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ON GRUNDY TOTAL DOMINATION NUMBER IN PRODUCT GRAPHS

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Budapest, Hungary***e-mail:** vizermate@gmail.com**Abstract**

A longest sequence (v_1, \dots, v_k) of vertices of a graph G is a Grundy total dominating sequence of G if for all i , $N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$. The length k of the sequence is called the Grundy total domination number of G and denoted $\gamma_{gr}^t(G)$. In this paper, the Grundy total domination number is studied on four standard graph products. For the direct product we show that $\gamma_{gr}^t(G \times H) \geq \gamma_{gr}^t(G)\gamma_{gr}^t(H)$, conjecture that the equality always holds, and prove the conjecture in several special cases. For the lexicographic product we express $\gamma_{gr}^t(G \circ H)$ in terms of related invariant of the factors and find some explicit formulas for it. For the strong product, lower bounds on $\gamma_{gr}^t(G \boxtimes H)$ are proved as well as upper bounds for products of paths and cycles. For the Cartesian product we prove lower and upper bounds on the Grundy total domination number when factors are paths or cycles.

Keywords: total domination, Grundy total domination number, graph product.

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1. INTRODUCTION

In [10] the Grundy domination number of a graph was introduced as the largest value obtained by a greedy domination procedure. When domination is replaced

by total domination (i.e., instead of closed neighborhoods, one considers open ones) the Grundy total domination number arises. The latter was introduced recently in [11], and was further studied in [9].

Let $S = (v_1, \dots, v_k)$ be a sequence of distinct vertices of G . The corresponding set $\{v_1, \dots, v_k\}$ of vertices from the sequence S will be denoted by \widehat{S} . The initial segment (v_1, \dots, v_i) of S will be denoted by S_i . A sequence $S = (v_1, \dots, v_k)$, where $v_i \in V(G)$, is a (*legal*) *open neighborhood sequence* if, for each i ,

$$(1) \quad N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset.$$

Note that if G is without isolated vertices and S is a maximal open neighborhood sequence of G , then \widehat{S} is a total dominating set of G (initially, total dominating sequences were introduced just for graphs with no isolated vertices).

If (1) holds, each v_i is said to be a *legal choice for Grundy total domination*. We will say that v_i *totally footprints* the vertices from $N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j)$, and that v_i is the *total footprinter* of any $u \in N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j)$. Any *maximal* legal open neighborhood sequence is called a *total dominating sequence*. For a total dominating sequence S any vertex in $V(G)$ has a unique total footprinter in \widehat{S} . An open neighborhood sequence S in G of maximum length is called a *Grundy total dominating sequence* or γ_{gr}^t -*sequence*, and the corresponding invariant the *Grundy total domination number* of G , denoted $\gamma_{gr}^t(G)$.

If condition (1) is replaced with

$$(2) \quad N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset,$$

then one speaks of a *legal choice for Grundy domination*. All the definitions from the above paragraph can now be restated just by omitting “total” everywhere. In particular, the maximum length of a legal dominating sequence in G is the *Grundy domination number* of G and denote it by $\gamma_{gr}(G)$. This invariant was studied for the first time in [10].

In [6] the Grundy domination number of grid-like, cylindrical and toroidal graphs was studied. More precisely, the four standard graph products of paths and/or cycles were considered, and exact formulas for the Grundy domination numbers were obtained for most of the products with two path/cycle factors. In this paper we follow this work and investigate the Grundy total domination number on the standard graph products. The main difference in proving the results for the Grundy total domination number comparing to related results from [6] is that the analogue of [6, Lemma 1] does not hold for Grundy total dominating sequences, therefore other techniques are required. In particular, as

in the case of total domination (cf. the monograph [14]), also in the Grundy total domination natural connections with hypergraphs are applicable.

In [7] a strong connection between the Grundy domination number and the zero forcing number of a graph was established. (The zero forcing number is in turn very useful in determining the minimum rank of a graph [2].) Lin [17] noticed a similar connection between the Grundy total domination number and the skew zero forcing number of graphs, where the skew zero forcing number was introduced in [15] and is denoted by $Z_-(G)$. (For some recent related results see [1, 3–5, 12, 16, 18].) More precisely, Lin proved that if G is a graph, then $\gamma_{gr}^t(G) = |V(G)| - Z_-(G)$. As a consequence, $\gamma_{gr}^t(G) \leq \text{mr}_0(G)$, where mr_0 is a variation of the minimum rank.

The vertex set of each of the four standard graph products of graphs G and H is equal to $V(G) \times V(H)$. In the *direct product* $G \times H$, vertices (g_1, h_1) and (g_2, h_2) are adjacent when $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. In the *lexicographic product* $G \circ H$ (also denoted in the literature by $G[H]$), vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $g_1g_2 \in E(G)$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$. In the *strong product* $G \boxtimes H$, vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever either $g_1g_2 \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Finally, in the *Cartesian product* $G \square H$, vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $g_1g_2 \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$. All these products are associative and, with the exception of the lexicographic product, also commutative. For more on the products see [13].

Let G and H be graphs and $*$ be one of the four graph products under consideration. If $h \in V(H)$, then the set $G^h = \{(g, h) \in V(G * H) : g \in V(G)\}$ is a *G-layer*. By abuse of notation we will also consider G^h as the corresponding induced subgraph. Clearly G^h is isomorphic to G unless $*$ is the direct product in which case it is an edgeless graph of order $|V(G)|$. For $g \in V(G)$, the *H-layer* gH is defined as ${}^gH = \{(g, h) \in V(G * H) : h \in V(H)\}$. We may again consider gH as an induced subgraph when appropriate.

The rest of the paper is organized as follows. In the next section we first observe that $\gamma_{gr}^t(G \times H) \geq \gamma_{gr}^t(G)\gamma_{gr}^t(H)$ holds for arbitrary graphs G and H and conjecture that actually the equality always holds. In the rest of the section we prove the conjecture for several special cases. In Section 3 we consider the lexicographic product and express $\gamma_{gr}^t(G \circ H)$ in terms of related invariant of the factors G and H . As a consequence, formulas for the Grundy total domination number of several special lexicographic products are obtained. In Section 4 lower bounds on $\gamma_{gr}^t(G \boxtimes H)$ are proved, while upper bounds are obtained for strong products of paths and cycles. The Cartesian product seems to be the most demanding with respect to the Grundy total domination number, a typical situation when domination problems are investigated on graph products, cf. [8]. In Section 5 we then give upper and lower bounds for Cartesian products of paths and cycles. In

the concluding section we briefly discuss two related invariants, L-Grundy domination number and Z-Grundy domination number and observe that some of the results derived in this paper extend to these two invariants.

2. DIRECT PRODUCT

We start this section with the following general bound on the Grundy total domination number of the direct product of graphs.

Lemma 2.1. *If G and H are graphs, then $\gamma_{gr}^t(G \times H) \geq \gamma_{gr}^t(G)\gamma_{gr}^t(H)$.*

Proof. Let $S = (x_1, \dots, x_k)$ be a Grundy total dominating sequence in G , let $S' = (y_1, \dots, y_\ell)$ be a Grundy total dominating sequence in H , and let x_i and y_i totally footprint x'_i and y'_i , respectively. Then $(x_1, y_1), (x_1, y_2), \dots, (x_1, y_\ell), (x_2, y_1), \dots, (x_k, y_\ell)$ is a total dominating sequence in $G \times H$ since (x_i, y_j) totally footprints (x'_i, y'_j) . In fact, $(x'_i, y'_j) \in N(x_i, y_j)$, since $x'_i \in N_G(x_i)$ and $y'_j \in N_H(y_j)$, and if $(x'_i, y'_j) \in N((x_m, y_n))$ for some m, n , then $m \geq i$ and $n \geq j$. ■

We conjecture that the inequality in Lemma 2.1 is in fact equality.

Conjecture 2.2. *If G and H are graphs, then $\gamma_{gr}^t(G \times H) = \gamma_{gr}^t(G)\gamma_{gr}^t(H)$.*

In [11] a correspondence between total dominating sequences in graphs and edge covering sequences in hypergraphs was established. In the latter we are given sets (hyperedges) E_1, \dots, E_k that cover the ground set V and we are looking for a maximum length ρ_{gr} of a sequence $E_{i_1}, \dots, E_{i_\ell}$ such that $E_{i_m} \not\subseteq \bigcup_{n < m} E_{i_n}$. This problem clearly generalizes the problem of total dominating sequences and dominating sequences in graphs when sets E_1, \dots, E_k are the open or the closed neighborhoods of vertices, respectively.

