

ON THE n -PARTITE TOURNAMENTS WITH EXACTLY
 $n - m + 1$ CYCLES OF LENGTH m

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Abstract

Gutin and Rafiey [*Multipartite tournaments with small number of cycles*, Australas J. Combin. 34 (2006) 17–21] raised the following two problems: (1) Let $m \in \{3, 4, \dots, n\}$. Find a characterization of strong n -partite tournaments having exactly $n - m + 1$ cycles of length m ; (2) Let $3 \leq m \leq n$ and $n \geq 4$. Are there strong n -partite tournaments, which are not themselves tournaments, with exactly $n - m + 1$ cycles of length m for two values of m ? In this paper, we discuss the strong n -partite tournaments D containing exactly $n - m + 1$ cycles of length m for $4 \leq m \leq n - 1$. We describe the substructure of such D satisfying a given condition and we also show that, under this condition, the second problem has a negative answer.

Keywords: multipartite tournaments, tournaments, cycles.

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1. INTRODUCTION

An n -partite or *multipartite tournament* is an orientation of a complete n -partite graph. A *tournament* is an n -partite tournament with exactly n vertices. A digraph D is *transitive* if, for every pair of arcs xy and yz in D such that $x \neq z$, the arc xz is also in D . It is easy to show that a tournament is transitive if and only if it is acyclic.

A digraph D is said to be *strong*, if for every pair of vertices x and y , D contains a path from x to y and a path from y to x . A directed path from x to y in D is denoted by an (x, y) -path. An l -cycle is a cycle of length l . A cycle or path in a digraph D is Hamiltonian if it includes all the vertices of D .

In 1966, Moon discussed the number of m -cycle in a strong tournament.

Theorem 1 (Moon [9]). *Let T be a strong tournament of order n . Then T contains at least $n - m + 1$ cycles of length m for $3 \leq m \leq n$.*

The tournaments which gain the lower bound in Theorem 1 were characterized by Burzio and Demaria [2] for $m = 3$, Douglas [3] for $m = n$ and Las Vergnas [8] for $4 \leq m \leq n - 1$. We list the result of Las Vergnas especially because we will use it to prove our main results.

Theorem 2 (Las Vergnas [8]). *Every strong tournament of order n having exactly $n - m + 1$ cycles of given length m with $4 \leq m \leq n - 1$ is isomorphic to Q_n , where Q_n is a tournament of order $n \geq 3$ obtained by reversing the arcs in the unique Hamiltonian path of a transitive tournament.*

In 2002, Volkmann extended Theorem 1 from tournaments to multipartite tournaments.

Theorem 3 (Volkmann [10]). *Let D be a strong n -partite tournament. Then D contains at least $n - m + 1$ cycles of length m for $3 \leq m \leq n$.*

It is notable that the bound in Theorem 3 is sharp which can be seen in [4] for $n = 3$ and in [6] for $4 \leq m \leq n$. In addition, Gutin and Rafiey [6] raised the following two interesting and natural problems.

Problem 4 (Gutin and Rafiey [6]). *Given $m \in \{3, 4, \dots, n\}$, find a characterization of strong n -partite tournaments having exactly $n - m + 1$ cycles of length m .*

Problem 5 (Gutin and Rafiey [6]). *Let $3 \leq m \leq n$ and $n \geq 4$. Are there strong n -partite tournaments, which are not themselves tournaments, with exactly $n - m + 1$ cycles of length m for two values of m ?*

Problem 4 seems to be especially interesting for the case $m = n$ which was already solved by Gutin *et al.* in [7]. In this paper, we investigate strong n -partite tournaments D , which are not themselves tournaments and contain exactly $n - m + 1$ cycles of length m for any given $4 \leq m \leq n - 1$. We prove that if D has an $(n - 1)$ -cycle with no pair of vertices from the same partite set, then D must contain some given multipartite tournament as its subdigraph.

