*Discussiones Mathematicae*
*Probability and Statistics 33 (2013) 121–125*
*doi:10.7151/dmps.1147*

# SIGNATURA OF MAGIC AND LATIN INTEGER SQUARES: ISENTROPIC CLANS AND INDEXING

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**Abstract**

The 2010 study of the Shannon entropy of order nine Sudoku and Latin square matrices by Newton and DeSalvo [Proc. Roy. Soc. A 2010] is extended to natural magic and Latin squares up to order nine. We demonstrate that decimal and integer measures of the Singular Value sets, here named SV clans, are a powerful way of comparing different integer squares.

Several complete sets of magic and Latin squares are included, including the order eight Franklin subset which is of direct relevance to magic square line patterns on chess boards. While early examples suggested that lower rank specimens had lower entropy, sufficient data is presented to show that some full rank cases with low entropy possess a set of singular values separating into a dominant group with the remainder much weaker. An effective rank measure helps understand these issues.

We also introduce a new measure for integer squares based on the sum of the fourth powers of the singular values which appears to give a useful method of indexing both Latin and magic squares. This can be used to begin cataloging a "library" of magical squares.

*Based on a video presentation in celebration of George Styan's 75th at LINSTAT2012 and IWMS-21 on 19 July, 2012 at Będlewo, Poland.*

**Keywords:** Shannon entropy, magic square, Latin square, singular value decomposition, singular value clan.

**2010 Mathematics Subject Classification:** 15B04, 15B06.

## References


[1] W.S. Andrews, *Magic Squares and Cubes*, 1st edition, The Open Court Publishing Company, New York; 1914 2nd edition with corrections and added chapters, 1908. See review by Miller [36] below.

[2] R.A. Bailey, P.J. Cameron and R.C. Connelly, *Sudoku, Gerechte Designs, Resolutions, Affine Space, Spreads, Reguli, and Hamming Code*s, Am. Math. Monthly **115** (2008) 383–404.

[3] W.S. Benson and O. Jacoby, New Recreations with Magic Squares, Dover Publications (New York, 1976).

[4] S.S. Block and S.A. Tavares, Before Sudoku – The World of Magic Squares (Oxford, 2009).

[5] C. Boyer, Magic squares of squares, 2012. http://www.multimagie.com/English/SquaresOfSquares.htm

[6] C. Bradgon, Frozen Fountain: Being Essays on Architecture and the Art of Design in Space, 1932 Alfred A. Knopf (New York, 1924).

[7] R. Brualdi, Introductory Combinatorics, 5th edition (Pearson, 2010).

[8] I. Cameron and P. Loly, Eigenvalues of an Algebraic Family of Compound Magic Squares of Order $n = 3^l, l = 2, 3, \ldots$, and Construction and Enumeration of their Fundamental Numerical Forms, Canadian Mathematical Society (Windsor, Ontario, 2009).

[9] I. Cameron, R. Rogers and P. Loly, Signatura of Integer Squares: Another Chapter in the Scientific Studies of Magical Squares, LINSTAT 2012, Będlewo, Poland, 16–20 July 2012. http://www.physics.umanitoba.ca/~icamern/Poland2012/

[10] W. Chan and P.D. Loly, *Iterative compounding of square matrices to generate large-order magic squares*, Mathematics Today **38** (2002), 113–118. (The Institute of Mathematics and its Applications, Southend-on-Sea, UK).

[11] R. Descombes, Les Carrés Magiques: Histoire, théorie et technique du carré magique, de l'Antiquité aux recherches actuelles, $2^e$ (Vuibert, 2000).

[12] J.H. Dinitz, P.R.J. Östergaard and D.R. Stinson, Packing Costas Arrays, arXiv: 1102.1332v1[math.CO] 7 Feb 2011.

[13] S.W. Drury, *There are no magic squares of rank* 2, 2007, personal communication.

[14] H.E. Dudeney, *The magic square of sixteen*, The Queen: The Lady's Newspaper and Court Chronicle, January 15, 125-126. See also his 1917 *Amusements in Mathematics*, p. 119–121, reprinted without change, 1958, 1970 Dover Publications (New York, 1910).

[15] C. Eggermont, Multimagic Squares, Thesis, Department of Mathematics, Radbout University of Nijmegen, 2007.

[16] A. van den Essen, Magische Vierkanten: Van Lo-Shu tot sudoku, De wonderbaarlijke geschiedenis, Veen Magazines (Diemen, 2006).