For sets X_1, X_2 of hyperedges covering V_1, V_2 , respectively, consider the product of $H_1 = (V_1, X_1)$, $H_2 = (V_2, X_2)$ as $H_1 \times H_2$ with hyperedges $\{E_1 \times E_2 : E_1 \in X_1, E_2 \in X_2\}$ covering $V_1 \times V_2$. Seemingly stronger conjecture in this setting is that $\rho_{gr}(H_1 \times H_2) = \rho_{gr}(H_1)\rho_{gr}(H_2)$. To see that the conjectures are in fact equivalent, for each of the given H_1, H_2 construct the bipartite (incidence) graph B_i whose vertices are $V_i \cup X_i$ and two vertices u, v are adjacent if $v \in V_i$, $u \in X_i$ and $v \in u$. It is not hard to see that since for $(v_1, v_2) \in B_1 \times B_2$ we have $N((v_1, v_2)) = N(v_1) \times N(v_2)$, $\gamma_{gr}^t(B_1 \times B_2)$ includes the questions of calculating ρ_{gr} on four independent product structures, namely $N(V_1) \times N(V_2)$, $N(V_1) \times N(X_2)$, $N(X_1) \times N(V_2)$ and $N(X_1) \times N(X_2)$, where $N(U)$ denotes the set of all the open neighborhoods of the vertices in a set U . The first product structure corresponds to $H_1 \times H_2$. If $\gamma_{gr}^t(B_1 \times B_2) = \gamma_{gr}^t(B_1)\gamma_{gr}^t(B_2)$, then the conjecture holds for all the four subproblems. In particular, it holds that $\rho_{gr}(H_1 \times H_2) = \rho_{gr}(H_1)\rho_{gr}(H_2)$.

In [10] a similar conjecture was posed stating that $\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H)$ and a construction connecting γ_{gr} and ρ_{gr} was introduced. Using the construction from the paper it can be shown that this conjecture is in fact equivalent to both of the above conjectures. Moreover, examining the arguments used above, it suffices to prove Conjecture 2.2 by only considering that both factors are bipartite graphs. In fact, the conjecture holds for all pairs of bipartite graphs on at most 10 vertices, which was checked using a computer.

Lemma 2.3. *Let E_1, \dots, E_k be subsets of the edge set $E(G)$ of a graph G such that $E_1 \cup \dots \cup E_k = E(G)$. Let G_1, \dots, G_k be the isolate-free graphs with edge sets E_1, \dots, E_k , respectively. Then $\gamma_{gr}^t(G) \leq \gamma_{gr}^t(G_1) + \dots + \gamma_{gr}^t(G_k)$.*

Proof. Let $S = (x_1, \dots, x_\ell)$ be a Grundy total dominating sequence in G . For each $x_i \in S$ choose y_i such that x_i totally footprints y_i . Each pair x_i, y_i induces an edge, thus it is in (at least) one of the graphs G_1, \dots, G_k . If it is in more than one of them, choose one arbitrarily. We claim that for each $1 \leq j \leq k$, the subsequence S_j of those vertices $x_i \in \widehat{S}$ that are (chosen) in G_j , is a legal open neighborhood sequence in G_j . Indeed, when x_i is chosen in G_j , it totally footprints y_i , if y_i was totally dominated in G_j by $x_r, r < i$, then y_i would be totally dominated also in G by x_r , a contradiction. Hence, $\gamma_{gr}^t(G) = |\widehat{S}| = |\widehat{S}_1| + \dots + |\widehat{S}_k| \leq \gamma_{gr}^t(G_1) + \dots + \gamma_{gr}^t(G_k)$. ■

Let $bc(G)$ be the smallest size of a covering of edges of G with complete bipartite graphs.

Corollary 2.4. *If G is a graph, then $\gamma_{gr}^t(G) \leq 2bc(G)$.*

Proof. The result follows from Lemma 2.3 by using the fact that the Grundy total domination number of complete bipartite graphs is 2. ■

Lemma 2.5. *Let $v \in V(H)$, and let S be a Grundy total dominating sequence in $G \times H$. Then $|\widehat{S} \cap G^v| \leq \gamma_{gr}^t(G)$.*

Proof. Note that the subsequence of vertices in S that lie in G^v forms a legal open neighborhood sequence of $G \times H$, and consequently, its projection to G is a legal open neighborhood sequence of G . Thus, $|\widehat{S} \cap G^v| \leq \gamma_{gr}^t(G)$. ■

It is known (see [9]) and easy to see that for any vertices v_1, v_2 with $N(v_1) = N(v_2)$ in G , we have $\gamma_{gr}^t(G) = \gamma_{gr}^t(G - v_2)$.

Lemma 2.6. *If G is a graph and k_1, k_2 are positive integers, then we have $\gamma_{gr}^t(G \times K_{k_1, k_2}) = 2\gamma_{gr}^t(G)$.*

Proof. Note that any two vertices u and v in K_{k_1, k_2} that are in the same bipartition set have the same open neighborhoods in K_{k_1, k_2} . Moreover, as $N_{G \times H}(u, v) = N_G(u) \times N_H(v)$, we infer that for any $g \in V(G)$, the vertices (g, u) and (g, v) have the same open neighborhoods in $G \times H$. By applying several times the observation before the lemma, we derive that $\gamma_{gr}^t(G \times K_{k_1, k_2}) = \gamma_{gr}^t(G \times K_2)$. By Lemma 2.1, $\gamma_{gr}^t(G \times K_2) \geq 2\gamma_{gr}^t(G)$, and by Lemma 2.5 the equality holds, that is, $\gamma_{gr}^t(G \times K_{k_1, k_2}) = \gamma_{gr}^t(G \times K_2) = 2\gamma_{gr}^t(G)$. ■

Theorem 2.7. *If G is a graph for which $\gamma_{gr}^t(G) = 2bc(G)$, then $\gamma_{gr}^t(G \times H) = \gamma_{gr}^t(G)\gamma_{gr}^t(H)$.*

Proof. Let E_1, \dots, E_k be a covering of the edge set of G with complete bipartite graphs, such that $k = bc(G)$. Then for $i \in [k]$ let F_i be the subgraph of G spanned by the edge set E_i . Then $E(F_1 \times H), \dots, E(F_k \times H)$ is a covering of the edge set of $G \times H$ such that the corresponding graphs G_1, \dots, G_k are isomorphic to the direct product of complete bipartite graphs with H . By Lemma 2.6, $\gamma_{gr}^t(G_i) = 2\gamma_{gr}^t(H)$, thus by Lemma 2.3, $\gamma_{gr}^t(G \times H) \leq 2bc(G)\gamma_{gr}^t(H) = \gamma_{gr}^t(G)\gamma_{gr}^t(H)$. By Lemma 2.1, the equality holds. ■

Corollary 2.8. *If T is a tree, then $\gamma_{gr}^t(T \times H) = \gamma_{gr}^t(T)\gamma_{gr}^t(H)$.*

Proof. By Theorem 2.7, it suffices to show that $\gamma_{gr}^t(T) = 2bc(T)$. By the result in [9], $\gamma_{gr}^t(T) = 2\beta(T)$, where $\beta(T)$ is the vertex cover number of T . Since the vertex cover can be interpreted as covering edges with stars, and stars are the only complete bipartite graphs in trees, it holds $\beta(T) = bc(T)$. ■

To find more graphs for which Theorem 2.7 applies, we first consider the following lemma, which is a straightforward consequence of definitions.

