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EVOLUTION EQUATIONS IN OSTENSIBLE METRIC SPACES: FIRST-ORDER EVOLUTIONS OF NONSMOOTH SETS WITH NONLOCAL TERMS

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Abstract

Similarly to quasidifferential equations of Panasyuk, the so-called mutational equations of Aubin provide a generalization of ordinary differential equations to locally compact metric spaces. Here we present their extension to a nonempty set with a possibly *nonsymmetric* distance. In spite of lacking any linear structures, a distribution-like approach leads to so-called right-hand forward solutions.

These extensions are mainly motivated by compact subsets of the Euclidean space whose evolution is determined by the nonlocal properties of both the current set and the normal cones at its topological boundary. Indeed, simple deformations such as isotropic expansions exemplify that topological boundaries do not have to evolve continuously in time and thus Aubin's original concept cannot be applied directly. Here neither regularity assumptions about the boundaries nor the inclusion principle are required. The regularity of compact reachable sets of differential inclusions is studied extensively instead.

This example of nonlocal set evolutions in the Euclidean space serves as an introductory motivation for extending ordinary differential equations (and evolution equations) beyond the traditional border of vector spaces – and for combining it with other examples in systems.

Keywords: mutational equations, quasidifferential equations, funnel equations, nonlocal geometric evolutions, reachable sets of differential inclusions, sets of positive erosion, sets of positive reach.

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