

DECOMPOSITION OF THE TENSOR PRODUCT
OF COMPLETE GRAPHS INTO CYCLES
OF LENGTHS 3 AND 6

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Abstract

By a $\{C_3^\alpha, C_6^\beta\}$ -decomposition of a graph G , we mean a partition of the edge set of G into α cycles of length 3 and β cycles of length 6. In this paper, necessary and sufficient conditions for the existence of a $\{C_3^\alpha, C_6^\beta\}$ -decomposition of $(K_m \times K_n)(\lambda)$, where \times denotes the tensor product of graphs and λ is the multiplicity of the edges, is obtained. In fact, we prove that for $\lambda \geq 1$, $m, n \geq 3$ and $(m, n) \neq (3, 3)$, a $\{C_3^\alpha, C_6^\beta\}$ -decomposition of $(K_m \times K_n)(\lambda)$ exists if and only if $\lambda(m-1)(n-1) \equiv 0 \pmod{2}$ and $3\alpha + 6\beta = \frac{\lambda m(m-1)n(n-1)}{2}$.

Keywords: cycle decomposition, tensor product.

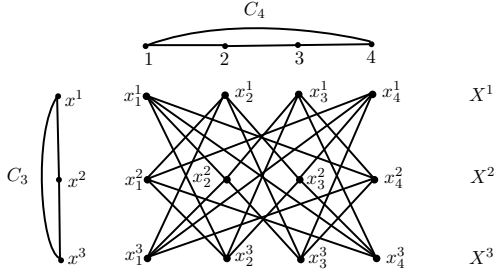
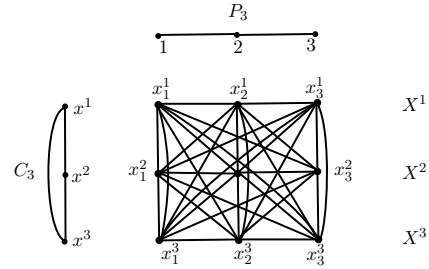
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1. INTRODUCTION

Throughout this paper, graphs are assumed to be loopless and finite. Let C_k denote the cycle of length k . The complete graph on n vertices is denoted by K_n . A graph G is said to be H -decomposable if the edge set $E(G)$ can be partitioned into E_1, E_2, \dots, E_k such that $\langle E_i \rangle \simeq H, 1 \leq i \leq k$. If a graph G can be decomposed into cycles of length k , then we say that G admits a C_k -decomposition and in this case we write $G = C_k \oplus C_k \oplus \dots \oplus C_k$; also we write it as $C_k \mid G$. A graph G is said to be $\{H_1, H_2\}$ -decomposable if the edge set of G can be partitioned into E_1, E_2, \dots, E_k such that $\langle E_i \rangle \simeq H_1$ or $\langle E_i \rangle \simeq H_2, 1 \leq i \leq k$ and $H_1, H_2 \in \{\langle E_1 \rangle, \langle E_2 \rangle, \dots, \langle E_k \rangle\}$. The graph obtained by replacing each edge of G by λ

parallel edges is denoted by $G(\lambda)$. For an integer k , kG denotes k disjoint copies of G . Definitions which are not given here can be found in [9].

For two simple graphs G_1 and G_2 their *tensor product*, denoted by $G_1 \times G_2$, has vertex set $V(G_1) \times V(G_2)$ in which $(x_1, y_1)(x_2, y_2)$ is an edge whenever x_1x_2 is an edge in G_1 and y_1y_2 is an edge in G_2 , see Figure 1. Similarly, the *wreath product* of the graphs G_1 and G_2 , denoted by $G_1 \circ G_2$, has vertex set $V(G_1) \times V(G_2)$ in which $(x_1, y_1)(x_2, y_2)$ is an edge whenever x_1x_2 is an edge in G_1 or, $x_1 = x_2$ and y_1y_2 is an edge in G_2 , see Figure 2. Note that, $(G_1 \times G_2)(\lambda) \simeq G_1(\lambda) \times G_2 \simeq G_1 \times G_2(\lambda)$. Let $V(G) = \{x^1, x^2, \dots, x^m\}$ and $V(H) = \{1, 2, \dots, n\}$. For $x^i \in V(G)$, $x^i \times V(H) = \{(x^i, j) \mid j \in \{1, 2, \dots, n\}\}$; we denote (x^i, j) by x_j^i . The set $X^i = \{x_1^i, x_2^i, \dots, x_n^i\} = x^i \times V(H)$ is called the i^{th} layer (of vertices) or i^{th} partite set of $G \times H$ (respectively $G \circ H$), corresponding to the vertex x^i , $1 \leq i \leq m$, of $V(G)$. Clearly, $K_m \circ \overline{K}_n$ is the complete m -partite graph in which each of its partite sets has n vertices. Further, $K_m \times K_n = K_m \circ \overline{K}_n - E(nK_m)$, where nK_m denotes n disjoint copies of K_m . As the tensor product is commutative, $K_m \times K_n \simeq K_n \times K_m$.

Figure 1. The graph $C_3 \times C_4$.Figure 2. The graph $C_3 \circ P_3$.

In the study of group divisible designs, complete multipartite graphs $K_m \circ \overline{K}_n$ are decomposed into complete subgraphs; but in a modified group divisible design the graph $K_m \times K_n$ is decomposed into complete subgraphs, see [3–6, 24]. In [5], Assaf used modified group divisible designs to construct covering and packing designs, and group divisible designs with block size 5. Further, a C_p -decomposition, p a prime, of the graph $K_m \times K_n$ was used to find a C_p -decomposition of $K_m \circ \overline{K}_n$, see [25]. Moreover, a resolvable $2k$ -cycle decomposition of $K_m \times K_n$ and a decomposition of $K_m \times K_n$ into closed trails of length k have been studied in [33, 34]. Besides that, Hamilton cycle decompositions of the graphs $K_m \times K_n$, $K_{m,m} \times K_n$, $K_{m,m} \times (K_r \circ \overline{K}_s)$ and $(K_m \circ \overline{K}_n) \times (K_r \circ \overline{K}_s)$ and the directed Hamilton cycle decompositions of the symmetric digraphs $(K_m \times K_n)^*$, $(K_{m,m} \times K_n)^*$, $(K_{m,m} \times (K_r \circ \overline{K}_s))^*$, $((K_m \times K_n) \times K_r)^*$, $((K_m \circ \overline{K}_n) \times K_r)^*$ and $((K_m \circ \overline{K}_n) \times (K_r \circ \overline{K}_s))^*$ are obtained in [8, 28–31, 35]. Hence $K_m \times K_n$ is proved to be an important proper spanning subgraph of the regular complete

multipartite graph $K_m \circ \overline{K}_n$.

Decompositions of complete graphs into specified subgraphs have been studied for a long time. Decompositions of complete graphs into cycles are well-studied. Decompositions of graphs into fixed length cycles and varying length cycles are completely settled for the complete graphs K_n and the complete multigraphs $K_n(\lambda)$. In [1, 21, 36], it is proved that if n is odd and $k \mid \binom{n}{2}$, $3 \leq k \leq n$, then $C_k \mid K_n$. Further, if n is even and $k \mid \frac{n(n-2)}{2}$, $3 \leq k \leq n$, then $C_k \mid K_n - I$, where I is a perfect matching of K_n . Bryant *et al.* [13, 14] completely settled the problem of decomposing $K_n(\lambda)$, $\lambda \geq 1$ into cycles of varying lengths.