Lemma 2.9. *The direct product $K_{k_1, k_2} \times K_{\ell_1, \ell_2}$ of two complete bipartite graphs is isomorphic to the disjoint union $K_{k_1\ell_1, k_2\ell_2} + K_{k_1\ell_2, k_2\ell_1}$.*

Lemma 2.10. *If G_1, G_2 are such that $\gamma_{gr}^t(G_i) = 2bc(G_i)$ for $i \in \{1, 2\}$, then $\gamma_{gr}^t(G_1 \times G_2) = 2bc(G_1 \times G_2)$.*

Proof. On one hand, $2bc(G_1 \times G_2) \geq \gamma_{gr}^t(G_1 \times G_2) \geq \gamma_{gr}^t(G_1)\gamma_{gr}^t(G_2) = 2bc(G_1)2bc(G_2)$. On the other hand, if $G_{1,1}, \dots, G_{1,k_1}$ are complete bipartite graphs with $bc(G_1) = k_1$ and $E(G_{1,1}), \dots, E(G_{1,k_1})$ is a covering of $E(G_1)$ and $G_{2,1}, \dots, G_{2,k_2}$ are complete bipartite graphs with $bc(G_2) = k_2$ and $E(G_{2,1}), \dots, E(G_{2,k_2})$ is a covering of $E(G_2)$, then by Lemma 2.9, each $G_{1,i} \times G_{2,j}$ is a disjoint union of two complete bipartite graphs. Thus $G_1 \times G_2$ has a partition of edges into $2bc(G_1)bc(G_2)$ complete bipartite graphs. This proves that the first inequality is in fact equality. ■

Notice that $\gamma_{gr}^t(P_\ell) = \ell$, $\gamma_{gr}^t(C_\ell) = \ell - 2$ if ℓ is even, and $\gamma_{gr}^t(P_\ell) = \ell - 1$, $\gamma_{gr}^t(C_\ell) = \ell - 1$ otherwise. Hence, by applying Corollary 2.8, we derive the following result.

Corollary 2.11. *If $k, \ell > 1$, then*

$$\gamma_{gr}^t(P_k \times P_\ell) = \begin{cases} k \cdot \ell, & k, \ell \text{ even,} \\ k \cdot (\ell - 1), & k \text{ even, } \ell \text{ odd,} \\ (k - 1) \cdot (\ell - 1), & k, \ell \text{ odd.} \end{cases}$$

If $k > 1$, $\ell > 2$, then

$$\gamma_{gr}^t(P_k \times C_\ell) = \begin{cases} k \cdot (\ell - 2), & k, \ell \text{ even,} \\ k \cdot (\ell - 1), & k \text{ even, } \ell \text{ odd,} \\ (k - 1) \cdot (\ell - 2), & k \text{ odd, } \ell \text{ even,} \\ (k - 1) \cdot (\ell - 1), & k, \ell \text{ odd.} \end{cases}$$

Theorem 2.12. $\gamma_{gr}^t(C_{n_1} \times C_{n_2}) = \gamma_{gr}^t(C_{n_1})\gamma_{gr}^t(C_{n_2})$.

Proof. Notice that with a permutation of vertices, open neighborhoods of C_ℓ can be presented as $\{0, 1\}, \{1, 2\}, \{2, 3\}, \dots, \{\ell - 1, 0\}$ in the case ℓ is odd and as a disjoint union of two such presentations in the case ℓ is even. For the sake of convenience, assume that both n_1 and n_2 are odd, we will deal with the other cases at the end of the proof. In this case the open neighborhoods of the product $C_{n_1} \times C_{n_2}$ are in the above presentation quadruples of the form $\{(0, 0), (1, 0), (0, 1), (1, 1)\}, \{(1, 0), (2, 0), (1, 1), (2, 1)\}, \dots, \{(n_1 - 1, n_2 - 1), (0, n_2 - 1), (n_1 - 1, 0), (0, 0)\}$. We will call a set of vertices $\{(i, 0), (i, 1), \dots, (i, n_2 - 1)\}$ a row and similarly a set of vertices $\{(0, i), (1, i), \dots, (n_1 - 1, i)\}$ a column.

Let $S = (v_1, \dots, v_n)$ be an optimal total dominating sequence in $C_{n_1} \times C_{n_2}$ of length n . For each $i \in [n] = \{1, \dots, n\}$ let $k_i = |\bigcup_{j=1}^i N(v_j)| - |\bigcup_{j=1}^{i-1} N(v_j)| - 1$. By definition of the total dominating sequence, $k_i \geq 0$. To prove that $\gamma_{gr}^t(C_{n_1} \times C_{n_2}) \leq \gamma_{gr}^t(C_{n_1})\gamma_{gr}^t(C_{n_2}) = n_1 n_2 - n_1 - n_2 + 1$ holds, we need to prove that $\sum_{j=1}^n k_j \geq n_1 + n_2 - 1$. By Lemma 2.1, this suffices to prove the statement.

For the convenience denote $D_i = \bigcup_{j=1}^i N(v_j)$ for each $i \in [n]$. Let $v_i \in \{v_2, \dots, v_n\}$ be such that it totally dominates a vertex which is in a row or a column such that no vertex in this row or a column is in D_{i-1} . We will say that this row or column is *newly dominated* by v_i . Let the number of columns or rows newly dominated by v_i be nd_i .

For each D_j , let G_j be a graph on vertices D_j such that two vertices in D_j are adjacent if they lie in the same column or the same row in $C_{n_1} \times C_{n_2}$. Notice that for $n \geq j_1 > j_2 \geq 1$, graph G_{j_2} is an induced subgraph of G_{j_1} . Let c_j be the number of connected components of G_j (with $c_0 = 0$). We would like to prove the following

Lemma 2.13. *For $i \geq 1$ we have*

$$k_i + (c_i - c_{i-1}) - nd_i \geq 0.$$

Proof. First let us suppose that $c_i - c_{i-1} \geq 0$.

If v_i newly dominates one column and no row, it totally dominates at least two vertices not in D_{i-1} , therefore $k_i \geq 1$. The same holds if v_i newly dominates one row and not any column, implying that in this two cases the sum is non-negative.

If v_i newly dominates one column and newly dominates one row, it totally dominates at least three vertices not in D_{i-1} , therefore $k_i \geq 2$. Similarly, if v_i newly dominates two columns (or rows) and one or zero rows (or columns), it totally dominates at least four vertices not in D_{i-1} , therefore $k_i = 3$. In all of the above cases when a row or a column is newly dominated, the corresponding k_i contributes a positive value for each row and column to the sum, making it non-negative.

What remains is to analyze the situation where there exists v_i that newly dominates two columns and two rows. Such vertex dominates four new vertices, hence $k_i = 3$. Moreover vertices in $N(v_i)$ form a new connected component implying $c_i - c_{i-1} = 1$. Thus the sum is non-negative.

Now we turn to the case when we have $c_i - c_{i-1} < 0$. By definition of the graphs, $c_{i-1} - c_i$ can be at most 3 since vertices of $N(v_i)$ lie in two columns and two rows, thus they can join at most four components into one.

First, suppose $c_{i-1} - c_i = 3$. In this case each of the four components has a vertex in exactly one distinct row or column that vertices of $N(v_i)$ lie in. In particular, this implies that v_i does not newly dominate any row or column. Moreover, no vertex in $N(v_i)$ is in D_{i-1} since otherwise the component of such a vertex would lie in a column and a row that vertices of $N(v_i)$ lie in, hence there could not be four components in G_{i-1} that are joined into one in G_i . Hence v_i must totally dominate four vertices not in D_{i-1} and $k_i = 3$.

In the case that $c_{i-1} - c_i = 2$ it is not hard to see by a simple check that either $k_i = 2$, and v_i does not dominate any new column or row or v_i newly dominates exactly one row or exactly one column, but in this case $k_i = 3$.

Finally if $c_{i-1} - c_i = 1$, then similarly the possible situations are: either $k_i = 1$, and v_i does not dominate any new column or row; or $k_i = 2$ and v_i newly dominates one column or one row; or $k_i = 3$, and v_i newly dominates two columns, two rows or a column and a row. \square

We conclude the proof of Theorem 2.12 with the following: By Lemma 2.13 we know that

$$\sum_{i=1}^n k_i + (c_i - c_{i-1}) - nd_i \geq 0.$$

Since we have $c_n = 1$, $c_0 = 0$ and $\sum_{i=1}^n nd_i = n_1 + n_2$, we have that $\sum_{j=1}^n k_j \geq n_1 + n_2 - 1$ which as stated proves the Theorem 2.12, when n_1, n_2 are odd.