As far as Problem 5, Gutin and Rafiey [6] gave a negative answer for two values $n - 1$ and n of m . This also implies that Problem 5 has a negative answer for $n = 4$. In this paper, we give a necessary condition to Problem 5 and show that if a strong n -partite tournament D , which is not itself a tournament, contains exactly $n - m + 1$ cycles of length m for two values of $m \in \{4, 5, \dots, n - 1\}$, then there is no an $(n - 1)$ -cycle with no pair of vertices from the same partite set in D .

2. TERMINOLOGY AND PRELIMINARIES

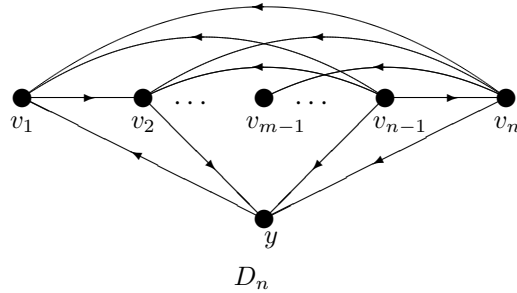
We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1].

Let D be a digraph with the vertex set $V(D)$ and the arc set $A(D)$. We call the number of vertices of D the *order* of D . A subdigraph induced by a subset $A \subseteq V(D)$ is denoted by $D\langle A \rangle$. We use $V(D) \setminus V(A)$ to stand for the set of vertices which are in $V(D)$ but not in $V(A)$.

If xy is an arc in D , then we say that x *dominates* y and write $x \rightarrow y$. For two disjoint subsets X and Y of $V(D)$, if every vertex of X dominates every vertex of Y , we say X *dominates* Y and write $X \rightarrow Y$. Furthermore, $X \Rightarrow Y$ denotes the property that there is no arc from Y to X .

The *out-neighborhood* $N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N^-(x)$ of a vertex x is the set of vertices dominating x . The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are the *outdegree* and *indegree* of x , respectively. The *global irregularity* of D is defined as $i_g(D) = \max\{\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} : x, y \in V(D)\}$. We denote by D^{-1} the inverse digraph of D .

In order to present our main results, we define a class of n -partite tournaments D_n of order $n + 1$ as described in the following figure, where $3 \leq m \leq n$, $\{v_2, \dots, v_{m-2}, v_m, \dots, v_n\} \rightarrow y \rightarrow v_1$, y and v_{m-1} belong to the same partite set and $v_i \rightarrow v_j$ for all $1 < j + 1 < i \leq n$.



The following two theorems on cycles in strong n -partite tournaments are very useful to prove our main results.

Theorem 6 (Guo and Volkmann [5]). *Every partite set of a strong n -partite tournament, $n \geq 3$, contains a vertex which lies on an m -cycle for each $m \in \{3, 4, \dots, n\}$.*

Theorem 7 (Gutin and Rafiey [6]). *Let D be a strong n -partite tournament containing exactly $n - m + 1$ cycles of length m for some $m \in \{3, 4, \dots, n\}$. Then every m -cycle of D has no pair of vertices from the same partite set.*

3. MAIN RESULTS

Before presenting the main results, we first prove the following lemma.

Lemma 8. *Let D be a strong n -partite tournament, $n \geq 5$, containing exactly $n - m + 1$ cycles of length m for some $3 \leq m \leq n - 1$. If D has an $(n - 1)$ -cycle C with no pair of vertices from the same partite set, then the following statements hold.*

- (a) *There are no two vertices u, w in $V(D) \setminus V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$.*
- (b) *There exists a vertex $v \notin V(C)$ such that $D \langle V(C) \cup \{v\} \rangle$ is strong.*

Proof. Let V_1, V_2, \dots, V_n be the partite sets of D . Suppose, without loss of generality, that $C = v_1 v_2 \cdots v_{n-1} v_1$ with $v_i \in V_i, i = 1, 2, \dots, n - 1$. By Theorem 1, $D \langle V(C) \rangle$ contains at least $(n - 1) - m + 1 = n - m$ cycles C_1, C_2, \dots, C_{n-m} of length m .

