

**”ON THE SHOULDERS OF GIANTS”
A BRIEF EXCURSION INTO THE HISTORY OF
MATHEMATICAL PROGRAMMING¹**

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Abstract

Similar to many mathematical fields also the topic of mathematical programming has its origin in applied problems. But, in contrast to other branches of mathematics, we don't have to dig too deeply into the past centuries to find their roots. The historical tree of mathematical programming, starting from its conceptual roots to its present shape, is remarkably short, and to quote ISAAK NEWTON, we can say:

”We are standing on the shoulders of giants”.

The goal of this paper is to describe briefly the historical growth of mathematical programming from its beginnings to the seventies of the last century and to review its basic ideas for a broad audience. During this process we will demonstrate that optimization is a natural way of thinking which follows some extremal principles.

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