To conclude the proof we need to cover the cases when at least one of the n_1, n_2 is even.

If, say, n_1 is odd and n_2 is even, then the open neighborhoods of the product $C_{n_1} \times C_{n_2}$ can be presented as a disjoint union of two copies of quadruples of the form $\{(0, 0), (1, 0), (0, 1), (1, 1)\}, \{(1, 0), (2, 0), (1, 1), (2, 1)\}, \dots, \{(n_1 - 1, \frac{n_2}{2} - 1), (0, \frac{n_2}{2} - 1), (n_1 - 1, 0), (0, 0)\}$. Hence by similar arguments as above,

$$\gamma_{gr}^t(C_{n_1} \times C_{n_2}) \leq 2(n_1 - 1) \left(\frac{n_2}{2} - 1 \right) = (n_1 - 1)(n_2 - 2) = \gamma_{gr}^t(C_{n_1})\gamma_{gr}^t(C_{n_2}).$$

Moreover, if both n_1 and n_2 are even, we have four disjoint copies of quadruples of the form $\{(0, 0), (1, 0), (0, 1), (1, 1)\}, \{(1, 0), (2, 0), (1, 1), (2, 1)\}, \dots, \{(\frac{n_1}{2} - 1, \frac{n_2}{2} - 1), (0, \frac{n_2}{2} - 1), (\frac{n_1}{2} - 1, 0), (0, 0)\}$ and get

$$\gamma_{gr}^t(C_{n_1} \times C_{n_2}) \leq 4 \left(\frac{n_1}{2} - 1 \right) \left(\frac{n_2}{2} - 1 \right) = (n_1 - 2)(n_2 - 2) = \gamma_{gr}^t(C_{n_1})\gamma_{gr}^t(C_{n_2}). \quad \blacksquare$$

Similar techniques (but more direct) can be used to show that Conjecture 2.2 holds if one of the factors is a cycle and one of them is a complete graph or both being complete graphs. This could possibly indicate why the conjecture holds for small graphs, since most of them could probably be partitioned into cycles, cliques, trees and complete bipartite graphs in the way of Lemma 2.3.

3. LEXICOGRAPHIC PRODUCT

As it turns out, the formula for $\gamma_{gr}^t(G \circ H)$ a bit surprisingly relies on (Grundy) dominating sequences of G . The results are very similar to the formula for $\gamma_{gr}(G \circ H)$ which was obtained in [6].

Given a dominating sequence $D = (d_1, \dots, d_k)$ in a graph G , let $a(D)$ denote the cardinality of the set of vertices d_i from D , which are not adjacent to any vertex from $\{d_1, \dots, d_{i-1}\}$.

Theorem 3.1. *For any graphs G and H , where H has no isolated vertices,*

$$\gamma_{gr}^t(G \circ H) = \max \{a(D)(\gamma_{gr}^t(H) - 1) + |\widehat{D}| : D \text{ is a dominating sequence of } G\}.$$

Proof. Let $D = (d_1, \dots, d_m)$ be a dominating sequence of G , and let (d'_1, \dots, d'_k) be a Grundy total dominating sequence of H . Then one can find a sequence S in $G \circ H$ of length $a(D)(\gamma_{gr}^t(H) - 1) + |\widehat{D}|$ as follows. Let S be the sequence that corresponds to D , and those vertices $d_i \in \widehat{D}$ which are not adjacent to any vertex from $\{d_1, \dots, d_{i-1}\}$ are repeated $\gamma_{gr}^t(H)$ times in a row, so that the

corresponding subsequence is of the form $((d_i, d'_1), \dots, (d_i, d'_k))$. On the other hand, the vertices $d_i \in \widehat{D}$ which are adjacent to some vertex from $\{d_1, \dots, d_{i-1}\}$ are projected only once from the vertices of S , i.e., there is a unique vertex (d_i, h) that belongs to S . It is easy to see that in either case the vertices in S are legally chosen. In the first case this is true because no vertex of ${}^d H$ is totally dominated yet at the point when (d_i, d'_1) is chosen, thus $((d_i, d'_1), \dots, (d_i, d'_k))$ is a legal open neighborhood subsequence. In the second case this is true because D is a legal closed neighborhood sequence in G , and so when d_i is chosen, there exists another vertex $t \in V(G)$, distinct from d_i , that d_i footprints. Hence, when (d_i, h) is chosen in S , it totally footprints vertices from ${}^t H$. Note that the length of S is $a(D)(\gamma_{gr}^t(H) - 1) + |\widehat{D}|$. This implies that $\gamma_{gr}^t(G \circ H) \geq \max \{a(D)(\gamma_{gr}^t(H) - 1) + |\widehat{D}| : D \text{ is a dominating sequence of } G\}$.

For the converse, let S be an arbitrary open neighborhood sequence in $G \circ H$. Let $S' = ((x_1, y_1), \dots, (x_n, y_n))$ be the subsequence of S , where $(x, y) \in S'$ if and only if (x, y) is the first vertex in S that belongs to ${}^x H$. We claim that the corresponding sequence of the first coordinates $T = (x_1, \dots, x_n)$ is a legal closed neighborhood sequence in G . Firstly, if x_i is not adjacent to any of $\{x_1, \dots, x_{i-1}\}$, then clearly x_i footprints itself. Otherwise, if x_i is adjacent to x_j , where $j \in \{1, \dots, i-1\}$, then there exists a vertex $(x_j, h) \in S$ that appears in S before any vertex from ${}^{x_i} H$. Thus when (x_i, y_i) is added to S , all vertices from ${}^{x_i} H$ are already totally dominated. Hence, since (x_i, y_i) must totally footprint some vertex, it can only be a vertex in ${}^x H$, where $x \in N_G(x_i)$. We infer that x_i footprints x with respect to T . Thus T is a legal closed neighborhood sequence.

Let $A(T)$ be the set of all vertices x_i in T , such that x_i is not adjacent to any vertex from $\{x_1, \dots, x_{i-1}\}$. (Note that $|A(T)| = a(T)$ by definition.) Two cases for a vertex $x_j \in \widehat{T}$ appear, which give different bounds on the number of vertices in ${}^{x_j} H \cap \widehat{S}$. If $x_j \notin A(T)$, then $|{}^{x_j} H \cap \widehat{S}| = 1$, because (x_j, y_j) totally dominates all neighboring H -layers, and ${}^{x_j} H$ is already totally dominated before (x_j, y_j) is added to S . On the other hand, if $x_j \in A(T)$, then clearly $|{}^{x_j} H \cap \widehat{S}| \leq \gamma_{gr}^t(G)$. We infer that $|\widehat{S}| \leq (|\widehat{T}| - a(T)) + a(T)\gamma_{gr}^t(H)$, where T is a closed neighborhood sequence. \blacksquare

Since any independent set of vertices yields a legal closed neighborhood sequence, we infer the following

Corollary 3.2. *For any graphs G and H with no isolated vertices,*

$$\gamma_{gr}^t(G \circ H) \geq \alpha(G)\gamma_{gr}^t(H).$$

To see that the inequality in Corollary 3.2 is not always equality, consider $P_4 \circ H$, where H is a graph with no isolated vertices. The bound in the corollary is $\alpha(P_4)\gamma_{gr}^t(H) = 2\gamma_{gr}^t(H)$, but $\gamma_{gr}^t(P_4 \circ H) \geq 2\gamma_{gr}^t(H) + 1$. Indeed, let u_1, u_2, u_3, u_4

be the vertices of P_4 with the natural adjacencies, and let S be the sequence that starts in the layer ${}^{u_1}H$, by legally picking $\gamma_{gr}^t(H)$ vertices. Then select an arbitrary vertex from ${}^{u_2}H$. Finally select $\gamma_{gr}^t(H)$ vertices in the layer ${}^{u_4}H$. This yields a legal sequence of the desired length.

Following the same arguments as in [6, Corollary 10], and using the fact that $\gamma_{gr}^t(H) \geq 2$ for any graph H with no isolated vertices, we derive the following results.