(a) Suppose to the contrary that there exist two vertices $u, w \in V(D) \setminus V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$. Obviously, $u \notin V_n$ or $w \notin V_n$. Assume, without loss of generality, that $w \in V_j$ for some $1 \leq j \leq n - 1$. Since $n \geq 5$, there exist at least two vertices v_k and v_t , such that u, w, v_k and v_t are in different partite sets. If $m = 3$, then $u w v_k u$ and $u w v_t u$ are two m -cycles different from C_1, C_2, \dots, C_{n-m} . This contradicts the fact that D contains exactly $n - m + 1$ cycles of length m . If $m \geq 4$, then $u w v_{j-1} v_j \cdots v_{j+m-4} u$ (if $u \notin V_{j+m-4}$) or $u w v_{j+1} v_{j+2} \cdots v_{j+m-2} u$ (if $u \in V_{j+m-4}$) is an m -cycle with $w, v_j \in V_j$ or $u, v_{j+m-4} \in V_{j+m-4}$ (where all indices are modulo $n - 1$). This is impossible by Theorem 7.

(b) Assume that there is no vertex $v \in V(D) \setminus V(C)$ such that $D \langle V(C) \cup \{v\} \rangle$ is strong. Let $S = \{x \in V(D) \setminus V(C) : C \Rightarrow x\}$ and $T = \{z \in V(D) \setminus V(C) : z \Rightarrow C\}$. Since D is strong, we have that S and T are non-empty and there are vertices $u \in S$ and $w \in T$ such that $u \rightarrow w$. Thus, we have $C \Rightarrow u \rightarrow w \Rightarrow C$, which contradicts (a). \blacksquare

Theorem 9. *Let D be a strong n -partite tournament which is not itself a tournament and contains exactly $n - m + 1$ cycles of length m for some $4 \leq m \leq n - 1$. If D has an $(n - 1)$ -cycle C with no pair of vertices from the same partite set, then D contains some D_i or D_i^{-1} as its subdigraph for $i \in \{n - 1, n\}$, where D_i is defined in Section 2.*

Proof. Let V_1, V_2, \dots, V_n be the partite sets of D and let $C = v_1 v_2 \cdots v_{n-1} v_1$, $v_i \in V_i, i = 1, 2, \dots, n - 1$. By Theorem 1, $D \langle V(C) \rangle$ contains at least $n - m$ cycles C_1, C_2, \dots, C_{n-m} of length m . By Theorem 6, there exists a vertex in V_n , say x , which lies on an m -cycle C_{n-m+1} different from C_1, C_2, \dots, C_{n-m} . We consider the following two cases.

Case 1. $D \langle V(C) \cup \{x\} \rangle$ is not strong. Since D contains exactly $n - m + 1$ cycles of length m , we have that $D \langle V(C) \rangle$ contains exactly $n - m$ cycles of length

m . By Theorem 2, $D\langle V(C) \rangle$ is isomorphic to Q_{n-1} . So we may assume that $v_i \rightarrow v_j$ for all $1 < j + 1 < i \leq n - 1$. Since $D\langle V(C) \cup \{x\} \rangle$ is not strong, we have that $C \rightarrow x$ or $x \rightarrow C$.

First we consider the case $C \rightarrow x$. Let $S = \{u \in V(D) \setminus (V(C) \cup \{x\}) : D\langle V(C) \cup \{u\} \rangle \text{ is strong} \}$. By Lemma 8(b), S is not empty. Since D is strong, there is a path from x to S . Let $P = x_1 x_2 \cdots x_t$ ($x_1 = x$) be such a path and assume that the P is of minimum length. That is, $x_t \in S$ and $D\langle V(C) \cup \{x_i\} \rangle$ is not strong for each $i \in \{1, 2, \dots, t - 1\}$. Since $C \rightarrow x_1$ and $x_1 \rightarrow x_2$, we have $x_2 \notin V(C)$. If $t > 2$, then by Lemma 8(a), we have $C \Rightarrow x_2$. Successively, we can get that $x_i \notin V(C)$ and $C \Rightarrow x_i$ for all $i \in \{2, 3, \dots, t - 1\}$ when $t > 2$.

