

CONGRUENCES AND HOEHNKE RADICALS ON GRAPHS

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Abstract

We motivate, introduce, and study radicals on classes of graphs. This concept, and the theory which is developed, imitates the original notion of a Hoehnke radical in universal algebra using congruences. It is shown how this approach ties in with the existing theory of connectednesses and disconnectednesses (= Kurosh-Amitsur radical theory).

Keywords: congruences and quotients of graphs, Hoehnke radicals of graphs, connectednesses and disconnectednesses of graphs, Kurosh-Amitsur radicals of graphs, subdirect representations of graphs.

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1. INTRODUCTION

Employing concepts and results from modern algebra and universal algebra, also when linked to the first-order logic of algebraic models [16], but transplanted from the algebraic context to the world of relational structures – and in particular the case of graphs – can deliver fruitful results. Two such concepts are those of tensor products [2] and of congruences [4], and another one is that of structural properties which are axiomatizable in an apt first-order logical language [3]. In this paper we introduce and study the notion of a *radical* in graph theory and show that it produces equally interesting results.

The concept of a radical in a mathematical structure first arose in ring theory in [14]. It is typically motivated by the idea of “improving” a ring by forming a quotient ring by collapsing to zero some ideal of “bad” elements of the ring. A “bad” element in a ring may, for instance, be a *nilpotent* one, of which some power is zero. Now there are many theories of radicals on mathematical structures in the literature, for example in the contexts of universal algebra (see for instance [12] and [13] by Hoehnke), of category theory (see for instance [17] by Veldsman), of ring theory (developed by Amitsur and Kurosh in the 1950’s, see for instance [11] by Gray or [10] by Gardner and Wiegandt and in the context of model theory (see for instance [5] and [6] by Buys and Heidema). The most general approach to radical theory is to be found in [15] by Marki, Mlitz and Wiegandt.

Radical theory for graphs is not new. Already in 1975 Fried and Wiegandt [9] developed a radical theory in the category of graphs which admit loops, there called connectednesses and disconnectednesses. Their approach was class based in the spirit of the general radical theory for groups and rings as developed by Kurosh and Amitsur. Our approach to radicals presented here will be one that imitates the one proposed by Hans-Jürgen Hoehnke in 1966 using congruences on universal algebras. Already congruences (and quotients) on graphs have been defined in [4] where it was introduced for graphs that do not admit loops (simple graphs). Such a class is too restrictive on the permissible mappings and the radical theory degenerates to triviality. We will redefine graph congruences for the more general case where a graph may have a loop at some vertices and develop the necessary tools (isomorphism theorems and subdirect products) required for the radical theory to follow.

For those notions on graphs in general not defined here, we refer the reader to [8]. Except when explicitly stated otherwise, all graphs considered are undirected and unlabelled, without multiple edges, and have non-empty vertex sets. A graph may have loops at some edges. There is, in general, no upper bound on the cardinalities of sets we use, except that – to ensure that the class of all graphs together constitute a proper set – they all lie below some fixed inaccessible cardinal.

If x and y are elements of some set, we shall denote an ordered pair formed by them by (x, y) and the unordered pair $\{x, y\}$ formed by them by $xy (= yx)$. As with ordered pairs of the form (x, x) , we do allow unordered pairs xx .

A graph G with vertex set V and edge set E will typically be denoted by $G = (V, E)$; when we are dealing with different graphs, we may use the notation V_G for V and, similarly, E_G for E . A (*graph*) *homomorphism* is an edge preserving mapping from the vertex set of a graph into the vertex set of a graph. A *strong homomorphism* is a homomorphism that sends "no edges" to "no edges" and if it is also a bijection, it is called an *isomorphism*. For a graph $G = (V_G, E_G)$, a *subgraph* $H = (V_H, E_H)$ of G is a graph with $V_H \subseteq V_G$ and $E_H \subseteq E_G$. When $E_H = \{ab \mid a, b \in V_H \text{ and } ab \in E_G\}$, then H is called a *strong subgraph* (or *induced subgraph*) of G . For a homomorphism $f : G \rightarrow H$, the *image graph* $f(G)$ will always be the induced subgraph of H on the vertex set $f(V_G)$. In general, unless mentioned otherwise, if a subset V_H of V_G is regarded as a graph, it will be the subgraph induced by G on V_H . There are two (non-isomorphic) one-vertex graphs, called the *trivial graphs*; the one with a loop T_0 and the one without a loop T .

2. CONGRUENCES ON GRAPHS

As motivation for the definition and much of the subsequent results on graph congruences, we acknowledge [4].

Definition 2.1. Let $G = (V_G, E_G)$ be a graph. A *congruence on G* is a pair $\theta = (\sim, \mathcal{E})$ such that:

- (i) \sim is an equivalence relation on V_G ;
- (ii) \mathcal{E} is a set of unordered pairs of elements from V with $E_G \subseteq \mathcal{E}$; and
- (iii) (*Substitution Property* of \mathcal{E} with respect to \sim) when $x, y, x', y' \in V_G$, $x \sim x'$, $y \sim y'$, and $xy \in \mathcal{E}$, then $x'y' \in \mathcal{E}$.

A *strong congruence on G* is a pair $\theta = (\sim, \mathcal{E})$ where \sim is an equivalence relation on V_G and $\mathcal{E} = \{xy \mid x, y \in V_G \text{ and there are } x', y' \in V_G \text{ with } x \sim x', y \sim y' \text{ and } x'y' \in E_G\}$.

It can easily be verified that a strong congruence is also a congruence. Congruences can be partially ordered by the relation "contained in". Indeed, for two congruences $\alpha = (\sim_\alpha, \mathcal{E}_\alpha)$ and $\beta = (\sim_\beta, \mathcal{E}_\beta)$ on G , α is *contained in* β , written as $\alpha \subseteq \beta$, if $\sim_\alpha \subseteq \sim_\beta$ and $\mathcal{E}_\alpha \subseteq \mathcal{E}_\beta$. Let \simeq denote the identity relation on V_G (i.e., $x \simeq y$ if and only if $x = y$). The congruence $\iota_G := (\simeq, E_G)$ on G , is called the *identity congruence* on G and is the smallest congruence on G . It is, in fact, a strong congruence on G . The *universal congruence* on G is the pair $\nu_G = (\rightsquigarrow, \mathcal{E})$ where \rightsquigarrow is the universal relation (i.e., $a \rightsquigarrow b$ for all $a, b \in V_G$)

and $\mathcal{E} = \{ab \mid a, b \in V_G\}$. Clearly, any congruence on G is contained in v_G . As will be seen later, the next example is the prototype of all graph congruences.

The kernel of a homomorphism. Given any graph homomorphism $f : G \rightarrow H$, we define a congruence on G , called the *kernel of f* and written as $\ker f = (\sim_f, \mathcal{E}_f)$, by $\sim_f = \{(x, y) \mid x, y \in V_G, f(x) = f(y)\}$ and $\mathcal{E}_f = \{uv \mid u, v \in V_G, f(u)f(v) \in E_H\}$. It is immediately clear that $\ker f$ is a congruence on G . With f is also associated the *strong kernel of f* , written as $\text{sker } f = (\sim_f, \mathcal{E}_{sf})$ with the same equivalence relation but $\mathcal{E}_{sf} = \{xy \mid x, y \in V_G \text{ and there are } x', y' \in V_G \text{ with } x \sim x', y \sim y' \text{ and } x'y' \in E_G\}$. This is a strong congruence on G and $\text{sker } f \subseteq \ker f$; in fact, if $\theta = (\sim_f, \mathcal{E})$ is any congruence on G for some \mathcal{E} , then $\text{sker } f \subseteq \theta$. If f is a strong homomorphism, then $\ker f = \text{sker } f$. As is to be expected, it can easily be shown that injectivity of a homomorphism is equivalent to the equivalence relation \sim_f coinciding with \simeq . Moreover, the kernel of f is the identity congruence on G if and only if f is an injective strong homomorphism. If f is a strong surjective homomorphism, then it is an isomorphism if and only if the kernel of f is the identity congruence.

