

UPPER AND LOWER SOLUTIONS METHOD FOR  
PARTIAL HADAMARD FRACTIONAL INTEGRAL  
EQUATIONS AND INCLUSIONS

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**Abstract**

In this paper we use the upper and lower solutions method combined with Schauder's fixed point theorem and a fixed point theorem for condensing multivalued maps due to Martelli to investigate the existence of solutions for some classes of partial Hadamard fractional integral equations and inclusions.

**Keywords:** functional integral equation, integral inclusion, Hadamard partial fractional integral, condensing multivalued map, existence, upper solution, lower solution, fixed point.

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