If there exist two vertices v_i, v_j on C such that $x_t \rightarrow \{v_i, v_j\}$, then, when $t > m - 1$, we have that $x_t v_i x_{t-(m-2)} x_{t-(m-3)} \cdots x_t$ (if $x_{t-(m-2)} \notin V_i$) or $x_t v_j x_{t-(m-2)} x_{t-(m-3)} \cdots x_t$ (if $x_{t-(m-2)} \in V_i$) is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction; when $t = m - 1$, it is clear that $x_t v_i x_1 \cdots x_t$ and $x_t v_j x_1 \cdots x_t$ are two m -cycles different from C_1, C_2, \dots, C_{n-m} , a contradiction; when $t \leq m - 2$, it is easy to see that $x_t v_i v_{i+1} \cdots v_{i+(m-1-t)} x_1 \cdots x_t$ and $x_t v_j v_{j+1} \cdots v_{j+(m-1-t)} x_1 \cdots x_t$ are two m -cycles different from C_1, C_2, \dots, C_{n-m} , a contradiction.

Therefore, x_t has only one out-neighbor on C . We will show that $x_t \rightarrow v_1$. In fact, if $x_t \rightarrow v_i$ and $i \geq 2$, then we have that $x_t v_i \cdots v_{i+m-2} x_t$ (when $i + m - 2 \leq n - 1$ and $x_t \notin V_{i+m-2}$) or $x_t v_i \cdots v_{i+m-3} v_{i-1} x_t$ (when $i + m - 2 \leq n - 1$ and $x_t \in V_{i+m-2}$) or $x_t v_i \cdots v_{n-1} v_1 \cdots v_{m-n+i-1} x_t$ (when $i + m - 2 \geq n$ and $x_t \notin V_{m-n+i-1}$) or $x_t v_i \cdots v_{n-1} v_2 \cdots v_{m-n+i} x_t$ (when $i + m - 2 \geq n$ and $x_t \in V_{m-n+i-1}$) is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction. So we have $\{v_2, v_3, \dots, v_{n-1}\} \Rightarrow x_t$. Furthermore, if $v_{m-1} \rightarrow x_t$, then $x_t v_1 v_2 \cdots v_{m-1} x_t$ is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction. So $x_t \in V_{m-1}$. Let $x_t = y$. Then D contains D_{n-1} as its subdigraph.

For the case $x \rightarrow C$, by considering the inverse of D , it is easy to see that D contains D_{n-1}^{-1} as its subdigraph.

Case 2. $D\langle V(C) \cup \{x\} \rangle$ is strong. In this case, $D\langle V(C) \cup \{x\} \rangle$ is a strong tournament of order n . By Theorem 1, $D\langle V(C) \cup \{x\} \rangle$ contains at least $n - m + 1$ cycles of length m . Note that D contains exactly $n - m + 1$ cycles of length m . We have that $D\langle V(C) \cup \{x\} \rangle$ contains exactly $n - m + 1$ cycles of length m . By Theorem 2, $D\langle V(C) \cup \{x\} \rangle$ is isomorphic to Q_n . So we may assume that $C' = v_1 v_2 \cdots v_n v_1$ is an n -cycle of $D\langle V(C) \cup \{x\} \rangle$ satisfying $v_i \in V_i$ and $v_i \rightarrow v_j$ for all $1 < j + 1 < i \leq n$. Obviously, $C_1 = v_1 v_2 \cdots v_m v_1, C_2 = v_2 v_3 \cdots v_{m+1} v_2, \dots, C_{n-m+1} = v_{n-m+1} v_{n-m+2} \cdots v_n v_{n-m+1}$ are $n - m + 1$ cycles of length m of D .