More examples. Given any equivalence \sim on the vertices of a graph G , the set $\mathcal{E} = \{xy \mid x, y \in V_G \text{ and there are } x', y' \in V_G \text{ with } x \sim x', y \sim y' \text{ and } x'y' \in E_G\}$ always gives a strong congruence $\rho = (\sim, \mathcal{E})$ on G . For another congruence on G , let $x \sim y$ be equality (i.e., $x \simeq y$) and $xy \in \mathcal{E}$ to mean that there is a (finite) path from x to y in G . Then (\simeq, \mathcal{E}) is a congruence on G and should one think of (V_G, \mathcal{E}) as a graph, then it is the graph obtained from $G = (V_G, E_G)$ by replacing each connected component of G by a complete graph on the vertex set of that component. More generally: the construction on a graph $G = (V_G, E_G)$ of leaving V_G intact while extending E_G to some edge set \mathcal{E} yielding the new graph (V_G, \mathcal{E}) corresponds to the congruence (\simeq, \mathcal{E}) on G . This correspondence is made clear in the next item.

Quotients. Given any congruence $\theta = (\sim, \mathcal{E})$ on a graph $G = (V_G, E_G)$, we define a new graph, denoted by $G/\theta = (V_{G/\theta}, E_{G/\theta})$ and called *the quotient of G modulo θ* , by taking $V_{G/\theta} := \{[x] \mid x \in V_G\}$ and $E_{G/\theta} := \{[x][y] \mid xy \in \mathcal{E}\}$. The natural or canonical mapping $p_\theta : G \rightarrow G/\theta$ given by $p_\theta(x) = [x]$ is a surjective homomorphism with $\ker p_\theta = \theta$. In particular, for $\theta = \iota_G$ we have G/ι_G isomorphic to G . If θ is a strong congruence, then p_θ is a strong homomorphism with $\text{sker } p_\theta = \theta$. In general, if we take $\theta = (\simeq, \mathcal{E})$ for some suitable \mathcal{E} to make θ a congruence on G , then G/θ is the graph with vertex set $V_{G/\theta} = V_G$ and edge set $E_{G/\theta} = \mathcal{E}$ (here we identify $[x] = \{x\}$ with x). If v_G is the universal congruence on G , then G/v_G is isomorphic to the trivial graph T_0 with a loop.

The lattice of congruences. For a given graph G , we denote the set of all congruences on G by $\mathcal{C}(G)$. We already know that $\mathcal{C}(G)$ is a partially ordered set with respect to containment \subseteq . We can say more. Any collection of congruences

$\{\theta_i = (\sim_i, \mathcal{E}_i) \mid i \in I\} \subseteq \mathcal{C}(G)$ has a greatest lower bound in $\mathcal{C}(G)$ given by $\bigcap_{i \in I} \theta_i = (\sim, \mathcal{E})$ where $a \sim b \Leftrightarrow a \sim_i b$ for all $i \in I$ and $ab \in \mathcal{E} \Leftrightarrow ab \in \mathcal{E}_i$ for all $i \in I$. Moreover, $\{\theta_i = (\sim_i, \mathcal{E}_i) \mid i \in I\}$ also has a least upper bound given by $\bigcup_{i \in I} \theta_i = (\sim, \mathcal{E})$ where $a \sim b \Leftrightarrow$ there are $i_1, i_2, \dots, i_n \in I$ and $a_{i_1}, a_{i_2}, \dots, a_{i_n} \in V, n \geq 2$, such that $a = a_{i_1} \sim_{i_1} a_{i_2} \sim_{i_2} a_{i_3} \sim_{i_3} \dots \sim_{i_{n-2}} a_{i_{n-1}} \sim_{i_{n-1}} a_{i_n} = b$ and $\mathcal{E} = \{ab \mid a, b \in V \text{ and there is an } i \in I \text{ and } a'b' \in \mathcal{E}_i \text{ with } a' \sim a \text{ and } b' \sim b\}$. It can easily be verified that $\bigcup_{i \in I} \theta_i$ is the least upper bound in $\mathcal{C}(G)$ for $\{\theta_i = (\sim_i, \mathcal{E}_i) \mid i \in I\}$.

When dealing with radicals, the basic tools are the appropriate versions of the algebraic isomorphism theorems for graph congruences.

Isomorphism theorems for congruences. Let $f : G \rightarrow H$ be a graph homomorphism with $\theta = (\sim, \mathcal{E})$ a congruence on G and $\alpha = (\sim_\alpha, \mathcal{E}_\alpha)$ a congruence on H . By $f(\theta)$ we mean the pair $(f(\sim), f(\mathcal{E}))$ where $f(\sim) = \{(f(a), f(b)) \mid a, b \in V_G, a \sim b\}$ and $f(\mathcal{E}) = \{f(a)f(b) \mid ab \in \mathcal{E}\}$. Of course, this need not be a congruence on H , but nevertheless it will be compared to α in the usual sense, meaning $f(\theta) \subseteq \alpha$ if and only if $f(\sim) \subseteq \sim_\alpha$ and $f(\mathcal{E}) \subseteq \mathcal{E}_\alpha$. We start with two auxiliary results.

Proposition 2.2. *Let $f : G \rightarrow H$ be a graph homomorphism. Then $f(\ker f) \subseteq \iota_H$ and if $\rho = (\sim_\rho, \mathcal{E}_\rho)$ is a congruence on G with $f(\rho) \subseteq \iota_H$, then $\rho \subseteq \ker f$.*

Proof. Let $a, b \in V_G$ with $a \sim_f b$. Then $f(a) = f(b)$ and so $f(\sim_f) \subseteq \simeq$. Let $ab \in \mathcal{E}_f$. Then $f(a)f(b) \in E_H$ and hence $f(\ker f) \subseteq \iota_H$. Let $\rho = (\sim_\rho, \mathcal{E}_\rho)$ be a congruence on G with $f(\rho) \subseteq \iota_H$. Then $f(\sim_\rho) \subseteq \simeq$ and hence, if $a, b \in V_G$ with $a \sim_\rho b$, then $f(a) = f(b)$ and thus $a \sim_f b$. Let $ab \in \mathcal{E}_\rho$. Since $f(\rho) \subseteq \iota_H$, $f(a)f(b) \in E_H$ and so $ab \in \mathcal{E}_f$. ■

Proposition 2.3. *Let $f : G \rightarrow H$ and $g : G \rightarrow K$ be surjective graph homomorphisms. Then $\ker f = (\sim_f, \mathcal{E}_f) \subseteq \ker g = (\sim_g, \mathcal{E}_g)$ if and only if there is a homomorphism $h : H \rightarrow K$ such that $h \circ f = g$.*

Proof. Suppose $\ker f = (\sim_f, \mathcal{E}_f) \subseteq \ker g = (\sim_g, \mathcal{E}_g)$, i.e., $\sim_f \subseteq \sim_g$ and $\mathcal{E}_f \subseteq \mathcal{E}_g$. Define $h : H \rightarrow K$ by $h(y) = g(x)$ where $x \in V_G$ with $f(x) = y$. This map is well-defined, for if $f(x') = y = f(x)$, then $x \sim_f x'$ and so $x \sim_g x'$ which gives $g(x) = g(x')$. Clearly $h \circ f = g$. Next it is shown that h preserves edges: Let $ab \in E_H$. By the surjectivity of f , there are $a', b' \in V_G$ with $f(a') = a$ and $f(b') = b$. Thus $f(a')f(b') \in E_H$ and so $a'b' \in \mathcal{E}_f \subseteq \mathcal{E}_g$ which gives $g(a')g(b') \in E_K$, i.e., $h(a)h(b) \in E_K$.

Conversely, suppose the homomorphism h with $h \circ f = g$ exists. If $f(a) = f(b)$, then $g(a) = h(f(a)) = h(f(b)) = g(b)$ and hence $\sim_f \subseteq \sim_g$. Let $ab \in \mathcal{E}_f$. Then $f(a)f(b) \in E_H$ and hence $h(f(a))h(f(b)) \in E_K$ which gives $g(a)g(b) \in E_K$. Thus $ab \in \mathcal{E}_g$ and $\mathcal{E}_f \subseteq \mathcal{E}_g$ follows. ■

Theorem 2.4 (*First Isomorphism Theorem*). *Let $f : G \rightarrow H$ be a homomorphism. Then $G/\ker f$ is isomorphic to $f(G)$ where $f(G)$ is the induced subgraph of H on $f(V_G)$. If f is surjective, then $G/\ker f$ is isomorphic to H . Moreover, if f is a surjective strong homomorphism, then $G/\ker f$ is isomorphic to H .*

Proof. Let $\theta = \ker f$ and let $p_\theta : G \rightarrow G/\theta$ be the canonical quotient map given by $p_\theta(x) = [x]$. Then we have two surjective homomorphisms $f : G \rightarrow f(G)$ and p_θ with $\ker f = \theta = \ker p_\theta$ and the previous proposition gives two homomorphisms $h : f(G) \rightarrow G/\theta$ and $k : G/\theta \rightarrow f(G)$ with $h \circ f = p_\theta$ and $k \circ p_\theta = f$. From this we conclude that k is an isomorphism. ■

