symmetrical convex polyhedrons. The vectors l_i for $i = 1, ..., m$, in Theorem 3 determine the normals to the faces of polyhedrons and the surfaces of polyhedrons are the level surfaces of the function $V_m(l,x)$.

Remark 1. The vectors assigning the Lyapunov function $V_m(l,x)$ in Theorem 3 can be used as vectors $l_i,\ i = 1,...,m$ for the Lyapunov function $V_{m,p}(l,x)$ in Theorem 2.

The parameters determining the class of Lyapunov functions $V_m(l,x)$, and $V_{m,p}(l,x)$ are the components of the vectors $l_i,\ i = 1,...,m$ and the numbers m and p.

The following proposition reduces the problem of asymptotic stability in the general case of inclusion (2) to the case of a convex compactum $\mathrm{co}(\mathcal{B})$.

Proposition 1. *The zero solution $x = 0$ of the differential inclusion* (2) *is asymptotically stable if and only if the zero solution of the differential inclusion*

$$\begin{gathered} \dot{x} \in F_c(x), \\ x(t_0) = x_0, \\ F_c(x) = \{y : y = Ax,\ A \in \mathrm{co}(\mathcal{B})\} \end{gathered} \tag{5}$$

is asymptotically stable.

Proof. Sufficiency of the proposition follows from the inclusion

$$F(x) \subset F_c(x).$$

Necessity follows from the equivalence of the closure of the solutions set of differential inclusion (2) and the solutions set of differential inclusion (5) [5, 6]. ∎

A convex compactum $\mathrm{co}(\mathcal{B})$ can be approximated by convex polyhedrons $\mathcal{B}_q$. We obtain the following assertion.

Proposition 2. *For the zero solution $x = 0$ of the differential inclusion* (2) *to be asymptotically stable it is necessary and sufficient that there exists a number $q \geq 1$ and a differential inclusion* (1) *whose zero solution $x = 0$ is asymptotically stable and*

$$F_c(x) \subset F_q(x). \tag{6}$$

Proof. Let $q \geq 1$ be a number such that $F_c(x) \subset F_q(x)$. Let the zero solution $x = 0$ of (1) be an asymptotically stable solution. The asymtotic stability of the zero solution of the differential inclusion (5) follows from condition (6). By Proposition 1 the zero solution of inclusion (2) is asymptotically stable, too.

Let the zero solution $x = 0$ of the inclusion $\dot{x} \in F(x)$ be asymptotically stable. It follows from the results of [2, 6] that there exists some $\epsilon > 0$ such that the zero solution of the differential inclusion

$$\dot{x} \in F_\epsilon(x),$$
$$x(t_0) = x_0,$$
$$F_\epsilon(x) = \{y : \ y = Ax, \ A \in \mathcal{B}_\epsilon\},$$

where $\mathcal{B}_\epsilon$ is a compactum and $\mathcal{B} \subset \mathcal{B}_\epsilon$, is asymptotically stable. This implies that the zero solution of the inclusion

$$\dot{x} \in F_{c,\epsilon}(x),$$
$$x(t_0) = x_0,$$
$$F_{c,\epsilon}(x) = \{y : \ y = Ax, \ A \in \mathrm{co}(\mathcal{B}_\epsilon)\}$$

is asymptotically stable (by Proposition 1). Then there must be some $q \geq 1$ such that

$$\mathrm{co}(\mathcal{B}) \subset \mathrm{co}(A_1, ..., A_q), \ A_s \in \mathrm{co}(\mathcal{B}_\epsilon), \ s = 1, ..., q.$$

By definition

$$\mathrm{co}(A_1, ..., A_q) \subset \mathrm{co}(\mathcal{B}_\epsilon), \ A_s \in \mathrm{co}(\mathcal{B}_\epsilon), \ s = 1, ..., q.$$

This means that there is some $q \geq 1$ such that the solution $x = 0$ of (1) is asymptotically stable and condition (6) holds. ∎

Remark 2. The sets of linear nonstationary systems

$$\dot{x} = A(t)x, \ A(t) = (a_{ij}(t))_{i,j=1}^n,$$
$$\alpha_{ij} \leq a_{ij}(t) \leq \beta_{ij}, \ i, j = 1, ..., n,$$

can be reduced to (1), where $\alpha_{ij}, \ \beta_{ij}$ are constans.

It is important to solve constructively the problem of stabilization of the zero solution for differential inclusion (1).