Corollary 3.3. *If H is a graph with no isolated vertices, then*

$$\gamma_{gr}^t(P_k \circ H) = \begin{cases} \frac{k}{2} \cdot \gamma_{gr}^t(H) + 1, & k \text{ is even, } k \neq 2, \\ \lceil \frac{k}{2} \rceil \cdot \gamma_{gr}^t(H), & k \text{ is odd.} \end{cases}$$

Corollary 3.4. *Let $k, \ell > 2$. If ℓ is even, then*

$$\gamma_{gr}^t(P_k \circ P_\ell) = \begin{cases} \frac{k}{2} \cdot \ell + 1, & k \text{ is even,} \\ \lceil \frac{k}{2} \rceil \cdot \ell, & k \text{ is odd.} \end{cases}$$

If ℓ is odd, then

$$\gamma_{gr}^t(P_k \circ P_\ell) = \begin{cases} \frac{k}{2} \cdot (\ell - 1) + 1, & k \text{ is even,} \\ \lceil \frac{k}{2} \rceil \cdot (\ell - 1), & k \text{ is odd.} \end{cases}$$

Corollary 3.5. *Let $k, \ell > 2$. If ℓ is even, then*

$$\gamma_{gr}^t(P_k \circ C_\ell) = \begin{cases} \frac{k}{2} \cdot (\ell - 2) + 1, & k \text{ is even,} \\ \lceil \frac{k}{2} \rceil \cdot (\ell - 2), & k \text{ is odd.} \end{cases}$$

If ℓ is odd, then

$$\gamma_{gr}^t(P_k \circ C_\ell) = \begin{cases} \frac{k}{2} \cdot (\ell - 1) + 1, & k \text{ is even,} \\ \lceil \frac{k}{2} \rceil \cdot (\ell - 1), & k \text{ is odd.} \end{cases}$$

Using Theorem 3.1 we also obtain the following result.

Corollary 3.6. *Let H be a graph without isolated vertices, and $k \geq 3$. Then*

$$\gamma_{gr}^t(C_k \circ H) = \begin{cases} \frac{k}{2} \cdot \gamma_{gr}^t(H), & k \text{ is even,} \\ \lceil \frac{k}{2} \rceil \cdot \gamma_{gr}^t(H) + 1, & k \text{ is odd.} \end{cases}$$

The value for Grundy total domination numbers of $C_k \circ C_\ell$ easily follows from the above corollary.

4. STRONG PRODUCT

Given a (closed neighborhood) dominating sequence $D = (d_1, \dots, d_k)$ in a graph G , let

- $C(D)$ denote the set of vertices d_i from D that footprint itself and at least one of its neighbors and let $c(D) = |C(D)|$;
- $B(D)$ denote the set of vertices d_i from D that does not footprint itself and footprint at least one of its neighbors and let $b(D) = |B(D)|$;
- $A(D)$ denote the set of vertices d_i from D that are not adjacent to any vertex from $\{d_1, \dots, d_{i-1}\}$ and let $a(D) = |A(D)|$.

Theorem 4.1. *If G and H are graphs, where G has no isolated vertices, then*

$$\gamma_{gr}^t(G \boxtimes H) \geq \max \{ (\gamma_{gr}(G) + 1) \cdot c(D) + \gamma_{gr}(G) \cdot b(D) + (|\widehat{D}| - b(D) - c(D)) \gamma_{gr}^t(G) : D \text{ is a dominating sequence of } H \}.$$

Proof. Let $D = (y_1, \dots, y_k)$ be an arbitrary legal closed neighborhood sequence of H . Let (x_1, \dots, x_ℓ) be a γ_{gr} -sequence of G and (g_1, \dots, g_j) a γ_{gr}^t -sequence of G . Let $x \in V(G)$ be a vertex that is footprinted by x_ℓ (note that x_ℓ can be chosen in such a way that $x \neq x_\ell$, since G has no isolated vertices). Now we construct a legal total dominating sequence $S = S_k$ of $G \boxtimes H$. First let S_0 be the empty sequence. For each $i \in [k]$ let $S_i = S_{i-1} \oplus ((x_1, y_i), (x_2, y_i), \dots, (x_{\ell-1}, y_i), (x, y_i), (x_\ell, y_i))$ if $y_i \in C(D)$, where \oplus means the concatenation operation on sequences. If $y_i \in B(D)$, let $S_i = S_{i-1} \oplus ((x_1, y_i), (x_2, y_i), \dots, (x_{\ell-1}, y_i), (x_\ell, y_i))$ otherwise $S_i = S_{i-1} \oplus ((g_1, y_i), (g_2, y_i), \dots, (g_j, y_i))$. ■

By symmetry we get the following.

Theorem 4.2. *If G and H are graphs, where H has no isolated vertices, then*

$$\gamma_{gr}^t(G \boxtimes H) \geq \max \{ (\gamma_{gr}(H) + 1) \cdot c(D) + \gamma_{gr}(H) \cdot b(D) + (|\widehat{D}| - b(D) - c(D)) \gamma_{gr}^t(H) : D \text{ is a dominating sequence of } G \}.$$

Theorem 4.3. *If G and H are graphs, then*

$$\gamma_{gr}^t(G \boxtimes H) \geq \max \{ \gamma_{gr}^t(G) \cdot a(D) + (|\widehat{D}| - a(D)) \gamma_{gr}(G) : D \text{ is a dominating sequence of } H \}.$$

Proof. Let $D = (y_1, \dots, y_k)$ be an arbitrary legal closed neighborhood sequence of H . Let (x_1, \dots, x_ℓ) be a γ_{gr} -sequence of G and (g_1, \dots, g_j) a γ_{gr}^t -sequence of G . Now we construct a legal total dominating sequence $S = S_k$ of $G \boxtimes H$. Let $S_0 = \emptyset$. For each $i \in [k]$ let $S_i = S_{i-1} \oplus ((g_1, y_i), (g_2, y_i), \dots, (g_j, y_i))$ if $y_i \in A(D)$, otherwise $S_i = S_{i-1} \oplus ((x_1, y_i), (x_2, y_i), \dots, (x_\ell, y_i))$. ■

By symmetry we also get the following.

Theorem 4.4. *If G and H are graphs, then*

$$\gamma_{gr}^t(G \boxtimes H) \geq \max \left\{ \gamma_{gr}^t(H) \cdot a(D) + (|\widehat{D}| - a(D))\gamma_{gr}(H) : \right. \\ \left. D \text{ is a dominating sequence of } G \right\}.$$

For the strong product of paths and/or cycles the above theorems imply the following lower bounds.

Corollary 4.5. *For any pair $k, \ell \geq 3$ of integers we have*

(i)

$$\gamma_{gr}^t(P_k \boxtimes P_\ell) \geq \begin{cases} k\ell - k - \ell + 2, & k, \ell \text{ are odd,} \\ k\ell - k - \ell + 3, & \text{otherwise.} \end{cases}$$

(ii)

$$\gamma_{gr}^t(P_k \boxtimes C_\ell) \geq \begin{cases} k\ell - 2k - \ell + 3, & k \text{ is odd, } \ell \text{ is even,} \\ k\ell - 2k - \ell + 4, & \text{otherwise.} \end{cases}$$

(iii)

$$\gamma_{gr}^t(C_k \boxtimes C_\ell) \geq \begin{cases} k\ell - 2k - 2\ell + 5, & k, \ell \text{ are even,} \\ k\ell - 2k - 2\ell + 6, & \text{otherwise.} \end{cases}$$

Let $V(P_k) = [k]$, $E(P_k) = \{12, 23, \dots, (k-1)k\}$. A subset I of consecutive elements of $[k]$ with $|I| \geq 2$ is called an *interval*. Let P_k^2 denote the graph with vertex set $[k]$ and edge set $\{(i, j) : |i - j| \leq 2\}$. *Strong components* of a subset $U \subseteq [k]$ are components of $P_k^2[U]$.

Proposition 4.6. *For $k \geq 3$ let U be a non-empty subset of $[k]$ with $|U| < k$. Then the number d of totally dominated vertices by U in P_k is either*

(i) *at least $|U| + 1$, or*

(ii) *$d = |U| - 1$ and k is odd and U consists of all odd numbers of $[k]$, or*

(iii) *$d = |U|$ and for the number i of intervals in U we have either*

1. *$i = 0$ and $U = \{1, 3, 5, \dots, 2l - 1 : 2l - 1 < k\}$, or $U = \{k - 2m, k - 2m + 2, \dots, k : k - 2m > 1\}$, or $U = \{1, 3, 5, \dots, 2l - 1\} \cup \{k - 2m, k - 2m + 2, \dots, k\}$ with $2l + 1 < k - 2m$, or*
2. *$i = 1$ and U consists of one single strong component and $1, k \in U$.*

Proof. We proceed by induction on k . If $k = 3$, then U is (isomorphic) either (to) $\{1\}$, $\{2\}$, $\{1, 2\}$ or $\{1, 3\}$ and the cases covered by the statement.