Let G be a graph with induced subgraph H . Then a congruence $\theta = (\sim, \mathcal{E})$ on G induces a congruence $H \cap \theta = (\sim_H, \mathcal{E}_H)$ on H with $\sim_H = (V_H \times V_H) \cap \sim = \{(a, b) \mid a, b \in V_H \text{ and } a \sim b\}$ and $\mathcal{E}_H = \{ab \mid a, b \in V_H\} \cap \mathcal{E} = \{ab \mid a, b \in V_H \text{ with } ab \in \mathcal{E}\}$. The mapping $f : H \rightarrow G/\theta$ defined by $f(a) = [a]$ for all $a \in V_H$ is a homomorphism with $\ker f = H \cap \theta$. Now $f(V_H)$ is a set of vertices of G/θ on which we form the induced subgraph of G/θ , denoted by $(H + \theta)/\theta$. Then, by the First Isomorphism Theorem, we have:

Theorem 2.5 (*Second Isomorphism Theorem*). *Let H be an induced subgraph of a graph G . Let θ be a congruence on G . Then $H \cap \theta$ as defined above is a congruence on H and $H/H \cap \theta \cong (H + \theta)/\theta$ where the latter graph is the induced subgraph of G/θ on the vertex set $\{[a] \mid a \in V_H\}$.*

Theorem 2.6 (*Third Isomorphism Theorem*). *Let G be a graph with two congruences $\theta_1 = (\sim_1, \mathcal{E}_1)$ and $\theta_2 = (\sim_2, \mathcal{E}_2)$ on G for which $\theta_1 \subseteq \theta_2$. Then $\theta_2/\theta_1 := (\sim, \mathcal{E})$ is a congruence on G/θ_1 where $[a]_1 \sim [b]_1 \Leftrightarrow a \sim_2 b$ and $[a]_1[b]_1 \in \mathcal{E} \Leftrightarrow ab \in \mathcal{E}_2$. Moreover, $(G/\theta_1)/(\theta_2/\theta_1)$ is isomorphic to G/θ_2 .*

Proof. Clearly \sim is an equivalence relation on G/θ_1 and $E_{G/\theta_1} = \{[a]_1[b]_1 \mid ab \in \mathcal{E}_1\} \subseteq \{[a]_1[b]_1 \mid ab \in \mathcal{E}_2\} = \mathcal{E}$ since $\theta_1 \subseteq \theta_2$. Moreover, to verify the Substitution Property of \mathcal{E} with respect to \sim , suppose $[a]_1[b]_1 \in \mathcal{E}$ with $[a]_1 \sim [a']_1$ and $[b]_1 \sim [b']_1$. This means $ab \in \mathcal{E}_2$ with $a \sim_2 a'$ and $b \sim_2 b'$. Thus $a'b' \in \mathcal{E}_2$ and so $[a']_1[b']_1 \in \mathcal{E}$. Hence θ_2/θ_1 is a congruence on G/θ_1 .

Define a mapping $f : G/\theta_1 \rightarrow G/\theta_2$ by $f([a]_1) = [a]_2$. It can be checked that this is a surjective homomorphism with $\ker f = \theta_2/\theta_1$. By the First Isomorphism Theorem we conclude that $(G/\theta_1)/(\theta_2/\theta_1) \cong G/\theta_2$. ■

A related result often used is:

Theorem 2.7. *Let G be a graph with θ a fixed congruence on G . Any congruence ξ of the graph G/θ is of the form α/θ for some congruence α on G with $\theta \subseteq \alpha$. Moreover, there is a one-to-one correspondence between $\{\alpha \mid \alpha \text{ is a congruence on } G \text{ with } \theta \subseteq \alpha\}$ and $\mathcal{C}(G/\theta)$ which preserves inclusions, intersections and unions of congruences.*

Proof. Let $\xi = (\sim_\xi, \mathcal{E}_\xi)$ be a congruence on G/θ . Define a congruence $\alpha = (\sim_\alpha, \mathcal{E}_\alpha)$ on G by $a \sim_\alpha b \Leftrightarrow [a]_\theta \sim_\xi [b]_\theta$ and $\mathcal{E}_\alpha = \{ab \mid [a]_\theta [b]_\theta \in \mathcal{E}_\xi\}$. It is straightforward to verify that $\theta \subseteq \alpha$ and that $\alpha/\theta = \xi$. Moreover, for two congruences α and β on G with $\theta \subseteq \alpha$ and $\theta \subseteq \beta$, the equality $\alpha/\theta = \beta/\theta$ holds if and only if $\alpha = \beta$ which establishes the required one-to-one correspondence. The preservation of inclusions, intersections and unions under this correspondence can easily be verified. ■

Products and subdirect products of graphs. For an index set I , let $G_i = (V_i, E_i)$ be a graph for all $i \in I$. The product $\prod_{i \in I} G_i$ of the graphs G_i is the graph $\prod_{i \in I} G_i = (\prod_{i \in I} V_i, E)$ where $\prod_{i \in I} V_i$ is just the usual Cartesian product of the sets V_i and $E = \{fg \mid f, g \in \prod_{i \in I} V_i \text{ with } f(i)g(i) \in E_i \text{ for all } i \in I\}$. For every $j \in I$, the j -th projection $\pi_j : \prod_{i \in I} G_i \rightarrow G_j$ defined by $\pi_j(f) = f(j)$ for all $f \in \prod_{i \in I} V_i$ is a surjective homomorphism. An induced subgraph H of $\prod_{i \in I} G_i$ is called a *subdirect product of the graphs $G_i, i \in I$* , provided the restriction of each projection π_j to H is a surjective mapping onto G_j . As in universal algebra, subdirect products can be expressed in terms of congruences and quotients.

Theorem 2.8. *For each $i \in I$, let θ_i be a congruence on a graph $G = (V_G, E_G)$ with $\theta = \bigcap_{i \in I} \theta_i$. Then G/θ is isomorphic to a subdirect product of the quotient graphs $G/\theta_i, i \in I$.*

Proof. For each $j \in I$, let $\theta_j = (\sim_j, \mathcal{E}_j)$ with $p_j : G \rightarrow G/\theta_j$ the canonical homomorphism, say $G_j = G/\theta_j = (V_j, E_j)$; $V_j = \{[a]_j \mid a \in V_G\}$ and $E_j = \{[a]_j [b]_j \mid ab \in \mathcal{E}_j\}$. Let $P = (V_P, E_P)$ be the product of the graphs $G_i = G/\theta_i, i \in I$ with $\pi_j : P \rightarrow G_j$ the j -th projection. This means $V_P = \{a \mid a : I \rightarrow \bigcup_{i \in I} G/\theta_i \text{ a function with } a(i) \in V_i \text{ for all } i \in I\}$, $E_P = \{ab \mid a, b \in V_P \text{ and } a(i)b(i) \in E_i \text{ for all } i \in I\}$ and $\pi_j(a) = a(j)$.

Define a function $f : G \rightarrow P$ by $f(a) = f_a$ where $f_a : I \rightarrow \bigcup_{i \in I} G/\theta_i$ is the function defined by $f_a(j) = [a]_j = p_j(a)$ for all $j \in I$. Thus $\pi_j \circ f = p_j$ for all j and it can easily be verified that f is a homomorphism with $\ker f = \theta = \bigcap_{i \in I} \theta_i$. ■

In particular, it then follows that:

Corollary 2.9. *A graph G is a subdirect product of graphs $G_i, i \in I$, if and only if for every $i \in I$ there are congruences θ_i on G with G_i isomorphic to G/θ_i and $\bigcap_{i \in I} \theta_i = \iota_G$.*

Proof. The necessity is clear in view of the theorem; so suppose that G is a subdirect product of the graphs $G_i = (V_{G_i}, E_{G_i})$ for all $i \in I$. If P denotes the product $P = \prod_{i \in I} G_i$ with $\pi_j : P \rightarrow G_j$ the j -th projection, then G is an induced subgraph of P and $\pi_j(G) = G_j$ for $j \in I$. Let $\theta_j = \ker \pi_j$, say $\theta_j = (\sim_j, \mathcal{E}_j)$. Thus $G/\theta_j \cong G_j$ for all $j \in I$. Let $\bigcap_{i \in I} \theta_i = (\sim, \mathcal{E})$. To complete the proof, we show $\bigcap_{i \in I} \theta_i = \iota_G$. For $a, b \in V_G$, $a \sim b \Leftrightarrow a \sim_j b$ for all $j \in I \Leftrightarrow a(j) = \pi_j(a) =$