[17] L. Euler, *Recherches sur une nouvelle espèce de quarrés magiques*, 1923 reprinted in Opera Omnia Series I volume VII (Teubner, Leipzig and Berlin, 1782), 291–392.

[18] R.P. Feyman, Lectures on Computation, edited by T. Hey and R.W. Allen (Westview, 1996).

[19] L.S. Frierson, A Mathematical Study of Magic Squares: A New Analysis, The Monist, XVII 272–293 (in Criticism and Discussion, signed L.S. Frierson, Frierson, LA), 1907 [Edited version in [1], pp. 129–145, signed L.S.F]

[20] H.G. Funkhouser, *A short account of the history of symmetric functions of roots of equations*, Am. Math. Monthly **37** (1930), 357–365. doi:10.2307/2299273

[21] F. Gaspalou, 2010. http://www.gaspalou.fr/magic-squares/, and personal communication.

[22] A. Girard, *Invention nouvelle en l'algèbre*, 1629. http://gallica.bnf.fr/ark:/12148/bpt6k5822034w.r=albert+girard+invention+nouvelle.langEN

[23] H.D. Heinz and J.R. Hendricks, *Magic Square Lexicon: Illustrated*, 2000 and H.D.H. 1999–2009 http://www.magic-squares.net/; Dudeney patterns: http://www.magic-squares.net/order4list.htm\#Introduction

[24] R.A. Horn and C.A. Johnson, Topics in Matrix Analysis (Cambridge University Press, 1991). doi:10.1017/CBO9780511840371

[25] S. Kirkland and M. Neumann, *Group inverses of M-matrices with associated nonnegative matrices having few eigenvalues*, Linear Algebra Appl. **220** (1995), 181–213. doi:10.1016/0024-3795(94)00301-S

[26] C. Knecht, *Topographical model*, 2011 knechtmagicsquare.paulscomputing.com

[27] K.-W. Lih, *A Remarkable Euler Square before Euler*, Mathematics Magazine, June 2010, 163–167. doi:10.4169/002557010X494805

[28] P.D. Loly, *The Invariance of the Moment of Inertia of Magic Squares*, The Mathematical Gazette **88** (2004), 151–153.

[29] P.D. Loly and M.J. Steeds, *A new class of pandiagonal squares*, International Journal of Mathematical Education in Science and Technology **36** (2005), 375–388.

[30] P. Loly, *Franklin Squares - A Chapter in the Scientific Studies of Magical Squares*, (version of a poster talk at NKS2006 (New Kind of Science), Washington, D.C., June 2006), Complex Systems **17** (2007), 143–161.

[31] P.D. Loly and D.G. Schindel,"A Simplified Demonstration of Counting the 880 Fourth Order Magic Squares Using Mathematica", 2007, New Kind of Science (NKS2006) Wolfram Science Conference, 2006. www.wolframscience.com/conference/2006/presentations/materials/loly.nb

[32] P.D. Loly, *Two small theorems for square matrices rotated a quarter turn*, Western Canada Linear Algebra Meeting (WCLAM2008), Winnipeg, 2008.

[33] P. Loly, I. Cameron, W. Trump and D. Schindel, *Magic square spectra*, Linear Algebra and Its Applications, **430** (2009), 2659–2680. doi:10.1016/j.laa.2008.10.032

[34] B.D. McKay and I.M. Wanless, *On the number of latin squares*, Ann. Combin. **9** (2005), 335–344. http://cs.anu.edu.au/~bdm/data/latin.html doi:10.1007/s00026-005-0261-7

[35] G.A. Miller, Historical Introduction to Mathematical Literature, The MacMillan Company, 1916, 1927.

[36] G.A. Miller, Review of [1], Bull. Amer. Math. Soc. **16** (1909), 85–87. doi:10.1090/S0002-9904-1909-01866-X

[37] C. Moler, *MATLAB's magical mystery tour*, The MathWorks Newsletter **7** (1993), 8–9.

[38] D. Morris, Best Franklin Squares, 2012. http://bestfranklinsquares.com/

[39] P.K. Newton and S.A. DeSalvo, *The Shannon entropy of Sudoku matrices*, Proc. R. Soc. A **466** (2010), 1957–1975.

[40] Dame K. Ollerenshaw and Sir H. Bondi, *Magic squares of order four*, Philosophical Transactions of the Royal Society of London, A **306** (1982), 443–532.