Chou *et al.* [16] obtained a necessary and sufficient condition for the existence of a decomposition of $K_{a,b}$ (respectively $K_{m,m} - I$, where $m \geq 3$ is odd and I denotes a perfect matching) into cycles of length 4, 6 and 8. In [17], Chou and Fu considered a $\{C_4^r, C_{2t}^s\}$ -decomposition of $K_{a,b}$ and $K_{m,m} - I$, where m is odd and I denotes a perfect matching. Later, Fu *et al.* [18] proved that the necessary conditions for the existence of a decomposition of $K_{m,m}$ (respectively $K_{m,m} - I$) into cycles of distinct lengths are sufficient whenever m is even (respectively odd) except $m = 4$. Recently, Asplund *et al.* [2] established a necessary and sufficient condition for the existence of a decomposition of $K_{a,b}(\lambda)$ into cycles of arbitrary lengths.

Billington *et al.* [12] proved the existence of a C_5 -decomposition of $(K_m \circ \overline{K}_n)(\lambda)$. Muthusamy and Shanmuga Vadivu [32] proved the existence of a C_{2k} -decomposition of $K_m \circ \overline{K}_n$. Very recently, irrespective of the parity of k , the authors of [15] actually solve the existence problem for a C_k -decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ whose cycle-set can be partitioned into 2-regular graphs containing all the vertices except those belonging to one part. A $\{C_4^\alpha, C_5^\beta\}$ -decomposition of $K_m \circ \overline{K}_n$ was given by Fu [22]. Moreover, Bahmanian and Šajna [7] showed that if $K_m(\lambda n)$ has a decomposition into cycles of lengths k_1, k_2, \dots, k_t (plus a perfect matching if $\lambda n(m-1)$ is odd), then $(K_m \circ \overline{K}_n)(\lambda)$ has a decomposition into cycles of lengths $k_1 n, k_2 n, \dots, k_t n$ (plus a perfect matching if $\lambda n(m-1)$ is odd).

Billington obtained necessary and sufficient conditions for the existence of a $\{C_3^\alpha, C_4^\beta\}$ -decomposition of the graph $K_{a,b,c}$ $a \leq b \leq c$, see [10]. Ganesamurthy and Paulraja proved that the existence of a $\{C_3^\alpha, C_6^\beta\}$ -decomposition of the graph $K_{a,b,c}$, $a \leq b \leq c$, see [19]. In [3], Assaf obtained a C_3 -decomposition of $(K_m \times K_n)(\lambda)$. For $p \geq 5$, p a prime, existence of C_p -decompositions of $K_m \times K_n$ and $K_m \circ \overline{K}_n$ were proved by Manikandan and Paulraja [25–27]. Existence of a C_k -decomposition of $K_m \times K_n$ is not yet known for general k . In this paper, we obtain a necessary and sufficient condition for the existence of a $\{C_3^\alpha, C_6^\beta\}$ -decomposition of $(K_m \times K_n)(\lambda)$.

Besides other results, the following main theorem is proved.

Theorem 1. For $\lambda \geq 1$, $m, n \geq 3$ and $(m, n) \neq (3, 3)$, the graph $(K_m \times K_n)$ (λ) admits a $\{C_3^\alpha, C_6^\beta\}$ -decomposition if and only if $\lambda(m-1)(n-1) \equiv 0 \pmod{2}$ and $3\alpha + 6\beta = \frac{\lambda m(m-1)n(n-1)}{2}$.

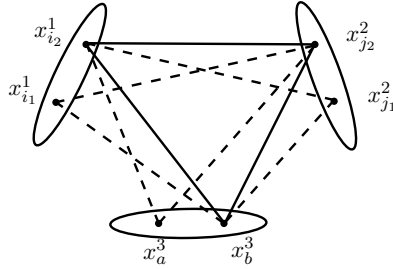
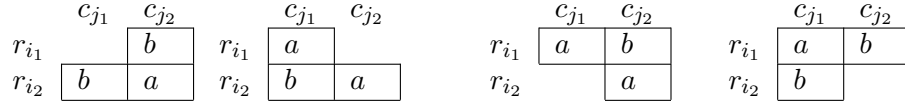
2. NOTATION AND TERMINOLOGY

A *latin square* of order n , denoted by L_n , is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \dots, n\}$ such that each row and each column of the array contains each of the symbols in $\{1, 2, \dots, n\}$ exactly once. As in [11], a cell (i, j) is termed “empty” if it contains no entry and “filled” otherwise. We represent a *partial latin square* L by a set of ordered triples (i, j, k) , where entry k occurs in row i and column j . In this sense (i, j, k) is an element of L . For our convenience, we avoid, if necessary, drawing empty cells of a partial latin square. A latin square is said to be *idempotent* if the cell (i, i) contains the symbol i , $1 \leq i \leq n$. A latin square of order k is *cyclic* if the 1st row entries are $a_1, a_2, a_3, \dots, a_k$, then the s^{th} row entries are $a_s, a_{s+1}, a_{s+2}, \dots, a_{s-1}$, in order.

Remark 2. Using a latin square, L_n , of order n , the complete tripartite graph $K_{n,n,n}$, $n \geq 2$, can be decomposed into C_3 's as follows. Let the partite sets of $K_{n,n,n}$ be $\{x_1^i, x_2^i, x_3^i, \dots, x_n^i\}$, $1 \leq i \leq 3$. For the $(i, j)^{\text{th}}$ cell of L_n with entry k , there corresponds a 3-cycle (x_i^1, x_j^2, x_k^3) in $K_{n,n,n}$. Since L_n has n^2 cells, we obtain n^2 cycles of length 3 which decompose $K_{n,n,n}$. Further, if we consider an idempotent latin square L_n of order n , $n \geq 3$, then the non-diagonal cells of L_n give a C_3 -decomposition of $K_3 \times K_n$, as $K_3 \times K_n = K_3 \circ \overline{K}_n - E(nK_3)$.

Remark 3. Consider a cyclic latin square C' of order $n \geq 3$ on the set $\{1, 2, \dots, n\}$, where n is an odd integer and the i^{th} row elements, in order, are $i, i+1, i+2, \dots, i-1$. Let $n = 2k+1$, $k \geq 1$. Now we rename the entries in C' by $j \rightarrow 1 + (j-1)k'$, where $k' = k+1$. The resulting latin square, I_n , is idempotent and commutative. Existence of an idempotent commutative latin square of order $2k+1$ is guaranteed in [23]. The entries in the cells in $T = \{(1, 2), (2, 3), \dots, (k-1, k), (k, 1)\}$ is a transversal of I_n . We can extend the latin square I_n to I_{n+1} , $n+1 = 2k+2$, $k \geq 1$, using the method of stripping the transversal T of I_n , see [23]. The resulting latin square I_{n+1} , is idempotent, see Appendix. Then for any $n \geq 3$, we can obtain an idempotent latin square of order n .