$\pi_j(b) = b(j)$ for all $j \in I \Leftrightarrow a = b$. Furthermore, $ab \in \mathcal{E} \Leftrightarrow a, b \in V_G$ and $ab \in \mathcal{E}_j$ for all $j \in I \Leftrightarrow a, b \in V_G$ and $\pi_j(a)\pi_j(b) \in E_G$ for all $j \in I \Leftrightarrow a, b \in V_G$ and $a_j b_j \in E_{G_j}$ for all $j \in I \Leftrightarrow a, b \in V_G$ and $ab \in E_P \Leftrightarrow ab \in E_G$ and we are done. ■

Although not required for what follows, it may be worthwhile to investigate a graph theoretical version of a classical result in Universal Algebra commonly known as Birkhoff's Theorem [1]. A graph G is called *subdirectly irreducible* if it is not a subdirect product of graphs (or equivalently, any intersection of non-identity congruences is not the identity congruence). Birkhoff's Theorem states that an algebra with more than one element is a subdirect product of subdirectly irreducible algebras. One may then want to investigate subdirectly irreducible graphs more closely, but these matters will not be pursued here.

It also makes sense to form the product of congruences. For each graph $G_i = (V_i, E_i)$, let $\theta_i = (\sim_i, \mathcal{E}_i)$ be a congruence on $G_i, i \in I$. Then $\prod_{i \in I} \theta_i = (\sim, \mathcal{E})$ is the congruence on $G = \prod_{i \in I} G_i$ defined by: For $a, b \in G, a \sim b \Leftrightarrow a(j) \sim_j b(j)$ for all $j \in I$ and $ab \in \mathcal{E} \Leftrightarrow a(j)b(j) \in \mathcal{E}_j$ for all $j \in I$. This congruence can be described in another way. Let $\pi_j : G \rightarrow G_j$ be the j -th projection and $p_j : G_j \rightarrow G_j/\theta_j$ the canonical quotient map. Then $f_j = p_j \circ \pi_j : G \rightarrow G_j/\theta_j$ is a homomorphism and by the universal property of the product $\prod_{i \in I} (G_i/\theta_i)$, there is a unique homomorphism $f : G \rightarrow \prod_{i \in I} (G_i/\theta_i)$ such that $\pi'_j \circ f = f_j$ for all $j \in I$ where $\pi'_j : \prod_{i \in I} (G_i/\theta_i) \rightarrow G_j/\theta_j$ is the j -th projection. More explicitly, f is given by $f(a) = \bar{a}$, say, where $a \in V_G$ and $\bar{a} : I \rightarrow \bigcup_{i \in I} G_i/\theta_i$ is the map $\bar{a}(j) = [a(j)]_{\theta_j}$. Then $\ker f = \prod_{i \in I} \theta_i$. Indeed, we record and prove this in:

Proposition 2.10. *For each $i \in I$, let $\theta_i = (\sim_i, \mathcal{E}_i)$ be a congruence on the graph G_i . Let $\prod_{i \in I} \theta_i = (\sim, \mathcal{E})$ be the product congruence on $G = \prod_{i \in I} G_i$ as defined above. For each $j \in I$, let $f_j : G \rightarrow G_j/\theta_j$ be the canonical surjective homomorphism as above with $f : G \rightarrow \prod_{i \in I} (G_i/\theta_i)$ the unique homomorphism for which $\pi'_j \circ f = f_j$ for all $j \in I$ where $\pi'_j : \prod_{i \in I} (G_i/\theta_i) \rightarrow G_j/\theta_j$ is the j -th projection. Then $\ker f = \prod_{i \in I} \theta_i$.*

Proof. Let $G' = \prod_{i \in I} (G_i/\theta_i)$ and $G'_j = G_j/\theta_j$. Let $\ker f = (\sim', \mathcal{E}')$ and choose $a, b \in G$. Then $a \sim' b \Leftrightarrow f(a) = f(b) \Leftrightarrow a(j) \sim_j b(j)$ for all $j \in I \Leftrightarrow a \sim b$. Furthermore, $ab \in \mathcal{E}' \Leftrightarrow f(a)f(b) \in E_{G'} \Leftrightarrow [a(j)]_{\theta_j}[b(j)]_{\theta_j} \in E_{G'}$ for all $j \in I \Leftrightarrow a(j)b(j) \in \mathcal{E}_j$ for all $j \in I \Leftrightarrow ab \in \mathcal{E}$. Hence $\ker f = \prod_{i \in I} \theta_i$. ■

3. THE HOEHNKE RADICAL OF A GRAPH

We now define a Hoehnke radical on graphs in terms of congruences. At the outset, we firstly need to specify the universum in which this is to be done. For radical theory this is usually in a prescribed universal class. A class \mathcal{W} of graphs is called a *universal class* if it is non-empty, closed under homomorphic

images and closed under the taking of subgraphs (= strongly hereditary). It should also be mentioned that we do not distinguish between isomorphic graphs (e.g., in the definition below, if G and H are isomorphic graphs in \mathcal{W} , then $\rho(G) = \rho(H)$). From the definition, it follows that \mathcal{W} contains a one-vertex graph, and consequently the class $\mathcal{T} = \{T_0, T\}$ of all trivial graphs in \mathcal{W} . Note that there is a unique congruence on T_0 ($\iota_{T_0} = \nu_{T_0}$) but on T it is possible to define two different congruences ($\iota_T \neq \nu_T$). All considerations relating to the radicals of graphs will be inside the class \mathcal{W} . A class of graphs \mathcal{M} in \mathcal{W} is an *abstract class* provided it contains the trivial graph T_0 and it is closed under isomorphic copies. All subclasses of \mathcal{W} under consideration will be assumed to be abstract, even though it may not always be explicitly stated. Since $G/\nu_G \cong T_0$ for any graph G , there is always at least one congruence θ on a graph G for which G/θ is in any abstract class. For a class \mathcal{M} in \mathcal{W} , we use $\overline{\mathcal{M}}$ to denote the *subdirect closure* of \mathcal{M} , i.e., the class of all graphs that are subdirect products of graphs from \mathcal{M} . Clearly $\mathcal{M} \subseteq \overline{\mathcal{M}}$ and we say \mathcal{M} is *subdirectly closed* if $\mathcal{M} = \overline{\mathcal{M}}$. In the definition below, we will need the image of a congruence under a homomorphism as defined in the lines preceding Proposition 2.2.

Definition 3.1. *A Hoehnke radical on \mathcal{W} is a function ρ that assigns to every graph G in \mathcal{W} a congruence $\rho(G) = \rho_G$ on G such that*

- (H1) *for every homomorphism $f : G \rightarrow H$, $f(\rho(G)) \subseteq \rho(f(G))$; and*
- (H2) *$\rho(G/\rho_G) = \iota_{G/\rho_G}$, the identity congruence on G/ρ_G .*

For a radical ρ , if $\rho(G) = \iota_G$, then G is called *semisimple* (actually, ρ -semisimple), the class $\mathcal{S}_\rho = \{G \in \mathcal{W} \mid \rho(G) = \iota_G\}$ is called the *associated semisimple class* and $\mathcal{R}_\rho = \{G \in \mathcal{W} \mid G/\rho_G \text{ is a trivial graph}\}$ is the *associated radical class*.

There are other types of general radicals as well, like Plotkin radicals, Kurosh-Amitsur radicals, etc. By radical we will mean a Hoehnke radical as defined above and if we need another type of radical, it will be called appropriately. The original motivation in ring theory for collapsing congruence classes of ring elements (modulo the radical ideal of the ring) to single elements of the quotient ring – namely some “badness” in their structural behaviour – can be transferred to radicals on graphs, but only with a pinch of salt. After all, in graphs there are no “zero” vertices, and hence no concept “nilpotency”. And yet, in the examples of graph radicals which follow, we suggest contextual reasons for squashing the individuality out of vertices which are in the same congruences class.

