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ON THE STAR CHROMATIC INDEX OF GENERALIZED PETERSEN GRAPHS

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Abstract

The star k-edge-coloring of graph G is a proper edge coloring using k colors such that no path or cycle of length four is bichromatic. The minimum number k for which G admits a star k-edge-coloring is called the star chromatic index of G, denoted by $\chi'_s(G)$. Let $\mathrm{GCD}(n,k)$ be the greatest common divisor of n and k. In this paper, we give a necessary and sufficient condition of $\chi'_s(P(n,k)) = 4$ for a generalized Petersen graph $P(n,k)$ and show that "almost all" generalized Petersen graphs have a star 5-edge-colorings. Furthermore, for any two integers k and n $(\geq 2k+1)$ such that $\mathrm{GCD}(n,k) \geq 3$, $P(n,k)$ has a star 5-edge-coloring, with the exception of the case that $\mathrm{GCD}(n,k) = 3$, $k \neq \mathrm{GCD}(n,k)$ and $\frac{n}{3} \equiv 1 \pmod 3$.

Keywords: star edge-coloring, star chromatic index, generalized Petersen graph.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected; for the terminologies and notations not defined here, we follow [3]. For any graph G, we denote by $V(G)$ and $E(G)$ the *vertex set* and the *edge set* of G, respectively. For any vertex v in G, a vertex $u \in V(G)$ is said to be a neighbor of v if $uv \in E(G)$. We use $N_G(v)$ to denote the set of neighbors of v. For positive integers n and k, let $\mathrm{GCD}(n,k)$ be the greatest common divisor of n and k.

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A *star k-edge-coloring* of a graph G is a proper edge-coloring using k colors such that at least three distinct colors are assigned to the edges of every path and cycle of length four. The minimum number k for which G admits a star k-edge-coloring is called the *star chromatic index* of G and is denoted by $\chi'_s(G)$.

The star edge-coloring was motivated by the vertex version [1, 4, 5, 7], which was first studied by Liu and Deng [8], who gave an upper bound on the star chromatic index of graph with maximum degree at least 7. Dvořák *et al.* [6] provided some upper and lower bounds for complete graphs. They also considered cubic graphs and showed that the star chromatic index of such graphs lies between 4 and 7.

Since there exist many cubic graphs with a star chromatic index equal to 6, e.g., $K_{3,3}$ or the Heawood graph, and no example of a subcubic graph with star chromatic index 7 is known, Dvořák *et al.* proposed the following conjecture.

Conjecture 1.1 [6]. *If G is a subcubic graph, then $\chi'_s(G) \leq 6$.*

Recently, Bezegová *et al.* [2] established tight upper bounds for trees and subcubic outerplanar graphs; they derived upper bounds for outerplanar graphs. In this paper, we obtain a necessary and sufficient condition of $\chi'_s(P(n,k)) = 4$, and present a construction of a star 5-edge-colorings of $P(n,k)$ for "almost all" values of n and k. Furthermore, we find that the generalized Petersen graph $P(n,k)$ with $n = 3$, $k = 1$ is the only graph with a star chromatic index of 6 among the investigated graphs. Based on these results, we conjecture that $P(3,1)$ is the unique generalized Petersen graph that admits no star 5-edge-coloring.

2. A NECESSARY AND SUFFICIENT CONDITION OF $\chi'_s(P(n,k)) = 4$

Let n and k be positive integers, $n \geq 2k+1$ and $n \geq 3$. The *generalized Petersen graph* $P(n,k)$, which was introduced in [9], is a cubic graph with $2n$ vertices, denoted by $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$, and all edges are of the form u_iu_{i+1}, u_iv_i, v_iv_{i+k} for $0 \leq i \leq n-1$. In the absence of a special claim, all subscripts of vertices of $P(n,k)$ are taken modulo n in the following.