3 A constructive algorithm for solving stabilization problems

Now we consider the problem of constructive derivation of Lyapunov function for system (1) in a bounded region G, $x \in G \subset \mathbf{R}^n$, $0 \in G$. The Lyapunov function in Theorem 2 has the form

$$V_{m,p}(l,x) = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} l_i^j x_j \Big)^{2p}, \ p \geq 1$$

or equivalently

$$V_p(z,x) = \sum_{r=1}^{N(p)} z_r \psi_r(x) = (z, \psi(x)),$$

where the $\psi_r(x),\ r = 1, ..., N(p)$ are standard monomials of degree $2p$

$$\psi(x) = x_1^{k_{1r}} \cdot ... \cdot x_n^{k_{nr}},$$

$z \in G_z \subset \mathbf{R}^{N(p)}$ are coefficients of monomials and $N(p) = C_{n+2p-1}^{2p}$ is the number of monomials. The derivative $W_p(z,x)$ of the function $V_p(z,x)$ along solutions of the differential inclusion (1) has the form

$$W_p(z,x) = \max_{y \in F_q(x)} \left(z, \frac{\partial \psi(x)}{\partial x} y \right) = \max_{\lambda \in \Theta} \sum_{\nu=1}^{q} \lambda_\nu \left(z, \frac{\partial \psi(x)}{\partial x} A_\nu x \right),$$

where $\lambda = (\lambda_1, \ldots, \lambda_q)^T$, $\Theta = \{\lambda : \sum_{j=1}^{q} \lambda_j = 1,\ \lambda_j \geq 0\}$ and $\frac{\partial \psi(x)}{\partial x} = (\frac{\partial \psi_r(x)}{\partial x_i})$ is a $(N(p) \times n)$-matrix for $r = 1, ..., N(p)$ and $i = 1, ..., n$.

Let

$$g_\nu(x) := \frac{\partial \psi(x)}{\partial x} A_\nu x,\ \nu = 1, ..., q$$

be a vector function. Let

$$X := ((x^1)^T, ..., (x^q)^T)^T,$$

where $x^\nu = (x_1^\nu, ..., x_n^\nu)^T,\ \nu = 1, ..., q$ and let

$$T_p(z,X) := \sum_{\nu=1}^{q} (z, g_\nu),\ \tilde{x} := 0.$$

Then we have

Proposition 3. *The inequality*

$$W_p(z,x) \ \le\ -\gamma||x||^{2p},\ \gamma>0, \quad z\in G_z$$

has a solution $\tilde z \in G_z$ *if and only if*

$$W_p(\tilde z, x) < W_p(\tilde z,\tilde x) \le W_p(z,\tilde x) \tag{7}$$

holds for all $z\in G_z,\ x\in G, x\neq 0.$

Proof. *Necessity:* $W_p(\tilde z,x)\ \le\ -\gamma||x||^{2p}$ and

$$W_p(\tilde z,x) < 0 = W_p(\tilde z,\tilde x) = W_p(z,\tilde x).$$

Sufficiency: $W_p(\tilde z,x) < 0 = W_p(\tilde z,\tilde x)$. Let

$$W_p(\tilde z,x) = \sum_{\nu=1}^{q} \hat\lambda_\nu\Big(z, \frac{\partial\psi(x)}{\partial x}A_\nu x\Big).$$

We get

$$W_p(\tilde z,x) = \sum_{\nu=1}^{q}\hat\lambda_\nu \sum_{\mu=1}^{n}\sum_{\tau=1}^{n}\tilde z_\tau \sum_{i=1}^{n} k_{i\tau}a^\nu_{i\mu}\prod_{j=1}^{n} x_j^{k_j\tau} < 0,$$

where $A_\nu = (a^\nu_{i\mu}),\ \mu, i = 1,...,n.$ Hence there exists $\gamma>0$ such that

$$W(\tilde z,x) + \gamma(x_1^2+...+x_n^2)^p < 0,\ x\neq 0.$$

The proof is complete. ■

In other words the pair $(\tilde z,\tilde x)$ is a saddle point of the function $W_p(z,x)$. Thus constructing the Lyapunov function $V_p(z,x)$ reduces to a problem of mathematical programming.

For finding the saddle points $(\tilde z,\ \tilde x)$ of function $W_p(z,x)$ (i.e. for finding vector $\tilde z$) we use the following algorithm:

a) find saddle points $(\tilde z_T,\tilde X)$ of the function $T_p(z,X)$,

b) put $\tilde z = \tilde z_T$.

For solving (a) we use a gradient method:

$$\dot{x}^\nu = \frac{\partial T_p(z,X)}{\partial x^\nu} = \Lambda_\nu^T(x^\nu)z, \tag{8}$$

$$\dot{z} = -\frac{\partial T_p(z,X)}{\partial z} = -\sum_{\nu=1}^{q} g_\nu(x_\nu), \tag{9}$$

where $\nu = 1, ..., q$. The

$$\Lambda_\nu(x^\nu) = \frac{\partial g_\nu(x^\nu)}{\partial x^\nu} = \left(\frac{\partial g_{\nu r}(x^\nu)}{\partial x_i^\nu}\right)_{r=1,i=1}^{N(p),n}$$
$$\nu = 1, ..., q$$

are $(N(p) \times n)$-matrices.