Another remark here is important. As mentioned earlier, a radical theory for universal classes of graphs like the ones under discussion here, has already been developed earlier in [9] by Fried and Wiegandt. In this theory, they defined and investigated the graph theoretical versions of the Kurosh-Amitsur (KA for short) radical classes called *connectednesses* and the semisimple classes called

disconnectedness. This terminology is highly suggestive of the examples of such classes. For example, the class of all connected graphs is a radical class (= connectedness), the corresponding radical congruence partitions any graph into its connected components and the resulting quotient graph reduces each component to a single vertex with a loop to end up with a rather disconnected graph in the corresponding semisimple class (= disconnectedness).

Hoehnke radicals are much more general than KA-radicals and what we will present here will not contribute anything new to the KA-radical theory of graphs. But what is new here, is the Hoehnke radicals of graphs and then the subsequent requirements on these radicals that will lead to the KA-radicals. Once this has been done (in Section 5), we will give examples.

For a Hoehnke radical ρ on \mathcal{W} , note that $\{T_0\} \subseteq \mathcal{S}_\rho \cap \mathcal{R}_\rho \subseteq \mathcal{T} \subseteq \mathcal{R}_\rho$ and the radical class \mathcal{R}_ρ is always homomorphically closed (and thus also strongly homomorphically closed). Next we really get to the essence of the radical and show that a Hoehnke radical is very general and is always of a prescribed form.

Theorem 3.2. *Let ρ be a mapping that assigns to any graph G in \mathcal{W} a congruence $\rho(G) = \rho_G$ on G . Then ρ is a Hoehnke radical on \mathcal{W} if and only if there is an abstract class of graphs \mathcal{M} in \mathcal{W} such that for all G in \mathcal{W} , $\rho(G) = \bigcap \{\theta \mid \theta \text{ is a congruence on } G \text{ for which } G/\theta \in \mathcal{M}\}$. Furthermore, $\mathcal{S}_\rho = \overline{\mathcal{M}}$.*

Proof. Let ρ be a Hoehnke radical. Then the semisimple class $\mathcal{S}_\rho = \{G \in \mathcal{W} \mid \rho(G) = \iota_G\}$ is an abstract class. We show that $\rho(G) = \varphi$ where $\varphi = \bigcap \{\theta \mid \theta \text{ is a congruence on } G \text{ for which } G/\theta \in \mathcal{S}_\rho\}$. By (H2), $\rho(G/\rho_G) \in \mathcal{S}_\rho$ and so $\varphi \subseteq \rho(G)$ since $\rho(G)$ is just one of these θ 's. Let θ be any congruence on G for which $G/\theta \in \mathcal{S}_\rho$. If $p_\theta : G \rightarrow G/\theta$ is the canonical quotient map, then by (H1) we have $p_\theta(\rho(G)) \subseteq \rho(p_\theta(G)) = \rho(G/\theta) = \iota_{G/\theta}$. This means $\rho(G) \subseteq \theta$, hence $\rho(G) \subseteq \varphi$ and $\rho(G) = \bigcap \{\theta \mid \theta \text{ is a congruence on } G \text{ for which } G/\theta \in \mathcal{S}_\rho\}$ follows.

We still have to show $\overline{\mathcal{S}_\rho} = \mathcal{S}_\rho$. Suppose a graph G is a subdirect product of graphs $G_i \in \mathcal{S}_\rho$, $i \in I$. By Theorem 2.8 we know that there are congruences α_i on G with $G/\alpha_i = G_i \in \mathcal{S}_\rho$ and $\bigcap_{i \in I} \alpha_i = \iota_G$. But then $\rho(G) = \bigcap \{\theta \mid \theta \text{ is a congruence on } G \text{ for which } G/\theta \in \mathcal{S}_\rho\} \subseteq \bigcap_{i \in I} \alpha_i = \iota_G$ and hence $G \in \mathcal{S}_\rho$.

Conversely, suppose $\rho(G) = \bigcap \{\theta \mid \theta \text{ is a congruence on } G \text{ for which } G/\theta \in \mathcal{M}\}$ for all G in \mathcal{W} where \mathcal{M} is some abstract class of graphs in \mathcal{W} . For (H1), let θ be a congruence on G and consider the canonical surjective homomorphism $f : G \rightarrow G/\theta$. We show that $f(\rho(G)) \subseteq \rho(f(G)) = \rho(G/\theta)$. Let α be a congruence on G/θ with $(G/\theta)/\alpha \in \mathcal{M}$. By the Third Isomorphism Theorem, α is of the form $\alpha = \beta/\theta$ for some congruence β on G with $\theta \subseteq \beta$ and G/β is isomorphic to $(G/\theta)/\alpha \in \mathcal{M}$. Hence $\rho(G) \subseteq \beta$ and so $f(\rho(G)) \subseteq \beta/\theta = \alpha$ from which it can be concluded that $f(\rho(G)) \subseteq \rho(G/\theta)$. For (H2), using the Third Isomorphism Theorem again, we have

$$\begin{aligned}
 \rho(G/\rho_G) &= \bigcap \{ \alpha \mid \alpha \text{ is a congruence on } G/\rho_G \text{ with } (G/\rho_G)/\alpha \in \mathcal{M} \} \\
 &= \bigcap \{ \beta/\rho_G \mid \beta \text{ is a congruence on } G, \rho_G \subseteq \beta \text{ and} \\
 &\quad G/\beta \cong (G/\rho_G)/(\beta/\rho_G) \in \mathcal{M} \} \\
 &= \rho_G/\rho_G = \iota_{G/\rho_G},
 \end{aligned}$$

which gives (H2).

Lastly we show $\mathcal{S}_\rho = \overline{\mathcal{M}}$. For this we first observe that $\mathcal{M} \subseteq \mathcal{S}_\rho$ since $G/\iota_G \cong G \in \mathcal{M}$ which gives $\rho(G) = \iota_G$. Then $\overline{\mathcal{M}} \subseteq \overline{\mathcal{S}_\rho} = \mathcal{S}_\rho$, the last equality follows from the first part of the proof. If $G \in \mathcal{S}_\rho$, then $\iota_G = \rho(G) = \bigcap \{ \theta \mid \theta \text{ is a congruence on } G \text{ for which } G/\theta \in \mathcal{M} \}$ which means G is a subdirect product of graphs from \mathcal{M} (Corollary 2.9) and hence in $\overline{\mathcal{M}}$. ■

To emphasize the salient features of the radical contained in the above result, we repeat them in the next corollary.

Corollary 3.3. (1) *The semisimple class of any Hoehnke radical is subdirectly closed.*

(2) *For a Hoehnke radical ρ , $\rho(G)$ is the smallest congruence on G for which $G/\rho(G)$ is semisimple (i.e., if θ is a congruence on G with $G/\theta \in \mathcal{S}_\rho$, then $\rho(G) \subseteq \theta$). Or, equivalently, $G/\rho(G)$ is the largest semisimple image of G (in the following sense: if $g : G \rightarrow H$ is a surjective homomorphism with $H \in \mathcal{S}_\rho$, then there is a homomorphism $h : G/\rho(G) \rightarrow H$ such that $h \circ p = g$ where $p : G \rightarrow G/\rho(G)$ is the canonical quotient map).*

(3) *For any abstract class of graphs \mathcal{M} in \mathcal{W} , define $\rho(G) = \bigcap \{ \theta \mid \theta \in \mathcal{C}(G) \text{ with } G/\theta \in \mathcal{M} \}$ for all $G \in \mathcal{W}$. Then ρ is a Hoehnke radical with $\mathcal{S}_\rho = \overline{\mathcal{M}}$ and every semisimple graph is a subdirect product of graphs from \mathcal{M} .*

Number (3) above is really the holy grail of radical theory. One would like to have a nice, well-behaved class of graphs \mathcal{M} and then, if a graph is semisimple with respect to the corresponding radical, it is a subdirect product of graphs from the class \mathcal{M} . Properties of the class or of the graphs inside the class \mathcal{M} may lead to stronger representations as is often seen, for example, in the radical theory of rings.

In general radical theory, hereditariness has proven to be a useful property. A class \mathcal{M} of graphs is said to be *hereditary* (respectively *strongly hereditary*) if $G \in \mathcal{M}$ implies all the induced subgraphs of G (respectively all the subgraphs of G) are in \mathcal{M} . Our next result shows that hereditariness is retained under subdirect closures.