Suppose now $k \geq 4$ and the statement is proved for $3, 4, \dots, k - 1$. Assume first U contains at least two strong components. Then there exists an $m \in [k]$ such $m, m + 1 \notin U$ but there are elements $l, r \in U$ with $l < m, m + 1 < r$. Then

let $[a + 1, b - 1]$ be the largest interval containing m that is disjoint with U (so $a, b \in U$ and $a + 1$ and $b - 1$ are totally dominated by U). If $a \geq 3$ and $b \leq k - 2$, then by induction applied to P_a and P_{k-b+1} we obtain that the number of vertices totally dominated by $U \cap [a]$ is at least $|U \cap [a]| - 1 + 1$ (the plus 1 stands for dominating $a + 1$) and that the number of vertices totally dominated by $U \cap [b, k]$ is at least $|U \cap [b, k]| - 1 + 1$ (the plus 1 stands for dominating $b - 1$). Therefore the total number of vertices dominated by U is at least $|U|$ and equality holds (again by induction) if $U = \{1, 3, \dots, a\} \cup \{b, \dots, k - 2, k\}$ as stated in the proposition. The cases when $a \leq 2$ or $b \geq k - 1$ can be covered similarly.

So from now on, we assume that U consists of a single strong component. If $i = 0$, then $U = \{j, j + 2, j + 4, \dots, j + 2l\}$ for some j and l . If $j \neq 1$ and $j + 2l \neq k$, then $d = |U| + 1$, while the other possibilities are included in the statement of the proposition. If $i = 1$ and 1 or k does not belong to U , then $d = |U| + 1$, while the case $1, k \in U$ is included in the statement of the proposition. Finally, if $i \geq 2$, then consider two maximal intervals $[a, b]$ and $[u, v]$ of U such that $[b + 1, u - 1] \cap U$ does not contain any intervals. Then the number of totally dominated vertices in $[a, v]$ is $|U \cap [a, v]| + 1$. Furthermore, by induction we obtain that the number of totally dominated vertices in $[1, a - 1]$ is at least $|U \cap [1, a - 1]|$ and the number of totally dominated vertices in $[v + 1, k]$ is at least $|U \cap [v - 1, k]|$. Therefore $d \geq |U| + 1$ holds as claimed. \blacksquare

Proposition 4.7. *Let $k \geq 3$ and U be a subset of $[2] \times [k]$, with $0 < n_1 := |U \cap (\{1\} \times [k])| < k$ and $0 < n_2 := |U \cap (\{2\} \times [k])| < k$. Let us denote by D the set of vertices totally dominated by U in $P_2 \boxtimes P_k$, and let $d_1 := |D \cap (\{1\} \times [k])|$, $d_2 := |D \cap (\{2\} \times [k])|$. Then we have*

- (a) $d_1 + d_2 \geq n_1 + n_2 + 2$,
- (b) $2d_1 + d_2 \geq 2n_1 + n_2 + 3$, unless $U = \{(i, j) : i \in [2], j \leq k - 2\} \cup \{(1, k)\}$ or $U = \{(i, j) : i \in [2], j \geq 3\} \cup \{(1, 1)\}$ in which cases $2d_1 + d_2 = 2n_1 + n_2 + 2$.

Proof. The proof of (a) follows from the observation that if a layer contains $1 \leq x < k$ vertices, then these vertices totally dominate at least $x + 1$ vertices in the neighboring layer.

To prove (b) we consider different cases. If $n_2 \geq n_1$, then again we are done by the above observation as $2d_1 + d_2 \geq 2(n_2 + 1) + (n_1 + 1) \geq 2n_1 + n_2 + 3$. So we may assume that $n_2 < n_1$. If $d_1 \geq n_1 + 1$, then we are done as using the above observation we have $2d_1 + d_2 \geq 2(n_1 + 1) + (n_1 + 1) > 2n_1 + n_2 + 3$.

Observe that if we have $d_1 = n_1 - 1$, then by Proposition 4.6 we know the structure of $U \cap (\{1\} \times [k])$, and we also know that none of the vertices of $U \cap (\{1\} \times [k])$ is dominated by any vertex of $U \cap (\{2\} \times [k])$. But then $U \cap (\{2\} \times [k])$ is an empty set. This cannot be the case as we assumed that $n_2 > 0$.

Finally, if $d_1 = n_1$, then according to Proposition 4.6 we either have $d_2 = 2n_1$ or $d_2 = k$. In the former case we have $2d_1 + d_2 = 4n_1 \geq 2n_1 + n_2 + 3$ as

$n_1 > n_2 \geq 1$ implies $n_1 \geq 2$ and $n_1 \geq n_2 + 1$. In the latter case we have $2d_1 + d_2 \geq 2n_1 + k \geq 2n_1 + n_2 + 2$ as $k > n_1 > n_2$ with equality if and only if $n_2 = k - 2$, $n_1 = k - 1$. Using Proposition 4.6 it is easy to see that this happens only if U is as stated. ■

With very similar proofs to those of Proposition 4.6 and Proposition 4.7 one can obtain the following statements.

Proposition 4.8. *For $k \geq 3$ let U be a non-empty subset of $[k]$ with $|U| \leq k - 2$. Then the number d of totally dominated vertices by U in C_k is either*

- (i) *at least $|U| + 2$, or*
- (ii) *$d = |U|$ and k is even and U is either the set of even numbers in $[k]$ or the set of odd numbers in $[k]$, or*
- (iii) *$d = |U| + 1$ and we have either*
 - 1. *$U = \{j, j+2, j+4, \dots, j+2l\}$, for some j and l with $2l < k$ and addition is modulo k , or*
 - 2. *$U = [a, b] \cup \{b+2, b+4, \dots, b+(k-b+a-2)\}$ for some $a < b$, $k-b+a$ even and addition is again modulo k .*

Proposition 4.9. *Let $k \geq 3$ and U be a subset of $[2] \times [k]$, with $0 < n_1 := |U \cap (\{1\} \times [k])| \leq k - 2$ and $0 < n_2 := |U \cap (\{2\} \times [k])| \leq k - 2$. Let us denote by D the set of vertices totally dominated by U in $P_2 \boxtimes C_k$, and let $d_1 := |D \cap (\{1\} \times [k])|$, $d_2 := |D \cap (\{2\} \times [k])|$. Then we have*

- (a) $d_1 + d_2 \geq n_1 + n_2 + 4$,
- (b) $2d_1 + d_2 \geq 2n_1 + n_2 + 6$ unless $U = \{(i, j) : i \in [2], j \neq j^* - 1, j^*, j^* + 1\} \cup \{(1, j^*)\}$ for some $j^* \in [k]$ and addition is modulo k .

With the above auxiliary results, we can prove the following upper bound on the strong product of paths and cycles.

Theorem 4.10. *For any pair $k, \ell \geq 3$ of integers we have*

- (i) $\gamma_{gr}^t(P_k \boxtimes P_\ell) \leq k\ell - \min\{k, \ell\} + 1$,
- (ii) $\gamma_{gr}^t(C_k \boxtimes C_\ell) \leq k\ell - \min\{2k, 2\ell\} + 1$,
- (iii) $\gamma_{gr}^t(P_k \boxtimes C_\ell) \leq k\ell - \min\{2k, \ell\} + 1$.

Proof. First we prove (i). Note that if an open neighborhood sequence intersects every P_k -layer in less than k vertices and every P_ℓ -layer in less than ℓ vertices, then the length of the sequence is at most $k\ell - \max\{k, \ell\}$.

Consider now the smallest m such that $D = (x_1, \dots, x_m)$ contains all vertices from a P_k -layer or all vertices from a P_ℓ -layer. If x_1, \dots, x_m would contain one complete P_k -layer and one complete P_ℓ -layer, then we would have $N(x_m) \subseteq \bigcup_{i=1}^{m-1} N(x_i)$, a contradiction.