Lemma 1 [6]. *If G is a simple cubic graph, then $\chi'_s(G) = 4$ if and only if G covers the graph of the 3-cube Q_3 (as shown in Figure 1), where a graph H is said to be covered by G if there is a locally bijective graph homomorphism from G to H.*

Theorem 2. *$\chi'_s(P(n,k)) = 4$ if and only if n is a multiple of 4 and k is an odd number.*

Proof. Consider an arbitrary generalized Petersen graph $P(n,k)$ with $n \equiv 0 \pmod 4$ and $k \equiv 1 \pmod 2$. We then prove that $P(n,k)$ covers Q_3. Define a

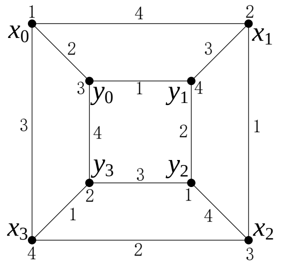

Figure 1. Cube Q_3 with a star 4-edge-coloring.

surjection $\phi : V(P(n,k)) \to V(Q_3)$ as follows: let $\phi(u_i) = x_{i \pmod 4}$ and $\phi(v_i) = y_{i \pmod 4}$, $i = 0, 1, \ldots, n-1$.

To show that ϕ is a covering map, we need to prove that for each $w \in V(P(n,k))$, the three neighbors of w in $P(n,k)$ map by ϕ to the three neighbors of $\phi(w)$ in Q_3. First, for each u_i, its three neighbors in $P(n,k)$ are u_{i+1}, u_{i-1}, v_i. By the structure of Q_3, the three neighbors of $\phi(u_i)$ $\big(= x_{i \pmod 4}\big)$ in Q_3 are $x_{i+1 \pmod 4}, x_{i-1 \pmod 4}$ and $y_{i \pmod 4}$. Therefore, $N_{Q_3}(\phi(u_i)) = \{\phi(u_{i+1}), \phi(u_{i-1}), \phi(v_i)\}$. Now, we consider a vertex v_i in $P(n,k)$. The three neighbors of v_i in $P(n,k)$ are u_i, v_{i+k}, v_{i-k}, and the three neighbors of $\phi(v_i)$ $\big(= y_{i \pmod 4}\big)$ in Q_3 are $x_{i \pmod 4}, y_{i+1 \pmod 4}, y_{i-1 \pmod 4}$. Observe that k is an odd number, which implies that $i+k \pmod 4 \neq i-k \pmod 4$, and $i+k \pmod 2 = i-k \pmod 2 \neq i \pmod 2$. Therefore, $\{\phi(v_{i+k}), \phi(v_{i-k})\} = \{y_{i+1 \pmod 4}, y_{i-1 \pmod 4}\}$, that is, $N_{Q_3}(\phi(v_i)) = \{\phi(u_i), \phi(v_{i+k}), \phi(v_{i-k})\}$. Hence, $P(n,k)$ covers Q_3, and $\chi'_s(P(n,k)) = 4$ by Lemma 1.

For the inverse implication, suppose that $P(n,k)$ has a star 4-edge-coloring f. For any vertex $w \in V(P(n,k))$, define a (vertex) 4-coloring f' of $P(n,k)$ by letting $f'(w)$ be the unique color that is missing on edges incident with w under f. Then, the three neighbors of any vertex are assigned to different colors under f'. Otherwise, assume that there exist some vertex w and its two neighbors w_1, w_2 in $P(n,k)$ satisfying $f'(w) = c_1, f'(w_1) = f'(w_2) = c_2$, $f(ww_1) = c_3$ and $f(ww_2) = c_4$, where $\{c_1, c_2, c_3, c_4\} = \{1, 2, 3, 4\}$. Then color c_4 appears on an edge incident with w_1, and c_3 appears on an edge incident with w_2. This creates a bichromatic path or cycle of length 4. Thus, if $f'(w) = c_1$, the incident edges and adjacent vertices of w are c_2, c_3, c_4 under f and f', respectively. There are exactly two possibilities as follows: either the edges incident with w colored c_2, c_3, c_4 lead to corresponding vertices ($w's$ neighbors) colored c_3, c_4, c_2, respectively, or to corresponding vertices colored c_4, c_2, c_3. These two possibilities are called the *local color pattern* at w. Then, f and f' induce a covering map Φ: $V(P(n,k)) \to V(Q_3)$ such that for each $w \in V(P(n,k))$, $f'(w) = f'(\Phi(w))$ (we use f' also for the vertex coloring of Q_3 shown in Figure 1), and w and $\Phi(w)$ have the same local color pattern.