Proposition 3.4. *If the class $\mathcal{M} \subseteq \mathcal{W}$ is hereditary, then so is its subdirect closure $\overline{\mathcal{M}}$.*

Proof. Let $G \in \overline{\mathcal{M}}$; say G is the subdirect product of the graphs $G/\theta_i \in \mathcal{M}$ where θ_i is a congruence on G for all $i \in I$ with $\bigcap_{i \in I} \theta_i = \iota_G$. Let H be an induced subgraph of G . Then $H \cap \theta_i$ is a congruence on H and by the Second Isomorphism Theorem, we have $H/H \cap \theta_i \cong (H + \theta_i)/\theta_i$ where the right hand side is the induced subgraph of $G/\theta_i \in \mathcal{M}$ on the vertex set $\{[a] \mid a \in H\}$. Since \mathcal{M} is hereditary, $H/H \cap \theta_i \in \mathcal{M}$. From $\bigcap_{i \in I} (H \cap \theta_i) = H \cap (\bigcap_{i \in I} \theta_i) = H \cap \iota_G = \iota_H$ and Corollary 2.9 we get $H \in \overline{\mathcal{M}}$. ■

4. RADICALS OF CONGRUENCES AND PRODUCTS

In this section we shall extend the radical concept from the radical of a graph to the radical of a congruence on a graph. For a Hoehnke radical ρ on \mathcal{W} and a congruence θ on a graph G , we know that $\rho(G/\theta)$ is a congruence on G/θ and consequently of the form θ^*/θ for some congruence θ^* on G . In order to get our hands on this congruence θ^* , we need the radical of a congruence. Recall that a *closure operator* on a poset $\langle P, \prec \rangle$ (see [7], p.145) is a mapping $c : P \rightarrow P$ which is extensive, order preserving and idempotent. We utilize this concept as it applies to the lattice $\langle \mathcal{C}(G), \subseteq \rangle$ of congruences of a graph.

Let ρ be a Hoehnke radical on the universal class \mathcal{W} . For $G \in \mathcal{W}$ and θ a congruence on G , we define the *radical ρ^* of the congruence θ* , denoted by $\rho^*(\theta)$, as the kernel of the composition of the canonical homomorphisms $G \rightarrow G/\theta \rightarrow (G/\theta)/(\rho(G/\theta))$. We do have a more explicit description of $\rho^*(\theta)$; in fact $\rho^*(\theta) = \theta^*$ where θ^* is the congruence on G with $\theta \subseteq \theta^*$ and $\rho(G/\theta) = \theta^*/\theta$. This follows easily from the properties of congruences treated in Section 2. Moreover, note that, by the Third Isomorphism Theorem, $G/\theta^* \cong (G/\theta)/(\theta^*/\theta) = (G/\theta)/\rho(G/\theta) \in \mathcal{S}_\rho$. In fact, in the theorem below we will see that θ^* is the smallest congruence on G with the properties $\theta \subseteq \theta^*$ and $G/\theta^* \in \mathcal{S}_\rho$. Another description of ρ^* as well as its main properties are given in the next result.

Theorem 4.1. *Let ρ be a Hoehnke radical on the universal class \mathcal{W} and let ρ^* be the associated radical of congruences. Then:*

- (1) *For any $G \in \mathcal{W}$ and congruence θ on G , $\rho^*(\theta) = \bigcap \{\alpha \mid \alpha \text{ is a congruence on } G \text{ with } \theta \subseteq \alpha \text{ and } G/\alpha \text{ semisimple}\}$.*
- (2) *The mapping $\rho^* : \mathcal{C}(G) \rightarrow \mathcal{C}(G)$ is a closure operator on the poset $\langle \mathcal{C}(G), \subseteq \rangle$.*
- (3) *For any G , $\rho^*(\iota_G) = \rho(G)$ and $\rho^*(\rho(G)) = \rho(G)$.*

Proof. (1) Let α be a congruence on G with $\theta \subseteq \alpha$ and G/α semisimple. Then α/θ is a congruence on G/θ and $(G/\theta)/(\alpha/\theta) \cong G/\alpha \in \mathcal{S}_\rho$; hence $\theta^*/\theta = \rho(G/\theta) \subseteq \alpha/\theta$ by Corollary 3.3(2). Thus $\rho^*(\theta) = \theta^* \subseteq \bigcap \{\alpha \mid \alpha \text{ is a congruence on } G \text{ with } \theta \subseteq \alpha \text{ and } G/\alpha \text{ semisimple}\}$. The equality then follows immediately since θ^* is just one of the α 's over which the intersection is taken.

(2) Let θ be a congruence on G . Already we know $\theta \subseteq \theta^* = \rho^*(\theta)$. Now $\theta^{**}/\theta^* = \rho(G/\theta^*) = \rho((G/\theta)/(\theta^*/\theta)) = \rho((G/\theta)/\rho(G/\theta)) = \iota_{G/\theta}$ by (H2); hence $\rho^*(\rho^*(\theta)) = \rho^*(\theta^*) = \theta^{**} = \theta^* = \rho^*(\theta)$. Lastly, for a closure operation, we have to show ρ^* is order preserving. Let α, β be two congruences on G with $\alpha \subseteq \beta$. Now β^* is a congruence on G with $\alpha \subseteq \beta^*$ and $G/\beta^* \in \mathcal{S}_\rho$. Thus, by (1), we get $\alpha^* \subseteq \beta^*$.

(3) The first equality follows immediately from Theorem 3.2 by taking $\theta = \iota_G$, for the second we call on the idempotency of ρ^* . ■

Corollary 4.2. $\rho^*(\mathcal{C}(G)) = \{\rho^*(\theta) \mid \theta \in \mathcal{C}(G)\}$ is a sub-meet-semilattice of $(\mathcal{C}(G), \subseteq)$, called the poset of ρ^* -radical congruences on G .

Proof. For $\alpha, \beta \in \mathcal{C}(G)$, we need to show $\rho^*(\alpha) \cap \rho^*(\beta) \in \rho^*(\mathcal{C}(G))$, i.e., $\alpha^* \cap \beta^* = \gamma^*$ for some $\gamma \in \mathcal{C}(G)$. Let $\gamma = \alpha^* \cap \beta^*$. Then γ is a congruence on G and $\gamma^*/\gamma = \rho(G/\gamma) = \iota_{G/\gamma}$. This last equality follows since $G/\gamma = G/(\alpha^* \cap \beta^*)$ which is a subdirect product of G/α^* and G/β^* . We know that both G/α^* and G/β^* are semisimple and that \mathcal{S}_ρ is closed under subdirect products; hence also G/γ is semisimple. We conclude that $\rho^*(\alpha) \cap \rho^*(\beta) = \alpha^* \cap \beta^* = \gamma = \gamma^* = \rho^*(\gamma) \in \rho^*(\mathcal{C}(G))$. ■

In general, using the same argument as in the above proof, it can be shown that for any collection of congruences $\theta_i, i \in I$, on a graph G , $\rho^*(\bigcap_{i \in I} \theta_i^*) = \rho^*(\bigcap_{i \in I} \rho^*(\theta_i)) = \bigcap_{i \in I} \rho^*(\rho^*(\theta_i)) = \bigcap_{i \in I} \rho^*(\theta_i) = \bigcap_{i \in I} \theta_i^*$. Another property we want to emphasize, already suggested by (3) in the above theorem and the proof of the corollary, is that for a congruence θ on G , $\rho^*(\theta) = \theta \Leftrightarrow \theta^* = \theta \Leftrightarrow \theta^*/\theta = \rho(G/\theta) = \iota_{G/\theta} \Leftrightarrow G/\theta$ is semisimple.

Radicals and products. We consider radicals on direct products of graphs. In particular, if ρ is a radical on \mathcal{W} and for all $i \in I$ we have $G_i \in \mathcal{W}$ with also $G = \prod_{i \in I} G_i$ in \mathcal{W} , we want to know the relationship between $\rho(G)$ and the $\rho(G_i)$'s. It will be shown that $\rho(G) \subseteq \prod_{i \in I} \rho(G_i)$. This corresponds to what is known for products and radical theory in general and no other general results are known.

Proposition 4.3. Let ρ be a radical on \mathcal{W} and for all $i \in I$, let $G_i \in \mathcal{W}$ with $G = \prod_{i \in I} G_i$ in \mathcal{W} . Then $\rho(G) \subseteq \prod_{i \in I} \rho(G_i)$.