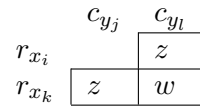
Remark 4. The edges of the triangles corresponding to the entries of each of the partial latin squares of Figure 3, define a graph isomorphic to $K_{2,2,2} - E(K_3)$ and it can be decomposed into three C_3 's or, a C_3 and a C_6 , see Figure 3, where r_{i_j} and c_{j_k} denote the row i_j and column j_k . Observe that in each case, in each of the three cells of the partial latin square, there are only two distinct symbols.



The subgraph of $K_{2,2,2}$ corresponding to the first partial latin square given above.
 Normal edges induce a C_3 and broken edges induce a C_6 .

Figure 3. $K_{2,2,2} - E(K_3) = C_3 \oplus C_6$, where $K_3 = \langle x_{i1}^1, x_{j1}^2, x_a^3 \rangle$.

1. An idempotent latin square of order n without its diagonal entries is denoted by $I_n - D$.
2. An ordered triple (i, j, k) , stands for the $(i, j)^{th}$ entry of a latin square is k .
3. At some places, we write the entries of a partial latin square by ordered triples; for example, the three triples (x_i, y_l, z) , (x_k, y_j, z) and (x_k, y_l, w) represent the partial latin square



where r_{x_i} represents the row x_i and similarly c_{y_j} represents the column y_j .

3. $\{C_3^\alpha, C_6^\beta\}$ -DECOMPOSITION OF $K_3 \times K_n$

In this section, we prove the existence of a decomposition of $K_3 \times K_n$ into α cycles of length 3 and β cycles of length 6.

The following lemma is a simple observation.

Lemma 5. *The graph $K_3 \times K_3$ cannot be decomposed into 4 copies of C_3 and a C_6 .*

Proof. The proof is left to the reader. ■

Lemma 6. *For $(\alpha, \beta) \neq (4, 1)$, the graph $K_3 \times K_3$ admits a $\{C_3^\alpha, C_6^\beta\}$ -decomposition.*

Proof. Let the vertex set of the three partite sets of $K_3 \times K_3$ be $\{x_1^i, x_2^i, x_3^i\}$, $1 \leq i \leq 3$. Observe that α is always even and the maximum value of α is 6.

(i) $(\alpha, \beta) = (6, 0)$. Consider the unique idempotent latin square I_3 ; the non-diagonal entries of I_3 give six edge disjoint copies of C_3 , see Remark 2.

(ii) $(\alpha, \beta) = (2, 2)$. A required set of cycles are (x_1^1, x_3^2, x_2^3) , (x_2^1, x_3^2, x_1^3) , $(x_1^1, x_3^3, x_2^2, x_1^2, x_3^1, x_2^2)$ and $(x_2^1, x_3^3, x_2^2, x_1^3, x_3^1, x_2^2)$.

(iii) $(\alpha, \beta) = (0, 3)$. A set of three cycles of length 6 is $(x_1^1, x_2^2, x_3^3, x_1^2, x_2^1, x_3^3)$, $(x_1^1, x_2^3, x_3^1, x_2^1, x_3^2, x_1^3)$ and $(x_2^1, x_3^3, x_2^2, x_1^3, x_2^2, x_3^3)$. ■

Lemma 7. *The graph $K_3 \times K_4$ has a $\{C_3^\alpha, C_6^\beta\}$ -decomposition.*

Proof. We consider only the possible values for α and β .

(i) $(\alpha, \beta) = (12, 0)$. The entries of the non-diagonal cells of an idempotent latin square I_4 give a C_3 -decomposition of $K_3 \times K_4$, see Remark 2.

(ii) $(\alpha, \beta) \in \{(10, 1), (8, 2), (6, 3), (4, 4)\}$.

Consider the following partial latin square $I_4 - D$ of I_4 .

	c_1	c_2	c_3	c_4
r_1		4	2	3
r_2	3		4	1
r_3	4	1		2
r_4	2	3	1	

The cells of $I_4 - D$ are partitioned into the following partial latin squares.

	c_2	c_3		c_1	c_4		c_1	c_4		c_2	c_3
r_1	4	2			3		4	2		1	
r_2		4		3	1		2			3	1
r_3											
r_4											

The edges of $K_3 \times K_4$ corresponding to each of these partial latin squares induces the subgraph isomorphic to $K_{2,2,2} - E(K_3)$, and it admits a decomposition consisting of three C_3 's or, a C_3 and a C_6 , see Figure 3. Depending on the value of α and β , we choose C_3 's or, a C_3 and a C_6 corresponding to each of these partial latin squares to get a $\{C_3^\alpha, C_6^\beta\}$ -decomposition of $K_3 \times K_4$.

(iii) $(\alpha, \beta) \in \{(2, 5), (0, 6)\}$. The graph

$$\begin{aligned} K_3 \times K_4 &= K_3 \times (K_3 \oplus K_{1,3}) \\ &= K_3 \times K_3 \oplus K_3 \times K_{1,3} \\ &= K_3 \times K_3 \oplus K_3 \times K_2 \oplus K_3 \times K_2 \oplus K_3 \times K_2. \end{aligned}$$

As the graph $K_3 \times K_2 \simeq C_6$, and the graph $K_3 \times K_3$ has a $\{C_3^r, C_6^s\}$ -decomposition for $(r, s) \neq (4, 1)$, we obtain a $\{C_3^\alpha, C_6^\beta\}$ -decomposition of $K_3 \times K_4$. ■

Lemma 8. *The graph $K_3 \times K_n$, $5 \leq n \leq 11$, admits a $\{C_3^\alpha, C_6^\beta\}$ -decomposition.*

Proof. If $(\alpha, \beta) = (n(n-1), 0)$, then the required decomposition exists by Remark 2. So we suppose that $\beta \neq 0$. First we consider $1 \leq \beta \leq n-1$. Consider an $I_n - D$, where I_n is obtained as in Remark 3; the idempotent latin squares $I_n, 5 \leq n \leq 11$, are given in Appendix. We use $n-1$ partial latin squares, each having three cells, of $I_n - D, 5 \leq n \leq 11$, to obtain C_6 's, $1 \leq \beta \leq n-1$; the three cells are chosen so that two cells are filled by a common symbol, (see Remark 4). According to our notation, each set of three triples in the following list of triples gives a partial latin square (of $I_n - D$) having three filled cells.

n = 5. $\{(r_1, c_3, 2)(r_1, c_4, 5)(r_2, c_3, 5)\}, \{(r_1, c_5, 3)(r_2, c_4, 3)(r_2, c_5, 1)\}, \{(r_3, c_1, 2)(r_3, c_2, 5)(r_4, c_1, 5)\}, \{(r_4, c_2, 3)(r_5, c_1, 3)(r_5, c_2, 1)\}$.

n = 6. $\{(r_1, c_2, 6)(r_1, c_3, 2)(r_2, c_3, 6)\}, \{(r_1, c_5, 3)(r_2, c_4, 3)(r_2, c_5, 1)\}, \{(r_3, c_1, 2)(r_3, c_2, 5)(r_4, c_1, 5)\}, \{(r_4, c_2, 3)(r_4, c_3, 1)(r_5, c_2, 1)\}, \{(r_5, c_3, 4)(r_6, c_2, 4)(r_6, c_3, 5)\}$.