Assume without loss of generality that the sequence x_1, \dots, x_m contains all vertices of a P_ℓ -layer ${}^j P_\ell$. Let n_i , $i \in [\ell]$, be the number of vertices from the i^{th} P_k -layer that are in x_1, \dots, x_m and let d_i be the number of vertices from the i^{th} P_k -layer that are totally dominated by D . Note that $0 < n_i < k$ for all $i \in [\ell]$. Adding up lower bounds of Proposition 4.7(a) on $d_i + d_{i+1}$ for $i = 2, 3, \dots, \ell - 1$ and lower bounds of Proposition 4.7(b) on $2d_1 + d_2$ and $d_{\ell-1} + 2d_\ell$ we obtain

$$2 \sum_{i=1}^{\ell} d_i \geq 2\ell - 2 + 2 \sum_{i=1}^{\ell} n_i = 2m + 2\ell - 2.$$

So D totally dominates at least $\ell - 1$ more vertices than its size. As this difference cannot decrease throughout the open neighborhood sequence and therefore $\gamma_{gr}^t(P_k \boxtimes P_\ell) \leq k\ell - \ell + 1$. This finishes the proof of (i).

The proof of (ii) and (iii) are similar too that of (i), so we just sketch them. To see (ii) observe first that if an open neighborhood sequence intersects every C_k -layer in less than $k - 1$ vertices and every P_ℓ -layer in less than $\ell - 1$ vertices, then the length of the sequence is at most $k\ell - 2 \max\{k, \ell\}$. So we can consider the smallest m such that $D = (x_1, x_2, \dots, x_m)$ contains $k - 1$ vertices from a C_k -layer or $\ell - 1$ vertices from a C_ℓ -layer. The assumption $k, \ell \geq 3$ implies that if $\{x_1, x_2, \dots, x_m\}$ contains all but one vertex from both a C_k and a C_ℓ -layer, then $N(x_m) \subseteq \cup_{i=1}^{m-1} N(x_i)$ holds. Therefore, we may assume that $\{x_1, \dots, x_m\}$ contains all but one vertex from a C_k -layer, at most one C_ℓ -layer is empty and for the number n_j of vertices of D in the j th C_ℓ -layer we have $n_j \leq \ell - 2$. If every C_ℓ -layer is non-empty, then we can apply Proposition 4.9(a) to all pairs of the j th and $(j + 1)$ st C_ℓ -layers to obtain (writing d_j for the number of vertices totally dominated by D in the j th C_ℓ -layer) $2 \sum_{j=1}^k d_j \geq 4k + 2 \sum_{j=1}^k n_j = 2m + 4k$. If the i th C_ℓ -layer is empty, then we apply Proposition 4.9(b) to the pair of the $(i + 1)$ st, $(i + 2)$ nd and $(i - 2)$ nd, $(i - 1)$ st C_ℓ -layer, while we apply Proposition 4.9(a) to all pairs of j th and $(j + 1)$ st C_ℓ -layers for $j = i + 2, i + 3, \dots, i - 3$ to obtain

$$2 \sum_{j=1}^k d_j \geq 4k - 2 + 2 \sum_{j=1}^k n_j = 2m + 4k - 2.$$

In both cases we obtained that the total number d of vertices totally dominated by D is at least $2k - 1$ more than the size of D . As this difference cannot decrease throughout the open neighborhood sequence, its length is at most $k\ell - 2k + 1$. This proves (ii).

In the proof of (iii) one needs to consider the smallest integer m such that the open neighborhood sequence $D = (x_1, x_2, \dots, x_m)$ either contains a P_k -layer completely or all but one vertices of a C_ℓ -layer. Depending on these two possibilities, the counting argument applies Proposition 4.9 or Proposition 4.7. Details are left to the reader. \blacksquare

5. CARTESIAN PRODUCT

Theorem 5.1. *For any pair $k, \ell \geq 3$ of integers we have*

- (i) $\gamma_{gr}^t(C_k \square C_\ell) \leq k\ell - \min\{k, \ell\}$,
- (ii) $\gamma_{gr}^t(P_k \square P_\ell) \leq k\ell - \min\{\lfloor k/2 \rfloor, \lfloor \ell/2 \rfloor\}$,
- (iii) $\gamma_{gr}^t(P_k \square C_\ell) \leq k\ell - \min\{k, \lceil \ell/2 \rceil\}$.

Proof. (i) Consider the directed graph G obtained from $C_k \square C_\ell$ by replacing every edge with two opposite arcs. We interpret the arc from u to v in G as a certificate that u can be added to an open neighborhood sequence x_1, \dots, x_m as $v \notin \cup_{i=1}^m N(x_i)$. As the open neighborhood sequence grows, we remove those arcs of G for which the above statement is no longer valid (see Figure 1). The number of arcs in G is $4k\ell$ out of which $2k\ell$ are in C_k -layers and the same number in C_ℓ -layers. We claim that whenever we add a vertex to an open neighborhood sequence, we have to remove at least two arcs in C_k -layers and at least two arcs in C_ℓ -layers. Indeed, since u is added to the open neighborhood sequence because it dominates a neighbor v that was not dominated before, then the arcs from all four neighbors of v to v should be removed.

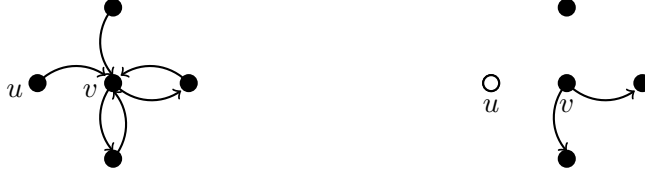


Figure 1. The left part of the figure presents the situation before u is added to the sequence, the right part presents the situation after u is added to the sequence.

Note that if an open neighborhood sequence intersects every C_k -layer in less than k vertices and every C_ℓ -layer in less than ℓ vertices, then the length of the sequence is at most $k\ell - \max\{k, \ell\}$. Moreover, if x_1, \dots, x_m would intersect one complete C_k -layer and one complete C_ℓ -layer with x_m being the intersection of these layers, then since $k, \ell \geq 3$ we would have $N(x_m) \subseteq \cup_{i=1}^{m-1} N(x_i)$, a contradiction.

Consider now the first moment m when x_1, \dots, x_m contains either all vertices from a C_k -layer or all vertices from a C_ℓ -layer. Assume without loss of generality that the former occurs and the sequence x_1, \dots, x_m contains all vertices of the j^{th} C_k -layer. Let $n_i, i \in [k]$, be the number of vertices from the i^{th} C_ℓ -layer that are in x_1, \dots, x_m . Note that all $2n_i$ arcs outgoing from these n_i vertices in the i^{th} C_ℓ -layer have been removed. Moreover, if $(v_i, u_s), \dots, (v_i, u_p)$ is a maximal sequence of consecutive vertices from the i^{th} C_ℓ -layer such that all of them are contained in $\{x_1, \dots, x_m\}$ and either $s \neq p$ or $s = j = p$ holds, then also the arcs

incoming to (v_i, u_s) and (v_i, u_p) have been deleted. As at least one vertex of this C_ℓ -layer is not in x_1, \dots, x_m , the number of arcs removed in the layer is at least $2n_i + 2$. Summing up we obtain that the number of arcs removed in C_ℓ -layers is at least $\sum_{i=1}^k (2n_i + 2) = 2(m + k)$. According to our observation above, in every later step at least two arcs of C_ℓ -layers will be removed, therefore the length of the complete open neighborhood sequence is at most

$$m + \frac{2k\ell - 2(m + k)}{2} = k\ell - k$$

and we are done with the proof of (i).

(ii) and (iii) We modify the proof above a little. We introduce the weighted directed graphs G' and G'' obtained from $P_k \square P_\ell$ and $P_k \square C_\ell$ respectively, by replacing every edge with two opposite arcs. All arcs have weight 1 except $((v_2, u_j), (v_1, u_j))$ and $((v_{k-1}, u_j), (v_k, u_j))$ for all $j \in [\ell]$ in both G' and G'' , and in G' also $((v_i, u_2), (v_i, u_1))$ and $((v_i, u_{\ell-1}), (v_i, u_\ell))$ for all $i \in [k]$. These arcs have weight 2. In this way, the total weight of the arcs in P_k -layers and P_ℓ -layers (respectively C_ℓ -layers) is $2k\ell$ both in G' and G'' . It remains also true that when adding a vertex to an open neighborhood sequence then both the arcs removed in P_k -layers and P_ℓ -layers (respectively C_ℓ -layers) have total weight at least 2.