Proof. We use Proposition 2.10 with $\theta_i = \rho(G_i)$ for all $i \in I$. Then $G_i/\theta_i = G_i/\rho(G_i) \in \mathcal{S}_\rho$. Any direct product is also a subdirect product, so by Corollary 3.3(1) we have $\prod_{i \in I} G_i/\theta_i \in \mathcal{S}_\rho$. Since $f : G \rightarrow \prod_{i \in I} G_i/\theta_i$ is a surjective homomorphism and $G/\ker f \cong \prod_{i \in I} G_i/\theta_i \in \mathcal{S}_\rho$, Corollary 3.3 gives $\rho(G) \subseteq \ker f = \prod_{i \in I} \theta_i = \prod_{i \in I} \rho(G_i)$ as required. ■

5. KUROSH-AMITSUR RADICALS: CONNECTEDNESSES AND
DISCONNECTEDNESSES

The basis for general radical theory is to be found in the fundamental work of Wedderburn, Köthe and Jacobson on the structure of algebras and rings, see [10]. These led to the axiomatization of radicals, independently formulated by Kurosh and Amitsur in the early 1950's. Subsequently these general radicals became known as Kurosh-Amitsur radicals and form the basis of general radical theory which spread to and found applications, not only in algebra, but also in topology, graphs, Petri nets, etc. Here we will discuss KA-radicals in the universal class \mathcal{W} of graphs. We will not pursue these radicals in detail here. We will only look at their relation to the Hoehnke radicals discussed above.

A class $\mathcal{R} \subseteq \mathcal{W}$ is a *connectedness* (= *KA-radical class*) if it satisfies the following condition. A graph $G \in \mathcal{W}$ is in \mathcal{R} if and only if every non-trivial homomorphic image of G has a non-trivial induced subgraph which is in \mathcal{R} .

A class $\mathcal{S} \subseteq \mathcal{W}$ is a *disconnectedness* (= *KA-semisimple class*) if it satisfies the following condition: A graph $G \in \mathcal{W}$ is in \mathcal{S} if and only if every non-trivial induced subgraph of G has a non-trivial homomorphic image which is in \mathcal{S} .

Unless mentioned otherwise, most of the statements to follow are standard radical theoretical results and will not be verified. For proofs, and many additional statements and properties, see for example Fried and Wiegandt [9]. The class of trivial graphs \mathcal{T} is always contained in any connectedness and also in any disconnectedness. It is easy to find examples of connectednesses and disconnectednesses. If $\mathcal{M} \subseteq \mathcal{W}$ is a hereditary class, then $\mathcal{UM} = \{G \in \mathcal{W} \mid \mathcal{G} \text{ has no non-trivial homomorphic image in } \mathcal{M}\}$ is a connectedness and if $\mathcal{H} \subseteq \mathcal{W}$ is a homomorphically closed class, then $\mathcal{DH} = \{G \in \mathcal{W} \mid G \text{ has no non-trivial induced subgraph in } \mathcal{H}\}$ is a disconnectedness.

Proposition 5.1 [9]. *Any connectedness is homomorphically closed and any disconnectedness is strongly hereditary and closed under subdirect products.*

From the preceding, we thus have: If \mathcal{R} is a connectedness, then \mathcal{DR} is a disconnectedness and if \mathcal{S} is a disconnectedness, then \mathcal{US} is a connectedness. Moreover, it can be shown that a class $\mathcal{R} \subseteq \mathcal{W}$ is a connectedness if and only if $\mathcal{R} = \mathcal{UDR}$ and a class $\mathcal{S} \subseteq \mathcal{W}$ is a disconnectedness if and only if $\mathcal{S} = \mathcal{DUS}$. For a Hoehnke radical ρ , one always has $\mathcal{R}_\rho = \mathcal{US}_\rho$. One of the features of KA-semisimple classes in general radical theory is that any object in the universal class under consideration has a maximal semisimple image. This is the main result from [9] which we recall here for our purposes. As noted earlier, whenever a subset of vertices of a graph is considered as a graph and nothing else is mentioned, it is the subgraph induced by the graph on this set of vertices.

Theorem 5.2 [9]. *Let \mathcal{R} be a connectedness with corresponding disconnectedness \mathcal{S} . Then*

- (a) *For every $G \in \mathcal{W}$, there is a strong homomorphism $s_G : G \rightarrow G_{\mathcal{S}}$ with $G_{\mathcal{S}} \in \mathcal{S}$ and if $f : G \rightarrow H$ is any surjective homomorphism with $H \in \mathcal{S}$, then there is a homomorphism $g : G_{\mathcal{S}} \rightarrow H$ such that $g \circ s_G = f$. (In categorical terms, this means \mathcal{S} is an epi-reflective subcategory of \mathcal{W} .) $G_{\mathcal{S}}$ is called the maximal \mathcal{S} -image of G .*
- (b) *For every $a \in G_{\mathcal{S}}$, $s_G^{-1}(a) \in \mathcal{R}$ and it is maximal in the sense that it is not contained in any other induced subgraph of G which is in \mathcal{R} .*
- (c) *If H is an induced subgraph of G with $H \in \mathcal{R}$, then there is an $a \in G_{\mathcal{S}}$ with $H \subseteq s_G^{-1}(a)$.*

On occasion we need to extend a congruence on a subgraph to a congruence on the graph leaving the equivalence classes on the subgraph intact. This will be done as follows. Let H be an induced subgraph of the graph G and suppose $\theta_H = (\sim_H, \mathcal{E}_H)$ is a congruence on H . Define a congruence $\overline{\theta}_H = (\sim_G, \mathcal{E}_G)$ on G as follows.

$$\text{For } a, b \in G, \text{ we let } a \sim_G b = \begin{cases} a \sim_H b, & \text{if both } a, b \in V_H, \\ a = b, & \text{if both } a, b \in V_G - V_H. \end{cases}$$

Considering the various cases, it can be checked that \sim_G is an equivalence on V_G with $[a]_G = \begin{cases} [a]_H, & \text{if } a \in V_H, \\ \{a\}, & \text{if } a \in V_G - V_H. \end{cases}$

Let $\mathcal{E}_G = \{ab \mid \text{there are } a', b' \in V_G \text{ with } a \sim_G a', b \sim_G b' \text{ and } a'b' \in \mathcal{E}_H \cup E_G\}$. Then $\mathcal{E}_H \cup E_G \subseteq \mathcal{E}_G$ and it can be shown that \mathcal{E}_G has the Substitution Property; hence $\overline{\theta}_H = (\sim_G, \mathcal{E}_G)$ is a congruence on G . This will be our canonical extension of a congruence on an induced subgraph to a congruence on the graph. Moreover, it follows readily that if θ_H is a strong congruence on H , then $\overline{\theta}_H$ is a strong congruence on G .

Two properties that a Hoehnke radical ρ on \mathcal{W} may satisfy are:

Completeness. If θ is a strong congruence on $G \in \mathcal{W}$ with $[a]_{\theta} \in \mathcal{R}_{\rho}$ for all $a \in V_G$, then $\theta \subseteq \rho_G$.

Idempotence. For $G \in \mathcal{W}$ and all $a \in V_G$, $[a]_{\rho_G} \in \mathcal{R}_{\rho}$.

Then we have

Theorem 5.3. *Let ρ be a Hoehnke radical on \mathcal{W} which is complete, idempotent and such that for all $G \in \mathcal{W}$, ρ_G is a strong congruence on G . Then \mathcal{S}_{ρ} is a disconnectedness and $\mathcal{R}_{\rho} = \mathcal{US}_{\rho}$ is a connectedness. Conversely, suppose \mathcal{S} is a disconnectedness in \mathcal{W} with corresponding connectedness \mathcal{R} . Then there is a Hoehnke radical ρ on \mathcal{W} which is complete, idempotent and for all $G \in \mathcal{W}$, ρ_G is a strong congruence on G . Moreover, $\mathcal{S}_{\rho} = \mathcal{S}$ and $\mathcal{R}_{\rho} = \mathcal{US} = \mathcal{R}$.*