n = 7. $\{(r_1, c_3, 2)(r_1, c_4, 6)(r_2, c_3, 6)\}, \{(r_1, c_5, 3)(r_2, c_4, 3)(r_2, c_5, 7)\}, \{(r_1, c_6, 7)(r_1, c_7, 4)(r_2, c_6, 4)\}, \{(r_3, c_1, 2)(r_3, c_2, 6)(r_4, c_1, 6)\}, \{(r_4, c_2, 3)(r_5, c_1, 3)(r_5, c_2, 7)\}, \{(r_6, c_1, 7)(r_6, c_2, 4)(r_7, c_1, 4)\}$.

n = 8. $\{(r_1, c_2, 8)(r_1, c_3, 2)(r_2, c_3, 8)\}, \{(r_1, c_5, 3)(r_2, c_4, 3)(r_2, c_5, 7)\}, \{(r_1, c_6, 7)(r_1, c_7, 4)(r_2, c_6, 4)\}, \{(r_3, c_1, 2)(r_3, c_2, 6)(r_4, c_1, 6)\}, \{(r_4, c_2, 3)(r_5, c_1, 3)(r_5, c_2, 7)\}, \{(r_6, c_2, 4)(r_6, c_3, 1)(r_7, c_2, 1)\}, \{(r_7, c_3, 5)(r_8, c_2, 5)(r_8, c_3, 6)\}$.

n = 9. $\{(r_1, c_3, 2)(r_1, c_4, 7)(r_2, c_3, 7)\}, \{(r_1, c_5, 3)(r_2, c_4, 3)(r_2, c_5, 8)\}, \{(r_1, c_6, 8)(r_1, c_7, 4)(r_2, c_6, 4)\}, \{(r_1, c_8, 9)(r_2, c_7, 9)(r_2, c_8, 5)\}, \{(r_3, c_1, 2)(r_3, c_2, 7)(r_4, c_1, 7)\}, \{(r_4, c_2, 3)(r_5, c_1, 3)(r_5, c_2, 8)\}, \{(r_6, c_1, 8)(r_6, c_2, 4)(r_7, c_1, 4)\}, \{(r_7, c_2, 9)(r_8, c_1, 9)(r_8, c_2, 5)\}$.

n = 10. $\{(r_1, c_2, 10)(r_1, c_3, 2)(r_2, c_3, 10)\}, \{(r_1, c_5, 3)(r_2, c_4, 3)(r_2, c_5, 8)\}, \{(r_1, c_6, 8)(r_1, c_7, 4)(r_2, c_6, 4)\}, \{(r_1, c_8, 9)(r_2, c_7, 9)(r_2, c_8, 5)\}, \{(r_3, c_1, 2)(r_3, c_2, 7)(r_4, c_1, 7)\}, \{(r_4, c_2, 3)(r_5, c_1, 3)(r_5, c_2, 8)\}, \{(r_6, c_1, 8)(r_6, c_2, 4)(r_7, c_1, 4)\}, \{(r_7, c_2, 9)(r_8, c_1, 9)(r_8, c_2, 5)\}, \{(r_9, c_2, 1)(r_9, c_3, 6)(r_{10}, c_2, 6)\}$.

n = 11. $\{(r_1, c_3, 2)(r_1, c_4, 8)(r_2, c_3, 8)\}, \{(r_1, c_5, 3)(r_2, c_4, 3)(r_2, c_5, 9)\}, \{(r_1, c_6, 9)(r_1, c_7, 4)(r_2, c_6, 4)\}, \{(r_1, c_8, 10)(r_2, c_7, 10)(r_2, c_8, 5)\}, \{(r_1, c_9, 5)(r_1, c_{10}, 11)(r_2, c_9, 11)\}, \{(r_3, c_1, 2)(r_3, c_2, 8)(r_4, c_1, 8)\}, \{(r_4, c_2, 3)(r_5, c_1, 3)(r_5, c_2, 9)\}, \{(r_6, c_1, 9)(r_6, c_2, 4)(r_7, c_1, 4)\}, \{(r_7, c_2, 10)(r_8, c_1, 10)(r_8, c_2, 5)\}, \{(r_9, c_1, 5)(r_9, c_2, 11)(r_{10}, c_1, 11)\}$.

Each of the subgraphs of $K_3 \times K_n$ corresponding to the above $n-1, 5 \leq n \leq 11$, partial latin squares is isomorphic to $K_{2,2,2} - E(K_3)$, see Figure 3, and it can be decomposed into C_3 's or, a C_3 and a C_6 and hence $K_3 \times K_n$ has a $\{C_3^\alpha, C_6^\beta\}$ -decomposition, when $(\alpha, \beta) = (n(n-1) - 2i, i), 5 \leq n \leq 11, 1 \leq i \leq n-1$. The filled cells of $I_n - D, 5 \leq n \leq 11$, which are not covered by the above $n-1$ partial latin squares partition the remaining edges of $K_3 \times K_n$ into 3-cycles, by Remark 2.

Now we complete the proof by induction on $n, n \geq 5$, for $\beta \geq n$. For $n = 5$, $K_3 \times K_5 = K_3 \times K_4 \oplus K_3 \times K_2 \oplus \cdots \oplus K_3 \times K_2$; we use Lemma 7 and the fact that

$K_3 \times K_2 \simeq C_6$ to complete the proof. The graph $K_3 \times K_{n+1} = K_3 \times (K_n \oplus K_{1,n}) = K_3 \times K_n \oplus K_3 \times K_2 \oplus \cdots \oplus K_3 \times K_2$. Now a required decomposition follows by induction applied to $K_3 \times K_n$ and the fact that $K_3 \times K_2 \simeq C_6$. ■

Lemma 9. *If $\beta \geq 4$, then the graph $K_3 \times (K_6 - e)$ has a $\{C_3^\alpha, C_6^\beta\}$ -decomposition.*

Proof. The graph $K_3 \times (K_6 - e) = K_3 \times (K_5 \oplus K_{1,4})$
 $= K_3 \times K_5 \oplus \underbrace{K_3 \times K_2 \oplus \cdots \oplus K_3 \times K_2}_{4\text{-copies}}.$

As $K_3 \times K_2 \simeq C_6$ and a $\{C_3^r, C_6^s\}$ -decomposition of $K_3 \times K_5$ follows by Lemma 8, we have the desired result. ■

Lemma 10. *If $\beta = 2$, then the graph $K_3 \times (K_6 - e)$ has a $\{C_3^\alpha, C_6^\beta\}$ -decomposition.*

Proof. The graph $K_3 \times (K_6 - e) = K_3 \times (K_3 \oplus K_3 \oplus K_3 \oplus K_3 \oplus K_2 \oplus K_2)$
 $= K_3 \times K_3 \oplus K_3 \times K_3 \oplus K_3 \times K_3 \oplus K_3 \times K_3 \oplus K_3 \times K_3$
 $\oplus K_3 \times K_2 \oplus K_3 \times K_2.$