Claim 10. *There exists a vertex $y \in V(D) \setminus V(C')$ such that $D\langle V(C') \cup \{y\} \rangle$ is strong.*

Proof. Assume that there is no vertex $y \in V(D) \setminus V(C')$ such that $D\langle V(C') \cup \{y\} \rangle$ is strong. Let $S = \{x \in V(D) \setminus V(C') : C' \Rightarrow x\}$ and $T = \{z \in V(D) \setminus V(C') : z \Rightarrow V(C')\}$. Since D is strong, we have that S and T are non-empty and there are vertices $u \in S$ and $w \in T$ such that $u \rightarrow w$. Suppose that $u \in V_i$ and $w \in V_j$ for $1 \leq i \neq j \leq n$. Then $u w v_{j+1} v_{j+2} \cdots v_{j+m-2} u$ (if $i \neq j + m - 2$) or $u w v_{j+2} v_{j+3} \cdots v_{j+m-1} u$ (if $i = j + m - 2$) is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$. Note that D contains exactly $n - m + 1$ cycles of length m . This is a contradiction. \square

By Claim 10, there are two vertices v_a, v_b ($1 \leq a, b \leq n$), such that $v_a \rightarrow y \rightarrow v_b$. Assume that v_k is the first vertex from v_1 to v_n dominating y .

Claim 11. $v_i \Rightarrow y$ for all $k \leq i \leq n$.

Proof. Otherwise, there exists some index t such that either $v_t \rightarrow y \rightarrow v_{t+1}$ ($k \leq t \leq n - 1$) or $y, v_{t+1} \in V_{t+1}$ but $v_t \rightarrow y \rightarrow v_{t+2}$ ($k \leq t \leq n - 2$). We still assume that t is such a minimum index.

If $t \leq n - m + 1$, then either $v_t y v_{t+1} \cdots v_{t+m-2} v_t$ or $v_t y v_{t+2} \cdots v_{t+m-1} v_t$ is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction.

If $n - m + 2 \leq t \leq n - 2$, then either $v_t y v_{t+1} \cdots v_n v_{n-m+2} \cdots v_t$ or $v_t y v_{t+2} \cdots v_n v_{n-m+1} \cdots v_t$ is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction.

If $t = n - 1$, then $y \rightarrow v_n$ and $v_{n-1} y v_n v_{n-m+2} \cdots v_{n-1}$ is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction. \square

Claim 12. $y \rightarrow v_1$.

Proof. If $y \in V_1$, then $y \rightarrow v_2$ (otherwise, $k = 2$ and $\{v_2, v_3, \dots, v_n\} \rightarrow y$ by Claim 11, which contradicts the assumption that $D\langle V(C') \cup \{y\} \rangle$ is strong). By Claim 11, we have $v_n \rightarrow y$. Therefore, $D\langle v_2, \dots, v_n, y \rangle$ is a strong tournament. Then y is in an m -cycle of $D\langle v_2, \dots, v_n, y \rangle$, which is different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction.

Therefore, $y \notin V_1$. If $v_1 \rightarrow y$, then $\{v_2, v_3, \dots, v_n\} \Rightarrow y$ by Claim 11, which contradicts the assumption that $D\langle V(C') \cup \{y\} \rangle$ is strong. So we have $y \rightarrow v_1$. \square

By Claim 12, we have that $2 \leq k \leq n$ and $y \Rightarrow v_{m-1}$. Otherwise, $y v_1 v_2 \cdots v_{m-1} y$ is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction.

If $k = 2$, then by $m \geq 4$ and Claim 11, we have $y \in V_{m-1}$, and hence, $\{v_2, v_3, \dots, v_n\} \Rightarrow y \rightarrow v_1$. Now, D contains D_n as its subdigraph.

If $2 < k < m - 1$, then $y \in V_{m-1}$, $y \rightarrow v_2$ and $v_m \rightarrow y$. Thus, $y v_2 \cdots v_m y$ is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction.

If $m - 1 \leq k \leq n - 1$, then $1 \leq k - m + 2 \leq k - 2$ and $y \Rightarrow v_{k-m+2}$. Now $v_k y v_{k-m+2} \cdots v_k$ (if $y \rightarrow v_{k-m+2}$) or $v_{k+1} y v_{k-m+3} \cdots v_{k+1}$ (if $y \in V_{k-m+2}$) is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction.