Proof. Let ρ be a Hoehnke radical on \mathcal{W} which fulfils the requirements as specified. By definition, to show that \mathcal{S}_ρ is a disconnectedness, we show that a graph $G \in \mathcal{W}$ is in \mathcal{S}_ρ if and only if every non-trivial induced subgraph of G has a non-trivial homomorphic image which is in \mathcal{S}_ρ . Let $G \in \mathcal{S}_\rho$, i.e., $\rho_G = \iota_G$. Let H be a non-trivial induced subgraph of G . It will be shown that H/ρ_H is non-trivial. If H/ρ_H is trivial, then $H \in \mathcal{R}_\rho$ and $V_H = [a]_{\rho_H}$ for arbitrary $a \in V_H$. Let $\overline{\rho_H}$ be the extension of ρ_H to a congruence on G . Since ρ_H is a strong congruence, so is $\overline{\rho_H}$. The $\overline{\rho_H}$ -equivalence classes in G are $H = [a]$ for any $a \in V_H$; otherwise they are $[b] = \{b\}$ for $b \in V_G - V_H$. In both cases, they are in \mathcal{R}_ρ and by the completeness we get the contradiction $\overline{\rho_H} \subseteq \rho_G = \iota_G$. Thus H/ρ_H is non-trivial and the canonical homomorphism $H \rightarrow H/\rho_H \in \mathcal{S}_\rho$ is the required map. Suppose now G is a graph for which every non-trivial induced subgraph has a non-trivial homomorphic image in \mathcal{S}_ρ . We need to show $G \in \mathcal{S}_\rho$, i.e., $\rho_G = \iota_G$. Let $\rho_G = (\sim_G, \mathcal{E}_G)$. For any $a \in V_G$, the idempotence of ρ gives $[a]_{\rho_G} \in \mathcal{R}_\rho$. If $[a]_{\rho_G}$ is non-trivial, then by the assumption it has a non-trivial homomorphic image, say H , in \mathcal{S}_ρ . But \mathcal{R}_ρ is homomorphically closed, hence $H \in \mathcal{S}_\rho \cap \mathcal{R}_\rho \subseteq \mathcal{T}$ which contradicts H non-trivial. Thus $[a]_{\rho_G} = \{a\}$ for all $a \in V_G$. This means the strong homomorphism $G \rightarrow G/\rho_G$ is a bijection and thus an isomorphism. Hence $\rho_G = \iota_G$. Thus \mathcal{S}_ρ is a disconnectedness and then \mathcal{R}_ρ a connectedness follows from the equality $\mathcal{R}_\rho = \mathcal{U}\mathcal{S}_\rho$.

Conversely, let \mathcal{S} be a disconnectedness in \mathcal{W} with corresponding connectedness $\mathcal{R} = \mathcal{U}\mathcal{S}$. For each $G \in \mathcal{W}$, let $s_G : G \rightarrow G_s$ be the maximal \mathcal{S} -image of G as given in Theorem 5.2. Amongst others, this means that s_G is a strong homomorphism and $G_s \in \mathcal{S}$. Then $\rho_G = \ker s_G$ is a strong congruence on G and $G/\rho_G \cong G_s \in \mathcal{S}$. Moreover, the canonical map $p_G : G \rightarrow G/\rho_G$ is a strong homomorphism and it is an isomorphism if and only if $\rho_G = \iota_G$ and hence, if and only if $G \in \mathcal{S}$. Next we show ρ is a Hoehnke radical. For (H1), let $f : G \rightarrow H$ be a surjective homomorphism. By the maximality of G_s , there is a homomorphism $h : G_s \rightarrow H_s$ such that $h \circ s_G = s_H \circ f$. Then $f(\rho_G) \subseteq \rho_H$. If $a, b \in V_G$ with $a \sim_{\rho_G} b$, then $s_G(a) = s_G(b)$ which gives $h(s_G(a)) = h(s_G(b))$ and thus $s_H(f(a)) = s_H(f(b))$. Hence $f(a) \sim_{\rho_H} f(b)$. If $ab \in \mathcal{E}_{\rho_G}$, then $s_G(a)s_G(b) \in E_{G_s}$ and so $s_H(f(a))s_H(f(b)) = h(s_G(a))h(s_G(b)) \in E_{H_s}$. This gives $f(a)f(b) \in \mathcal{E}_{\rho_H}$. The validity of (H2) is clear since $G/\rho_G \cong G_s \in \mathcal{S}$ and the remarks above. Thus ρ is a Hoehnke radical on \mathcal{W} with $\mathcal{S}_\rho = \mathcal{S}$ and $\mathcal{R} = \mathcal{U}\mathcal{S} = \mathcal{U}\mathcal{S}_\rho = \mathcal{R}_\rho$.

ρ is complete: Let $\theta = (\sim_\theta, \mathcal{E}_\theta)$ be a strong congruence on G with $[a]_\theta \in \mathcal{R}_\rho$ for all $a \in V_G$. It must be shown that $\theta \subseteq \rho_G = \ker s_G = (\sim_{s_G}, \mathcal{E}_{s_G})$. By Theorem 5.2(c), for every $a \in V_G$, there is a $t_a \in V_{G_s}$ with $s_G([a]_\theta) = t_a$. Thus $a \sim_\theta b \Rightarrow s_G(a) = s_G(b)$ and so $a \sim_{s_G} b$. We still need $\mathcal{E}_\theta \subseteq \mathcal{E}_{s_G}$. Let $ab \in \mathcal{E}_\theta$. Then there are $a', b' \in V_G$ with $a \sim_\theta a', b \sim_\theta b'$ and $a'b' \in E_G$. Hence $s_G(a)s_G(b) = s_G(a')s_G(b') \in E_{G_s}$ which gives $ab \in \mathcal{E}_{s_G}$.

ρ is idempotent: By definition, $\rho_G = \ker s_G = (\sim_{\rho_G}, \mathcal{E}_{\rho_G})$. For any $a \in V_G$, let $s_G(a) = t_a$, say, for some $t_a \in G_s$. Then $s_G^{-1}(t_a) = [a]_{\rho_G}$. Indeed, $b \in [a]_{\rho_G} \Leftrightarrow t_a = s_G(a) = s_G(b) \Leftrightarrow b \in s_G^{-1}(t_a)$. By Theorem 5.2 we know $s_G^{-1}(t_a) \in \mathcal{R} = \mathcal{R}_\rho$ from which the idempotence follows. ■

In view of this result, a Hoehnke radical ρ which is complete, idempotent and for which ρ_G is a strong congruence for all G , is called a *KA-radical*. We conclude with a number of examples. In view of the above result, for a disconnectedness \mathcal{S} , the kernel of the maximal \mathcal{S} -image of any graph gives a KA-radical which is also a Hoehnke radical. However, it may be more instructive to rather explicitly give examples of Hoehnke radicals (which could give rise to KA-radicals).

Examples. In the examples below, the universal class \mathcal{W} is the class of all undirected graphs which admit loops. The first two examples are trivial but should be mentioned.

(1) For each graph G , let $\rho_G = \iota_G$. Then ρ is a KA-radical with $\mathcal{R}_\rho = \mathcal{T}$ and $\mathcal{S}_\rho = \mathcal{W}$.

(2) For each graph G , let $\rho_G = \nu_G$. Then ρ is a complete and idempotent Hoehnke radical, but not a KA-radical since ν_G need not be a strong congruence. Here $\mathcal{R}_\rho = \mathcal{W}$ and $\mathcal{S}_\rho = \{T_0\}$.

(3) For each graph G , let $\rho_G = (\simeq, \mathcal{E}_{\rho_G})$ where $\mathcal{E}_{\rho_G} = \{ab \mid a, b \in V_G \text{ and there is a path from } a \text{ to } b\}$. It can easily be verified that ρ_G is a congruence on G , but it is not a strong congruence. Because a homomorphism preserves edges, condition (H1) follows. For (H2), note that there is a path from $[a]$ to $[b]$ in $G/\rho_G \Leftrightarrow$ there is a path from a to b in $G \Leftrightarrow ab \in \mathcal{E}_{\rho_G} \Leftrightarrow [a][b] \in E_{G/\rho_G}$ from which we get $\rho(G/\rho_G) = \iota_{G/\rho_G}$. It can also be verified that ρ is complete and idempotent even though not a KA-radical. Here we have $\mathcal{R}_\rho = \mathcal{T}$ and $\mathcal{S}_\rho = \{G \mid \text{every connected component of } G \text{ is complete}\}$.

(4) Let $\rho_G = (\sim_{\rho_G}, \mathcal{E}_{\rho_G})$ where $a \sim_{\rho_G} b \Leftrightarrow a = b$ or there is a path from a to b and $\mathcal{E}_{\rho_G} = \{ab \mid \text{there are } a', b' \in V_G \text{ with } a \sim_{\rho_G} a', b \sim_{\rho_G} b' \text{ and } a'b' \in E_G\}$. Then ρ_G is a strong congruence on G and ρ is a Hoehnke radical with $\mathcal{R}_\rho = \{G \mid \text{if } G \text{ is not trivial, then it is connected}\}$ and $\mathcal{S}_\rho = \{G \mid \text{the only edges of } G \text{ are loops}\}$. It is then easy to see that ρ is complete and idempotent; hence ρ is a KA-radical, \mathcal{R}_ρ is a connectedness and \mathcal{S}_ρ a disconnectedness.

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