The rest of the proof is very similar to that of part (i). Let m be the first moment when a complete P_k -layer or P_ℓ -layer (respectively C_ℓ -layer) of $P_k \square P_\ell$ or $P_k \square C_\ell$ belongs to the open neighborhood sequence x_1, \dots, x_m . Note that this time it is possible that the sequence x_1, \dots, x_m contains both a P_k -layer and a P_ℓ -layer (respectively C_ℓ -layer).

Case I. x_1, \dots, x_m contains only a P_k -layer or a P_ℓ -layer (respectively C_ℓ -layer). If in $P_k \square C_\ell$ the sequence x_1, \dots, x_m contains a P_k -layer, then with the same proof as in (i) we obtain $\gamma_{gr}^t(P_k \square C_\ell) \leq k\ell - k$. So we can assume that the sequence x_1, \dots, x_m contains a C_ℓ -layer and we will prove $\gamma_{gr}^t(P_k \square C_\ell) \leq k\ell - \lceil \ell/2 \rceil$. (The case of P_ℓ -layer for $P_k \square P_\ell$ is analogous.)

If the number of vertices in the j^{th} P_k -layer is m_j , then we have $1 \leq m_j < k$ for all $1 \leq j \leq \ell$. Therefore the total weight of arcs removed in P_ℓ -layers from G' or G'' is at least $2m_j + 1$ with equality if and only if the m_j vertices form a subpath of the path containing one of its endpoints. Summing up for all P_k -layers we obtain that the total weight of removed arcs in P_k -layers is at least $2m + \ell$. Therefore the length of the complete open neighborhood sequence is at most

$$m + \frac{2k\ell - (2m + \ell)}{2}.$$

Case II. x_1, \dots, x_m contains both a P_k -layer and a P_ℓ -layer (respectively C_ℓ -layer). We consider the open neighborhood sequence x_1, \dots, x_{m-1} . As x_1, \dots, x_m

contains both a P_k -layer and a P_ℓ -layer (respectively C_ℓ -layer), we know that every P_k -layer and P_ℓ -layer (respectively C_ℓ -layer) contains at least one vertex from x_1, \dots, x_{m-1} and one vertex not from x_1, \dots, x_{m-1} . Therefore the argument can be finished as in the previous case. ■

Theorem 5.2. *For any pair $k, \ell \geq 3$ we have*

1. $\gamma_{gr}^t(P_k \square P_\ell) \geq k\ell - \min\{k, \ell\}$,
2. $\gamma_{gr}^t(P_k \square C_\ell) \geq k\ell - \min\{2k, \ell\}$,
3. $\gamma_{gr}^t(C_k \square C_\ell) \geq k\ell - \min\{2k, 2\ell\}$.

Proof. Let the vertex set of $X \square Y$ be the following:

$$V(X \square Y) := \{(a, b) : 0 \leq a \leq k-1, 0 \leq b \leq \ell-1\},$$

and let the edge set be

$$\begin{aligned} E(X \square Y) := \{((a, b), (c, d)) \in V(X \square Y) \times V(X \square Y) : \\ (a = c \pm 1 \text{ (} a \equiv c \pm 1 \pmod{k} \text{ if } X = C_k \text{) and } b = d) \text{ or} \\ (a = c \text{ and } b = d \pm 1 \text{ (} b \equiv d \pm 1 \pmod{\ell} \text{ if } Y = C_\ell \text{)})\}. \end{aligned}$$

We also define the lexicographic and antilexicographic ordering on $V(X \square Y)$. For any $(a, b), (c, d) \in V(X \square Y)$ let

$$(a, b) \prec_{lex} (c, d) \text{ if and only if } a < c \text{ or } (a = c \text{ and } b < d),$$

$$(a, b) \prec_{alex} (c, d) \text{ if and only if } b < d \text{ or } (b = d \text{ and } a < c).$$

Let $v_1, \dots, v_{k\ell}$ be the lexicographic ordering of vertices of $X \square Y$, with $|V(X)| = k, |V(Y)| = \ell$. Furthermore, let $u_1, u_2, \dots, u_{k\ell}$ be the antilexicographic ordering of vertices of $X \square Y$. Then $(v_1, v_2, \dots, v_{(k-1)\ell})$ and $(u_1, u_2, \dots, u_{(\ell-1)k})$ are legal total dominating sequences of $P_k \square P_\ell$. Thus $\gamma_{gr}^t(P_k \square P_\ell) \geq k\ell - \min\{k, \ell\}$.

On the other hand $(v_1, \dots, v_{(k-1)\ell})$ and $(u_1, \dots, u_{(\ell-2)k})$ are legal total dominating sequences of $P_k \square C_\ell$. Thus $\gamma_{gr}^t(P_k \square C_\ell) \geq k\ell - \min\{2k, \ell\}$.

Finally, $(v_1, \dots, v_{(k-2)\ell})$ and $(u_1, \dots, u_{(\ell-2)k})$ are legal total dominating sequences of $C_k \square C_\ell$. Thus $\gamma_{gr}^t(C_k \square C_\ell) \geq k\ell - \min\{2k, 2\ell\}$. ■

We note that the following is known [12, 15] for the Grundy total domination number of the Cartesian product of two paths of the same length: for $k \geq 1$ we have

$$\gamma_{gr}^t(P_k \square P_k) = k^2 - k.$$

On the other hand, for an odd k it holds that $C_k \times C_k \cong C_k \square C_k$, hence by Theorem 2.12 it holds $\gamma_{gr}^t(C_k \times C_k) = k^2 - 2k + 1$.

6. CONCLUDING REMARKS

In [7] two additional versions of dominating sequences were introduced and studied. When the definition (1) is modified to read as

$$(3) \quad N(v_i) \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset,$$

we get the so-called Z -sequences, and when

$$(4) \quad N[v_i] \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset,$$

the so-called L -sequences are defined provided that vertices in such a sequence are distinct. The corresponding invariants obtained from the longest possible (Z - or L -) sequences are denoted by $\gamma_{gr}^Z(G)$ and $\gamma_{gr}^L(G)$, respectively. As it turns out, these two invariants have natural counterparts also in the zero-forcing and minimum-rank world, see [17].

With a little effort, by using almost the same proofs as in this paper, one can prove some of the results for the remaining two invariants.

- Version of Theorem 3.1 holds also for Z - and L -Grundy domination numbers of lexicographic products,

Theorem 6.1. *For any $i \in \{Z, L\}$ and graphs G and H with no isolated vertices,*

$$\gamma_{gr}^i(G \circ H) = \max \{a(D)(\gamma_{gr}^i(H) - 1) + |\widehat{D}| : D \text{ is a dominating sequence of } G\}.$$

Clearly, one can also derive exact values of these invariants in lexicographic products of paths and/or cycles.

- Lemma 2.1 (and its proof) holds also if Grundy total domination number is replaced with the Z -Grundy domination number, i.e.,

$$\gamma_{gr}^Z(G \times H) \geq \gamma_{gr}^Z(G)\gamma_{gr}^Z(H)$$

holds for any graphs G and H . However, for γ_{gr}^Z the left-hand side can be strictly greater, as demonstrated by $K_3 \times K_3$, where $\gamma_{gr}^Z(K_3) = 1$ while $\gamma_{gr}^Z(K_3 \times K_3) = 4$.

- Also, Lemma 2.3 can be proved in the setting of any of the four Grundy domination invariants.

Lemma 6.2. *Let E_1, \dots, E_k be the subsets of the edge set of a graph G . Let G_1, \dots, G_k be graphs on E_1, \dots, E_k , respectively. Then $\gamma_{gr}^x(G) \leq \gamma_{gr}^x(G_1) + \dots + \gamma_{gr}^x(G_k)$ for $x \in \{Z, t, L, \emptyset\}$.*

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