If $k = n$, then $v_n \rightarrow y \Rightarrow \{v_1, v_2 \cdots v_{n-1}\}$ by the choice of k . It is easy to see that $y \in V_{n-m+2}$, as otherwise $yv_{n-m+2} \cdots v_n y$ is an m -cycle different from $C_1, C_2, \dots, C_{n-m+1}$, a contradiction. Now D contains D_n^{-1} as its subdigraph. ■

Theorem 13. *Let D be a strong n -partite tournament, $n \geq 5$, which is not itself a tournament. If D contains an $(n - 1)$ -cycle with no pair of vertices from the same partite set, then D does not contain exactly $n - m + 1$ cycles of length m for two values of $m \in \{4, 5, \dots, n - 1\}$.*

Proof. Let m and m_1 be two distinct values from the set $\{4, 5, \dots, n - 1\}$ and assume that D has exactly $n - m + 1$ cycles of length m . Let V_1, V_2, \dots, V_n be the partite sets of D . By Theorem 9, D contains some D_i or D_i^{-1} as its subdigraph for $i \in \{n - 1, n\}$.

If D contains D_{n-1} (D_{n-1}^{-1}) as its subdigraph, then let $C = v_1 v_2 \cdots v_{n-1} v_1$ be an $(n - 1)$ -cycle of D_{n-1} (D_{n-1}^{-1}) with $v_i \in V_i$ ($i = 1, 2, \dots, n - 1$), $v_i \rightarrow v_j$ for all $1 < j + 1 < i \leq n - 1$, $y \in V_{m-1}$, $\{v_2, v_3, \dots, v_{n-1}\} \Rightarrow y \rightarrow v_1$ ($y \in V_{n-m+1}$ and $v_{n-1} \rightarrow y \Rightarrow \{v_1, v_2, \dots, v_{n-2}\}$). By Theorem 1, $D\langle V(C) \rangle$ contains at least $(n - 1) - m_1 + 1 = n - m_1$ cycles of length m_1 . Note that $yv_1 v_2 \cdots v_{m_1-1} y$ ($v_{n-1} y v_{n-(m_1-1)} v_{n-(m_1-2)} \cdots v_{n-1}$) is another m_1 -cycle of D_{n-1} (D_{n-1}^{-1}). In addition, there exists a vertex in V_n , say x , which is in an m_1 -cycle of D different from the above m_1 -cycles. Thus, D contains at least $n - m_1 + 2$ cycles of length m_1 .

If D contains D_n (D_n^{-1}) as its subdigraph, then let $C = v_1 v_2 \cdots v_n v_1$ be an n -cycle of D_n (D_n^{-1}) with $v_i \in V_i$ ($i = 1, 2, \dots, n$), $v_i \rightarrow v_j$ for all $1 < j + 1 < i \leq n$, $y \in V_{m-1}$, $\{v_2, v_3, \dots, v_n\} \Rightarrow y \rightarrow v_1$ ($y \in V_{n-m+2}$, $v_n \rightarrow y \Rightarrow \{v_1, v_2, \dots, v_{n-1}\}$). By Theorem 1, $D\langle V(C) \rangle$ contains at least $n - m_1 + 1$ cycles of length m_1 . It is easy to see that $yv_1 v_2 \cdots v_{m_1-1} y$ ($v_n y v_{n-(m_1-2)} v_{n-(m_1-3)} \cdots v_n$) is another m_1 -cycle of D_n (D_n^{-1}). Then D contains at least $n - m_1 + 2$ cycles of length m_1 . The theorem is complete. ■

In 2004, Winzen [11] showed that an n -partite tournament D with $n \geq 4$ and $i_g(D) \leq 2$ contains a strong subtournament of order p for every $p \in \{3, 4, \dots, n - 1\}$. So D contains an $(n - 1)$ -cycle with no pair of vertices from the same partite set, which yields the following result.

Corollary 14. *If D is a strong n -partite tournament with $n \geq 5$ and $i_g(D) \leq 2$, which is not itself a tournament, then D does not contain exactly $n - m + 1$ cycles of length m for two values of $m \in \{4, 5, \dots, n - 1\}$.*

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