Let X_i and Y_i denote the set of vertices of $P(n,k)$ that are mapped to x_i and y_i, respectively, under Φ, $i = 0,1,2,3$. Thus, under f' vertices in X_0 and Y_2 are colored with 1, in X_1 and Y_3 vertices are colored with 2, in X_2 and Y_0 vertices are colored with 3, and in X_3 and Y_1 vertices are colored with 4.

Claim. $|X_i| = |Y_j| = \frac{n}{4}$ *for* $i, j \in \{0,1,2,3\}$.

Proof. Observe that by the definition of Φ, for every vertex $w \in X_0$, there is exactly one neighbor of w that belongs to Y_0; for every vertex $w' \in Y_0$, there is exactly one neighbor of w' that belongs to X_0. This implies that there is a bijection between X_0 and Y_0. Therefore, $|X_0| = |Y_0|$. Analogously, we have $|X_0| = |X_1| = |X_3|$, $|X_1| = |X_2| = |Y_1|$, $|X_2| = |X_3| = |Y_2|$, and $|X_3| = |Y_3|$. Therefore, $|X_i| = |Y_j|$, and $|V(P(n,k))| = 2n = 8|X_i|$, which indicates that $n = 4|X_i|$, and the claim holds. □

Clearly, n is a multiple of 4 by the above claim. In what follows, we show k is an odd number.

From the definition of covering projections, we see that every cycle of length ℓ in $P(n,k)$ is mapped to a cycle of length ℓ' in Q_3 such that $\ell = m\ell'$ for some nonnegative integer m. Therefore, the cycle $C = u_0u_1 \cdots u_{n-1}u_0$ is mapped to a cycle C' of length 4 or 8. Note that Q_3 is a bipartite graph that does not contain any cycle with odd number of vertices. In addition, if C' is a 6-cycle, then with a similar analysis as below, the subgraph of Q_3 induced by vertices corresponding to $v_0, v_1, \ldots, v_{n-1}$ consists of two paths with length 1 and a contraction.

If C' is a cycle of length 4, without loss of generality, it is assumed that $C' = x_0x_1y_1y_0x_0$, and then any 4 consecutive vertices on C are mapped to x_0, x_1, y_1, y_0 in one order of (x_0, x_1, y_1, y_0), (x_1, y_1, y_0, x_0), (y_1, y_0, x_0, x_1) or (y_0, x_0, x_1, y_1). In this way, we can assume the following without the loss of generality

$$\Phi(u_i) = \begin{cases} x_0, i \equiv 0 \pmod 4, \\ x_1, i \equiv 1 \pmod 4, \\ y_1, i \equiv 2 \pmod 4, \\ y_0, i \equiv 3 \pmod 4. \end{cases}$$

Then,

$$\Phi(v_i) = \begin{cases} x_3, i \equiv 0 \pmod 4, \\ x_2, i \equiv 1 \pmod 4, \\ y_2, i \equiv 2 \pmod 4, \\ y_3, i \equiv 3 \pmod 4, \end{cases}$$

$x_3y_2 \notin E(Q_3)$ and $x_2y_3 \notin E(Q_3)$, so the vertex mapped to x_3 (or x_2) is not adjacent to the vertex mapped to y_2 or x_3 (or y_3 or x_2) in $P(n,k)$. Therefore, k is an odd number in this case.

If C' is a cycle of length 8, then n is a multiple of 8, and C' is a Hamilton cycle such as $C' = x_0x_1x_2x_3y_3y_2y_1y_0x_0$. Clearly, any 8 consecutive vertices on C are mapped to $x_0, x_1, x_2, x_3, y_3, y_2, y_1, y_0$, preserving the adjacent relation in C'. Without loss of generality, we assume

$$\Phi(u_i) = \begin{cases} x_0, i \equiv 0 \pmod 8, \\ x_1, i \equiv 1 \pmod 8, \\ x_2, i \equiv 2 \pmod 8, \\ x_3, i \equiv 3 \pmod 8, \\ y_3, i \equiv 4 \pmod 8, \\ y_2, i \equiv 5 \pmod 8, \\ y_1, i \equiv 6 \pmod 8, \\ y_0, i \equiv 7 \pmod 8. \end{cases}$$