[41] Dame K. Ollerenshaw, *On 'most perfect' or 'complete' 8x8 pandiagonal magic squares*, Proc. R. Soc. Lond. A **407** (1986), 259–281. doi:10.1098/rspa.1986.0096

[42] K. Ollerenshaw and D.S. Brée, *Most-perfect pandiagonal magic squares: their construction and enumeration*, The Institute of Mathematics and its Applications, Southend-on-Sea, UK, 1998.

[43] K. Ollerenshaw, *Constructing pandiagonal magic squares of arbitrarily large size*, Mathematics Today, Parts 1 and 2, Feb. 23–29; Part 3, Apr. 66–69, (The Institute of Mathematics and its Applications, Southend-on-Sea, UK), 2006.

[44] P.C. Pasles, Benjamin Franklin's Numbers: An Unsung Mathematical Odyssey (Princeton, 2008).

[45] K. Pinn and C. Wieczerkowski, *Number of magic squares from parallel tempering Monte Carlo*, International Journal of Modern Physics C **9** (1998), 541–546.

[46] R. Rao, Computing a rosetta stone for the indus script, *TED talk*: http://www.ted.com/talks/rajesh_rao_computing_a_rosetta_stone_for_the_indus_script.html; Rao, R. *et al.* 2009, *Entropic evidence for linguistic structure in the indus script*, Science **324** (2009), 1165.

[47] A. Rogers, 2004; A. Rogers, P. Loly and G.P.H. Styan, 2008, *Sums of Kronecker Products for Compound Magic Squares: Eigenproperties*, Western Canada Linear Algebra Meeting (WCLAM2008), Winnipeg; A. Rogers, P. Loly and Styan 2012, preprint.

[48] O. Roy and M. Vetterli, The Effective Rank: A Measure of Effective Dimensionality, EUSIPCO (EURASIP), Poznań, 2007.

[49] W.A. Sammons, *Magic squares and groups*, IMA Bulletin 27 (August) (1991), 161–172.

[50] D.G. Schindel, M. Rempel and P.D. Loly, *Enumerating the bent diagonal squares of Dr Benjamin Franklin FRS*, Proceedings of the Royal Society A: Physical, Mathematical and Engineering **462** (2006), 2271–2279. The electronic supplementary material of the 4320 set is available at: rspa.royalsocietypublishing.org/content/suppl/2009/02/11/462.2072.2271.DC1/rspa20061684supp2.txt

[51] J. Sesiano, Les Carrés Magiques dans les Pays Islamic, Presses Poltechniques et Universitaires Romandes, 2004.

[52] R. Schroeppel, *The order* 5 *magic squares*, Written by Michael Beeler with assistance from Schroeppel – see M. Gardner's 1976 column on Mathematical Games, Scientific American **234** (1971), 118–123.

[53] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, 2012. http://www.research.att.com/~njas/sequences/

[54] G.P.H. Styan, *An illustrated introduction to Caïssan squares: the magic of chess*, Acta et Commentationes Universitatis Tartuensis de Mathematica **16** (2012), 97–143. Online at www.math.ut.ee/acta/

[55] M. Suzuki, Archived web pages about magic squares, especially a database and algorithms: http://mathforum.org/te/exchange/hosted/suzuki/MagicSquare.html and mathforum.org/te/exchange/hosted/suzuki/MagicSquare.wasan.html

[56] F.J. Swetz, Legacy of the Luoshu – The 4000 Year Search for the Meaning of the Magic Square of Order Three (Chicago, Open Court, 2002).

[57] G.G. Szpiro, A Mathematical Medley: Fifty Easy Pieces on Mathematics, AMS, 2010. See chapters 30 and 31.

[58] A.C. Thompson, *Odd magic powers*, Am. Math. Monthly **101** (1994), 339–342. doi:10.2307/2975626

[59] D. Trenkler and G. Trenkler, Magic Squares, Melancholy and The Moore-Penrose Inverse, IMAGE 27, October 2001, 3–10.

[60] W.H. Thompson, *On magic squares*, The Quarterly Journal of Pure and Applied Mathematics **X** (1869), 186–202 [XI, 57, 123, 213]

[61] W. Trump, Notes on Magic Squares and Cubes, 2003. http://www.trump.de/magic-squares/ and *Estimate of the number of magic squares of order* 6, http://www.trump.de/magic-squares/normal-6/index.html

[62] E.W. Weisstein, Latin Square, From MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com/LatinSquare.html

## 1. **Electronic files**

1. shannonData.txt: Matrix elements for squares cited in tables, where not easily found in a reference.

Received 15 April 2013
Revised 23 October 2013