Difference approximation for (8) and (9) gives a numerical procedure of finding a vector z

$$\begin{gathered} x^\nu(\tau+1) = x^\nu(\tau) + h\Lambda^T z(\tau), \\ z(\tau+1) = z(\tau) - h\sum_{\nu=1}^{q} g_\nu(x^\nu(\tau)), \\ x(0) \in G,\ z(0) \in G_z, \\ \nu = 1, ..., q, \end{gathered}$$

i.e., a procedure of finding the Lyapunov function of Theorem 2.

All the above results can be simply extended to the difference inclusions

$$\begin{gathered} x(k+1) \in F_q(x(k)), \\ k = 0, 1, ..., \end{gathered} \tag{10}$$

where $F_q(0) = \{0\}$. The function $F_q(x)$ is defined as in (1).

We introduce the Lyapunov function $V(z,x)$ from the class of homogeneous forms of order $2p$ and formulate the stabilization criterion of system (10) in a constructive form as follows.

Let $\tilde{x} := 0$. The function

$$W(z,x) = \max_{y \in F_q(x)} V(z,y) - V(z,x)$$

is a difference derivative of the Lyapunov function along solutions of the difference inclusion (10).

Proposition 4. *The inequality*

$$W(z,x) \le 0, \; x \in G, \; z \in G_z$$

has a solution $\tilde{z} \in G_z$ *if and only if the inequalities*

$$W(\tilde{z},x) < W(\tilde{z},\tilde{x}) \le W(z,\tilde{x})$$

are fulfilled for all $z \in G_z$ *and* $x \in G, \; x \neq 0$.

4 Implementation of the constructive algorithm

We consider nonlinear nonstationary systems of the type

$$\dot{x} = Ax + \sum_{r=1}^{M} b^r \phi_r(\sigma_r, t), \tag{11}$$

$$\sigma_r = (c^r, x), \; \phi_r(0,t) \equiv 0, \; r = 1, ..., M,$$

where $b^r \in \mathbf{R}^n, \; c^r \in \mathbf{R}^n$ are constant vectors. The nonlinear functions $\phi(\sigma,t) = (\phi_r(\sigma_r,t))_{r=1}^M$ satisfy the conditions

$$0 \le \phi_r(\sigma_r)\sigma_r \le k_r \sigma_r^2,$$

$$0 < k_r < \infty, \; r = 1, ..., M.$$

It is well known [7] that the system (11) is equivalent to the differential inclusion

$$\dot{x} \in F_\phi(x), \tag{12}$$

$$F_\phi(x) = \{y : \; y = Ax + \sum_{r=1}^{\nu} b^r \lambda_r(c^r, x), \quad 0 \le \lambda_r \le k_r, \quad r = 1, ..., M\}.$$

Evidently (12) is a special case of (1) if

$$A_\nu = A + \sum_{r=1}^{M} h_{\nu_r} b^r (c^r)^T,$$

$$q = 2^M, \; \nu = 1, ..., q, \; r = 1, ..., M,$$

where $h_{\nu_r} = 0$ or $h_{\nu_r} = k_r, \; \nu = 1, ..., q$ and $r = 1, ..., M$.

Example 1. Consider a second order system

$$\dot{x}_1 = x_2 \tag{13}$$

$$\dot{x}_2 = -2x_1 - x_2 - \phi(x_1, t) \tag{14}$$

where the matrix $A = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$ of the linear portion of the system is of Hurwitz type and $\phi(x_1, t)$ is a nonlinear function belonging to the class of functions that satisfy the inequalities

$$0 \leq \phi(x_1, t) \leq kx_1^2,$$

where $0 < k < \infty$. We derive the Lyapunov function complying with the inequalities $(x \neq 0)$

$$V(x) > 0, \tag{15}$$

$$\left(\frac{\partial V}{\partial x}, (Ax)\right) < 0, \tag{16}$$

$$\left(\frac{\partial V}{\partial x}, (Ax + (0,1)^T kx_1)\right) < 0, \tag{17}$$

at $k = 3.73$. We construct the Lyapunov function from the class of the quadratic forms:

$$V_2(x) = x_1^2 + 0.303x_1x_2 + 0.251x_2^2.$$

In the class of fourth order forms, we construct the Lyapunov function satisfying the inequalities (15) – (17) for $k = 6.40$, namely

$$V_4(x) = x_1^4 + 0.267x_1^3x_2 + 0.559x_1^2x_2^2 + 0.159x_1x_2^3 + 0.046x_2^4.$$

We use the conditions of absolute stability (15) – (17) for the system (13) – (14), which follow from Theorems 2 and 3 (see [7]). The functions $V_2(x)$ and $V_4(x)$ were construct on a grid.

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