As $K_3 \times K_2 \simeq C_6$, the result follows by Lemma 6. ■

Lemma 11. *If $\beta \neq 1$, then the graph $K_3 \times (K_7 - E(K_3))$ has a $\{C_3^\alpha, C_6^\beta\}$ -decomposition.*

Proof. The graph $K_3 \times (K_7 - E(K_3)) = K_3 \times \underbrace{(K_3 \oplus K_3 \oplus \cdots \oplus K_3)}_{6\text{-copies}}$
 $= K_3 \times K_3 \oplus \cdots \oplus K_3 \times K_3$

Now the result follows by Lemma 6. ■

Lemma 12. *The cells of the first two rows of $I_n - D$, where $n = 2k + 2$, can be partitioned into $\lfloor \frac{4k+2}{3} \rfloor$ partial latin squares, each of which is one of the form given in Figure 3, together with one or two filled cells depending on n .*

Proof. Let $n = 2k + 2$, $k \geq 1$. Obtain the idempotent latin square I_n and the partial latin square $I_n - D$, as in Remark 3. The entries of the first two rows of $I_n - D$ are shown in Figure 4, see Appendix for I_n , $5 \leq n \leq 11$. We partition the cells of these two rows of $I_n - D$ into $\lfloor \frac{4k+2}{3} \rfloor$ 3-subsets as shown in Figures 5, 6 and 7 according to $n \equiv 0, 2$ or $4 \pmod{6}$, respectively. Each of the subsets has three filled cells having two distinct elements as shown in Remark 4.

	c_1	c_2	c_3	c_4	c_5	\dots	c_{2k-2}	c_{2k-1}	c_{2k}	c_{2k+1}	c_{2k+2}
r_1		$2k+2$	2	$k+3$	3	\dots	$2k$	k	$2k+1$	$k+1$	$k+2$
r_2	$k+2$		$2k+2$	3	$k+4$	\dots	k	$2k+1$	$k+1$	1	$k+3$

Figure 4. First two rows of $I_n - D$.

$n \equiv 0 \pmod{6}$:

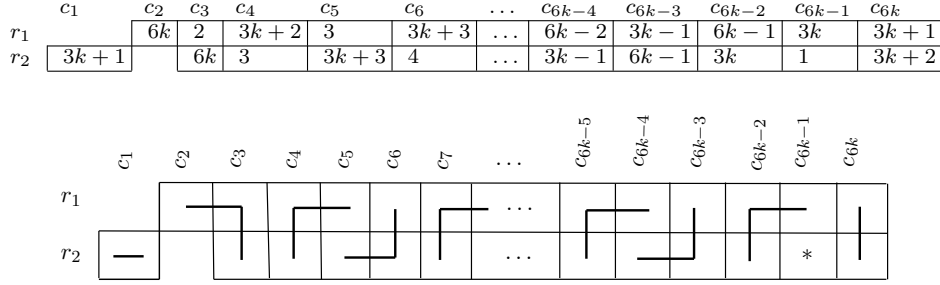


Figure 5. Except the cell with *, all other cells are partitioned into 3 cells as shown above, where the last column cells are combined with the first cell of the second row.

$n \equiv 2 \pmod{6}$:

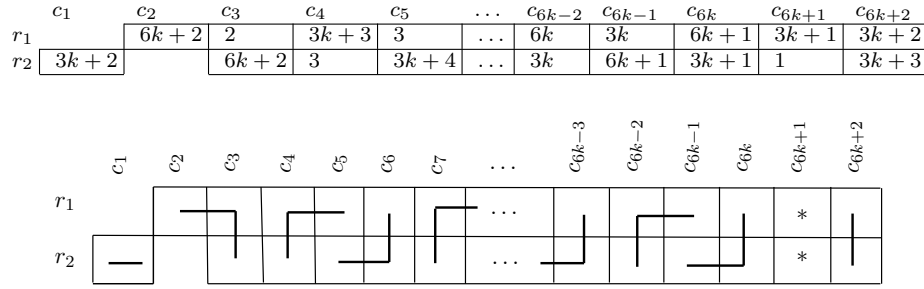


Figure 6. Except the two cells with *, all other cells are partitioned into 3 cells as shown above, where the last column cells are combined with the first cell of the second row.

$n \equiv 4 \pmod{6}$:

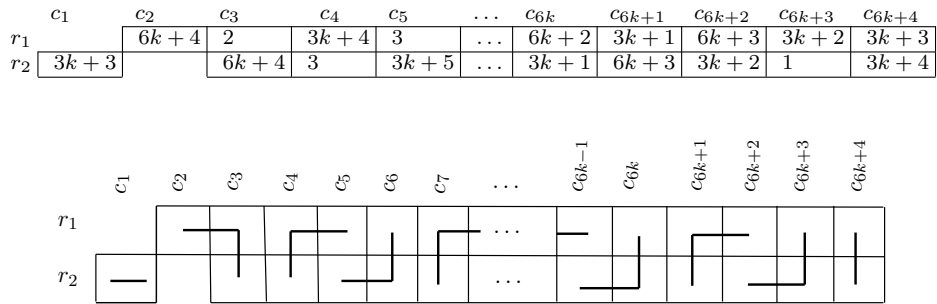


Figure 7. The two cells of the last column cells are combined with the first cell of the second row.

■

We apply following theorem to prove Theorem 14.

Theorem 13 [19]. *Let $K_{a,b,c}$ be the complete tripartite graph with $a \leq b \leq c$ and let $K_{a,b,c} \neq K_{1,1,c}$, when $c \equiv 1 \pmod{6}$ and $c > 1$. If $a \equiv b \equiv c \pmod{6}$, then $K_{a,b,c}$ admits a $\{C_3^\alpha, C_6^\beta\}$ -decomposition for any $\alpha \equiv a \pmod{2}$, with $0 \leq \alpha \leq ab$.*

Theorem 14. *The graph $K_3 \times K_n$, $n \geq 4$, admits a $\{C_3^\alpha, C_6^\beta\}$ -decomposition.*

Proof. Since the graph $K_3 \times K_n$ has a C_3 -decomposition, we assume that $\beta \geq 1$. Because of Lemmas 7 and 8, we assume that $n \geq 12$.

Case (i): $n \equiv 0 \pmod{4}$. Let $n = 4k, k \geq 3$. The graph $K_3 \times K_n = K_3 \times (kK_4 \oplus K_k \circ \overline{K_4}) = k(K_3 \times K_4) \oplus K_3 \times (K_k \circ \overline{K_4}) = G_1 \oplus G_2$, where $G_1 = k(K_3 \times K_4)$ and $G_2 = K_3 \times (K_k \circ \overline{K_4})$.

The graph $G_2 = K_3 \times (K_k \circ \overline{K_4}) = (K_3 \times K_k) \circ \overline{K_4} = (K_3 \oplus K_3 \oplus \cdots \oplus K_3) \circ \overline{K_4} = (K_{4,4,4} \oplus K_{4,4,4} \oplus \cdots \oplus K_{4,4,4})$, since $K_3 | K_3 \times K_n$. Now invoke Theorem 13 and Lemma 7 to the graphs $K_{4,4,4}$ and G_1 , respectively, to complete the proof of this case.

Case (ii): $n \equiv 1 \pmod{4}$. Let $n = 4k + 1, k \geq 3$. The graph $K_3 \times K_n = K_3 \times (\underbrace{K_5 \oplus K_5 \oplus \cdots \oplus K_5}_{k\text{-copies}} \oplus K_k \circ \overline{K_4}) = (K_3 \times K_5) \oplus (K_3 \times K_5) \oplus \cdots \oplus (K_3 \times K_5) \oplus K_3 \times (K_k \circ \overline{K_4}) = G_1 \oplus G_2$, where $G_1 = (K_3 \times K_5) \oplus (K_3 \times K_5) \oplus \cdots \oplus (K_3 \times K_5)$ and $G_2 = K_3 \times (K_k \circ \overline{K_4}) = (K_3 \times K_k) \circ \overline{K_4} = (K_3 \oplus K_3 \oplus \cdots \oplus K_3) \circ \overline{K_4}$. As in Case (i), G_2 is isomorphic to $K_{4,4,4} \oplus \cdots \oplus K_{4,4,4}$.

Now apply Theorem 13 and Lemma 8 to the graphs $K_{4,4,4}$ and G_1 , respectively, to complete the proof of this case.

Case (iii): $n \equiv 2 \pmod{4}$. Let $n = 4k + 2, k \geq 3$. First we prove for the case $\beta < 2(k - 1) = 2k - 2$. Out of the $\lfloor \frac{8k+2}{3} \rfloor$ partial latin squares, each having 3 cells, described in Lemma 12, consider $2k - 3$ partial latin squares. The edge induced subgraph of $K_3 \times K_n$, corresponding to each of these $2k - 3$ partial latin squares admits three copies of C_3 or, a C_3 and a C_6 and the cells not covered by these partial latin squares, give a C_3 -decomposition of the remaining subgraph of $K_3 \times K_n$. Thus we obtain a $\{C_3^\alpha, C_6^\beta\}$ -decomposition of $K_3 \times K_n$.

Next consider the case $\beta \geq 2(k - 1)$. The graph $K_3 \times K_n = K_3 \times K_{4k+2} = K_3 \times (K_6 \oplus K_6 - e \oplus K_6 - e \oplus \cdots \oplus K_6 - e \oplus K_k \circ \overline{K_4}) = K_3 \times K_6 \oplus K_3 \times K_6 - e \oplus \cdots \oplus K_3 \times K_6 - e \oplus K_3 \times (K_k \circ \overline{K_4}) = G_1 \oplus G_2 \oplus G_3$, where $G_1 = K_3 \times K_6$, $G_2 = (K_3 \times K_6 - e) \oplus (K_3 \times K_6 - e) \oplus \cdots \oplus (K_3 \times K_6 - e)$ and $G_3 = K_3 \times (K_k \circ \overline{K_4})$. The result follows by Lemmas 8, 9 and 10 as the graph G_3 is isomorphic to the graph G_2 considered in Case (i) above.

Case (iv): $n \equiv 3 \pmod{4}$. Let $n = 4k + 3, k \geq 3$. If $\beta = 1$, then consider the cells $\{(r_1, c_3, 2)(r_1, c_4, 2k + 4)(r_2, c_3, 2k + 4)\}$ of $I_{(4k+3)} - D$; the subgraph of

$K_3 \times K_n$ corresponding to these three cells is a C_3 and a C_6 , and each of the remaining cells of $I_{4k+3} - D$ gives a C_3 .

If $\beta \geq 2$, then $K_3 \times K_n = K_3 \times K_{4k+3} = K_3 \times (K_7 \oplus (K_7 - E(K_3)) \oplus \cdots \oplus (K_7 - E(K_3)) \oplus K_k \circ \overline{K_4}) = K_3 \times K_7 \oplus K_3 \times (K_7 - E(K_3)) \oplus \cdots \oplus K_3 \times (K_7 - E(K_3)) \oplus (K_3 \times (K_k \circ \overline{K_4})) = G_1 \oplus G_2 \oplus G_3$, where $G_1 = K_3 \times K_7$, $G_2 = K_3 \times (K_7 - E(K_3)) \oplus \cdots \oplus K_3 \times (K_7 - E(K_3))$ and $G_3 = K_3 \times (K_k \circ \overline{K_4})$. Now apply Lemma 8 to G_1 and Lemma 11 to G_2 ; the graph G_3 is isomorphic to the graph G_2 in Case (i). ■

4. $\{C_3^\alpha, C_6^\beta\}$ -DECOMPOSITION OF $(K_m \times K_n)(\lambda)$

In this section we prove the existence of a $\{C_3^\alpha, C_6^\beta\}$ -decomposition of $(K_m \times K_n)(\lambda)$. We need some lemmas to prove the main theorem.

Lemma 15. *The graph $K_{1,3} \times K_5$ has a decomposition into ten C_6 's.*

Proof. Let $V(K_{1,3}) = \{x^1, x^2, x^3, x^4\}$ with the center x^1 and $V(K_5) = \{1, 2, 3, 4, 5\}$. Let $V(K_{1,3} \times K_5) = \bigcup_{i=1}^4 X^i$, where X^i is as defined in the introduction. Let $C = (x_1^1, x_3^3, x_4^4, x_2^2, x_5^5, x_1^4)$ and $C' = (x_1^1, x_2^4, x_5^1, x_1^2, x_4^1, x_3^2)$. Then $\{C, \rho(C), \dots, \rho^4(C), C', \rho(C'), \dots, \rho^4(C')\}$ is a C_6 -decomposition, where $\rho = (12345)$ and its powers are the permutations acting on the subscripts of the vertices of the cycles C and C' , where $\rho(C)$ stands for $(x_{\rho(1)}^1, x_{\rho(3)}^3, x_{\rho(4)}^1, x_{\rho(2)}^2, x_{\rho(5)}^1, x_{\rho(4)}^4)$. ■

Assaf proved the existence of a C_3 -decomposition of $(K_m \times K_n)(\lambda)$ whenever the obvious necessary conditions are satisfied, see [3]. The proof of it uses a C_3 -decomposition of $K_4 \times K_5$; but the C_3 -decomposition of $K_4 \times K_5$ given in Lemma 3.4 of [3] contains a typo. The next lemma contains a proof of C_3 -decomposition of $K_4 \times K_5$.

Lemma 16. *The graph $K_4 \times K_5$ has a $\{C_3^\alpha, C_6^\beta\}$ -decomposition.*

Proof. Let $V(K_4) = \{x^1, x^2, x^3, x^4\}$ and $V(K_5) = \{1, 2, 3, 4, 5\}$. Let vertex set of $K_4 \times K_5$ be as defined in Lemma 15. The eight cycles C^i , $1 \leq i \leq 8$, given below and $\rho, \rho^2, \rho^3, \rho^4$ applied to the subscripts of vertices of the C^i , which we denote by $\rho^j(C^i)$, decompose $K_4 \times K_5$ into 3-cycles, that is, $C^1, \rho(C^1), \dots, \rho^4(C^1), C^2, \rho(C^2), \dots, \rho^4(C^2), \dots, C^8, \rho(C^8), \dots, \rho^4(C^8)$ is a C_3 -decomposition of $K_4 \times K_5$, where $\rho(C)$ is defined as in the previous lemma.

$$\begin{aligned} C^1 &= (x_1^1, x_2^3, x_4^4) & C^2 &= (x_1^1, x_3^3, x_5^4) & C^3 &= (x_3^2, x_2^3, x_5^4) \\ C^4 &= (x_1^1, x_2^2, x_5^3) & C^5 &= (x_2^2, x_4^3, x_4^4) & C^6 &= (x_1^1, x_3^2, x_4^3) \\ C^7 &= (x_1^1, x_5^2, x_4^4) & C^8 &= (x_1^1, x_4^2, x_4^4). \end{aligned}$$

First we consider the proof for the case $1 \leq \beta \leq 10$. Let $G_i = C^{3i-2} \cup C^{3i-1} \cup C^{3i}$, $1 \leq i \leq 2$, be the subgraph of $K_4 \times K_5$, where cycles $C^j, 1 \leq j \leq 8$, denote the above 3-cycles. Observe that the edge induced subgraph $G_i, 1 \leq i \leq 2$, is isomorphic to $K_{2,2,2} - E(K_3)$, see Figure 8.

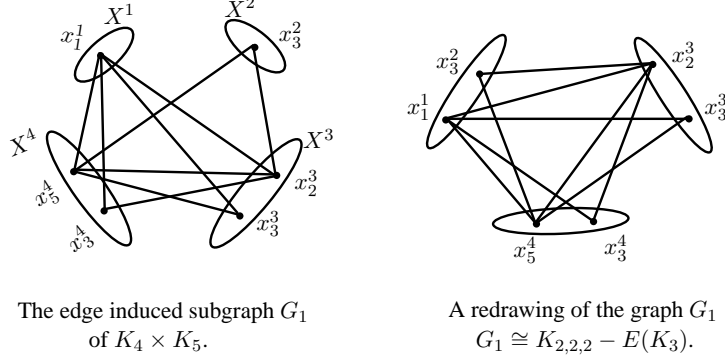


Figure 8.

Let $\rho = (12345)$ be the permutation on $V(K_5) = \{1, 2, 3, 4, 5\}$. Allow $\rho, \rho^2, \rho^3, \rho^4$ to act on the subscripts of the vertices of $G_i, 1 \leq i \leq 2$, and $C^j, 7 \leq j \leq 8$, which we denote by $G_i, \rho(G_i), \rho^2(G_i), \rho^3(G_i), \rho^4(G_i), C^j, \rho(C^j), \rho^2(C^j), \rho^3(C^j), \rho^4(C^j), 1 \leq i \leq 2, 7 \leq j \leq 8$. For $i = 1, 2, G_i, \rho(G_i), \rho^2(G_i), \rho^3(G_i), \rho^4(G_i)$ give ten copies of $K_{2,2,2} - E(K_3)$ and for $j = 7, 8, C^j, \rho(C^j), \rho^2(C^j), \rho^3(C^j), \rho^4(C^j)$, give ten copies of C_3 in $K_4 \times K_5$. As each $K_{2,2,2} - E(K_3)$ is decomposable into three copies of C_3 or, a C_3 and a C_6 , these ten copies of $K_{2,2,2} - E(K_3)$ give β cycles of length 6, where $1 \leq \beta \leq 10$ and the rest into C_3 's.

Next we consider the proof for the case $\beta \geq 11$. As the graph $K_4 \times K_5 = (K_3 \oplus K_{1,3}) \times K_5 = K_3 \times K_5 \oplus K_{1,3} \times K_5$, the lemma follows by Lemmas 8 and 15. \blacksquare

Lemma 17. *The graph $K_6 \times K_5$ admits a $\{C_3^\alpha, C_6^\beta\}$ -decomposition.*

Proof. Let $V(K_6) = \{x^1, x^2, \dots, x^6\}$ and $V(K_5) = \{1, 2, 3, 4, 5\}$. A set of 20 base cycles for a C_3 -decomposition of $K_6 \times K_5$ is given below.

$$\begin{array}{lll}
 C^1 = (x_1^1, x_4^3, x_2^6) & C^2 = (x_1^1, x_3^2, x_5^5) & C^3 = (x_3^2, x_1^3, x_2^6) \\
 C^4 = (x_1^3, x_4^4, x_3^6) & C^5 = (x_2^1, x_4^4, x_5^6) & C^6 = (x_2^1, x_5^2, x_3^1) \\
 C^7 = (x_2^2, x_4^3, x_5^4) & C^8 = (x_4^3, x_5^5, x_3^6) & C^9 = (x_5^4, x_4^5, x_3^6) \\
 C^{10} = (x_1^1, x_3^3, x_4^4) & C^{11} = (x_2^1, x_3^3, x_5^5) & C^{12} = (x_2^1, x_4^4, x_5^5) \\
 C^{13} = (x_3^1, x_4^2, x_5^5) & C^{14} = (x_1^2, x_5^3, x_4^5) & C^{15} = (x_5^3, x_2^4, x_3^5) \\
 C^{16} = (x_2^1, x_3^2, x_4^5) & C^{17} = (x_2^1, x_4^5, x_1^6) & C^{18} = (x_4^1, x_3^2, x_1^6) \\
 C^{19} = (x_1^2, x_5^5, x_3^6) & C^{20} = (x_4^2, x_3^4, x_5^6). &
 \end{array}$$

First we consider the proof for the case $\beta \leq 30$. Let $G_i = C^{3i-2} \cup C^{3i-1} \cup C^{3i}$, $1 \leq i \leq 6$; clearly the edge induced subgraph $G_i, 1 \leq i \leq 6$, of $K_6 \times K_5$, is isomorphic to $K_{2,2,2} - E(K_3)$.

Let $\rho = (12345)$ be a permutation on $V(K_5) = \{1, 2, 3, 4, 5\}$. Then $G_i, \rho(G_i), \rho^2(G_i), \rho^3(G_i), \rho^4(G_i), C^j, \rho(C^j), \rho^2(C^j), \rho^3(C^j), \rho^4(C^j), 1 \leq i \leq 6, 19 \leq j \leq 20$, where $\rho^s(G_i)$ and $\rho^r(C^j)$ have the same meaning as in the proof of Lemma 16, give 30 copies of $K_{2,2,2} - E(K_3)$ and 10 copies of C_3 in $K_6 \times K_5$. Each copy of $K_{2,2,2} - E(K_3)$ is decomposable into C_3 's or, a C_3 and a C_6 and using this decomposition of $K_{2,2,2} - E(K_3)$ suitably, we can achieve a required $\{C_3^\alpha, C_6^\beta\}$ -decomposition of $K_6 \times K_5$, for $\beta \leq 30$.

Next let $\beta \geq 31$. Clearly, $K_6 \times K_5 = (K_4 \oplus K_3 \oplus K_{1,3} \oplus K_{1,3}) \times K_5 = (K_4 \times K_5) \oplus (K_3 \times K_5) \oplus (K_{1,3} \times K_5) \oplus (K_{1,3} \times K_5)$. By Lemmas 8, 15 and 16, the lemma follows. \blacksquare

We quote the following results to prove our main Theorem 1.

Theorem 18 [23]. (i) *If $n \equiv 1$ or $3 \pmod{6}$, then K_n can be decomposed into cycles of length 3.*

(ii) *If $n \equiv 5 \pmod{6}$, then K_n can be decomposed into K_3 's and a K_5 .*

Lemma 19 [20]. *If $n \equiv 0$ or $1 \pmod{3}$, then K_n can be decomposed into K_3 's, K_4 's and K_6 's.*

Theorem 20 [20]. *Let λ and $m \geq 3$ be positive integers. There exists a K_3 -decomposition of $K_m(\lambda)$ if and only if $\lambda(m-1) \equiv 0 \pmod{2}$ and $\lambda m(m-1) \equiv 0 \pmod{6}$.*

Proof of Theorem 1. $\lambda = 1$. The proof of the necessity is obvious and we prove the sufficiency. If $m = 3$ or $n = 3$, then the result follows by Theorem 14. Since $(m, n) \neq (3, 3)$, we assume that m and n are at least 4. As m or n is odd and the tensor product is commutative, we assume that m is odd. Then $m \equiv 1, 3$ or $5 \pmod{6}$. If $m \equiv 1$ or $3 \pmod{6}$ then the graph

$$\begin{aligned} K_m \times K_n &= (K_3 \oplus K_3 \oplus \cdots \oplus K_3) \times K_n, \text{ by Theorem 18,} \\ &= K_3 \times K_n \oplus K_3 \times K_n \oplus \cdots \oplus K_3 \times K_n. \end{aligned}$$

Now by Theorem 14 the result follows. If $m \equiv 5 \pmod{6}$, let $m = 6k + 5$. Since $K_m = K_5 \oplus K_3 \oplus \cdots \oplus K_3$, by Theorem 18, $K_m \times K_n = K_5 \times K_n \oplus K_3 \times K_n \oplus K_3 \times K_n \oplus \cdots \oplus K_3 \times K_n$, $n \geq 4$. Because of Theorem 14, it is enough to show that the graph $K_5 \times K_n$ has a $\{C_3^\alpha, C_6^\beta\}$ -decomposition. By the divisibility condition, $n \equiv 0$ or $1 \pmod{3}$. Since $n \equiv 0$ or $1 \pmod{3}$, K_n can be decomposed into K_3 's, K_4 's and K_6 's, by Lemma 19. Then $K_5 \times K_n$ is the edge disjoint union of the graphs $K_5 \times K_3, K_5 \times K_4$ and $K_5 \times K_6$, and now apply Lemmas 8, 16 and 17 to complete the proof.

Next we consider the case $\lambda = 2$. By hypothesis, either $m \equiv 0$ or $1 \pmod{3}$ or $n \equiv 0$ or $1 \pmod{3}$. Without loss of generality, assume that $m \equiv 0$ or $1 \pmod{3}$, as the tensor product is commutative. The graph

$$(K_m \times K_n)(2) \simeq K_m(2) \times K_n = (K_3 \oplus K_3 \oplus \cdots \oplus K_3) \times K_n, \text{ by Theorem 20} \\ = (K_3 \times K_n \oplus K_3 \times K_n \oplus \cdots \oplus K_3 \times K_n).$$

The result follows by Theorem 14. Now we consider the case $\lambda = 3$. As λ is odd, either m or n is odd; we assume that m is odd. $(K_m \times K_n)(3) \simeq K_m(3) \times K_n = (K_3 \oplus \cdots \oplus K_n) \times K_n$, by Theorem 20. Now apply Theorem 14, the result follows. The last case is $\lambda = 6$. Edge divisibility condition is satisfied for all m and n and again by applying Theorem 20, the desired result is obtained. This completes the proof. \square

APPENDIX

	c_1	c_2	c_3	c_4	c_5
$I_5 : r_1$	1	4	2	5	3
r_2	4	2	5	3	1
r_3	2	5	3	1	4
r_4	5	3	1	4	2
r_5	3	1	4	2	5

	c_1	c_2	c_3	c_4	c_5	c_6
$I_6 : r_1$	1	6	2	5	3	4
r_2	4	2	6	3	1	5
r_3	2	5	3	6	4	1
r_4	5	3	1	4	6	2
r_5	6	1	4	2	5	3
r_6	3	4	5	1	2	6

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
$I_7 : r_1$	1	5	2	6	3	7	4
r_2	5	2	6	3	7	4	1
r_3	2	6	3	7	4	1	5
r_4	6	3	7	4	1	5	2
r_5	3	7	4	1	5	2	6
r_6	7	4	1	5	2	6	3
r_7	4	1	5	2	6	3	7

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$I_8 : r_1$	1	8	2	6	3	7	4	5
r_2	5	2	8	3	7	4	1	6
r_3	2	6	3	8	4	1	5	7
r_4	6	3	7	4	8	5	2	1
r_5	3	7	4	1	5	8	6	2
r_6	7	4	1	5	2	6	8	3
r_7	8	1	5	2	6	3	7	4
r_8	4	5	6	7	1	2	3	8

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
$I_9 : r_1$	1	6	2	7	3	8	4	9	5
r_2	6	2	7	3	8	4	9	5	1
r_3	2	7	3	8	4	9	5	1	6
r_4	7	3	8	4	9	5	1	6	2
r_5	3	8	4	9	5	1	6	2	7
r_6	8	4	9	5	1	6	2	7	3
r_7	4	9	5	1	6	2	7	3	8
r_8	9	5	1	6	2	7	3	8	4
r_9	5	1	6	2	7	3	8	4	9

		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	
I_{10} :	r_1	1	10	2	7	3	8	4	9	5	6	
	r_2	6	2	10	3	8	4	9	5	1	7	
	r_3	2	7	3	10	4	9	5	1	6	8	
	r_4	7	3	8	4	10	5	1	6	2	9	
	r_5	3	8	4	9	5	10	6	2	7	1	
	r_6	8	4	9	5	1	6	10	7	3	2	
	r_7	4	9	5	1	6	2	7	10	8	3	
	r_8	9	5	1	6	2	7	3	8	10	4	
	r_9	10	1	6	2	7	3	8	4	9	5	
	r_{10}	5	6	7	8	9	1	2	3	4	10	
		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}
I_{11} :	r_1	1	7	2	8	3	9	4	10	5	11	6
	r_2	7	2	8	3	9	4	10	5	11	6	1
	r_3	2	8	3	9	4	10	5	11	6	1	7
	r_4	8	3	9	4	10	5	11	6	1	7	2
	r_5	3	9	4	10	5	11	6	1	7	2	8
	r_6	9	4	10	5	11	6	1	7	2	8	3
	r_7	4	10	5	11	6	1	7	2	8	3	9
	r_8	10	5	11	6	1	7	2	8	3	9	4
	r_9	5	11	6	1	7	2	8	3	9	4	10
	r_{10}	11	6	1	7	2	8	3	9	4	10	5
	r_{11}	6	1	7	2	8	3	9	4	10	5	11

Idempotent latin squares of orders 5, 6, . . . , 11 are given above.

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