Then, it follows that

$$\Phi(v_i) = \begin{cases} x_3, i \equiv 0 \pmod 8, \\ y_1, i \equiv 1 \pmod 8, \\ y_2, i \equiv 2 \pmod 8, \\ x_0, i \equiv 3 \pmod 8, \\ y_0, i \equiv 4 \pmod 8, \\ x_2, i \equiv 5 \pmod 8, \\ x_1, i \equiv 6 \pmod 8, \\ y_3, i \equiv 7 \pmod 8. \end{cases}$$

Since in Q_3, x_3 is not adjacent to y_2, y_0, x_1 or x_3 itself, it follows that the vertex mapped to x_3 is not adjacent to the vertex mapped to y_2, y_0, x_1 or x_3, in $P(n, k)$. Therefore, k is an odd number, which completes the proof. ∎

3. Construction of Star 5-Edge-Colorings for $P(n, k)$

A *list* L of a graph G is a mapping from a finite set of colors (positive integers) to each vertex of G. For any $V' \subseteq V(G)$, $L(V')$ denotes the set of colors that are assigned to the vertices of V', i.e., $L(V') = \{L(v)|v \in V'\}$. A proper edge-coloring f of G is called an *irlist-edge-coloring* if $f(e) \notin L(u) \cup L(v)$ for any edge $e(= uv) \in E(G)$. An edge-coloring of G is *strong* if any two edges within distance two apart receive different colors.

Let $C = v_1v_2 \dots v_mv_1$ be a cycle of length m, $m \geq 3$. We call C a *listed-cycle* if C has a list L and refer to the colors in $L(V(C))$ as *listed-colors* of C. In particular, if there are exactly two consecutive vertices v_i, v_{i+1} satisfying $L(v_i)$ (respectively, $L(v_{i+1})$) $\neq L(v_j)$ and $L(v_j)$=$L(v_{j'})$ for all $j, j' \in \{1, 2, \dots, m\} \setminus \{i, i+1\}$, then we say C is *quaint* and v_i and v_{i+1} are *the quaint vertices of* C, where $v_{m+1} = v_m$.

Lemma 3. *Let $C = v_1v_2 \cdots v_mv_1$ be a cycle, $m \geq 3$ and $m \neq 5$. Then, C has a star 3-edge-coloring. Particularly, when $m \equiv 0 \pmod 3$, C has a strong edge-coloring using three colors.*

Proof. We construct our desired colorings as follows. When $m \equiv 0 \pmod 3$, we color edges $v_1v_2, v_2v_3, \ldots, v_mv_1$ with three colors $1, 2, 3$, repeatedly. When $m \equiv 1 \pmod 3$, we color edges $v_1v_2, v_2v_3, \ldots, v_{m-1}v_m$ with three colors $1, 2, 3$, repeatedly, and v_mv_1 with color 2. When $m \equiv 2 \pmod 3$, it follows that $m \geq 8$. We color edges v_1v_2, $v_2v_3, \ldots, v_{m-5}v_{m-4}$ with three colors $1, 2, 3$, repeatedly, and color $v_{m-4}v_{m-3}$, $v_{m-3}v_{m-2}$,$v_{m-2}v_{m-1}$, $v_{m-1}v_m$ and v_mv_1 with 1, 2, 1, 3 and 2, respectively. ∎

Lemma 4. *Let $C = v_1v_2 \cdots v_mv_1$, $m \geq 3$, be a quaint listed-cycle with list L such that $|L(v)| = 2$ for every $v \in V(C)$. Suppose that v_{m-1} and v_m are the two quaint vertices of C. If $L(v_i) \nsubseteq (L(v_{m-1}) \cup L(v_m))$ for $i \in \{1, 2, \ldots, m-2\}$, then*

(1) *when $m \equiv 1 \pmod 3$, C has a strong irlist-edge-coloring using at most two non-listed-colors;*

(2) *when $m \equiv 2 \pmod 3$, C has an irlist-edge-coloring using at most two non-list-colors, for which any three consecutive edges receive different colors except $v_{m-2}v_{m-1}, v_{m-1}v_m$ and v_mv_1.*

Proof. Let $L(v_i) = \{c_1, c_1'\}$, $i \in \{1, 2, \ldots, m-2\}$, and $L(v_{m-1}) = \{c_2, c_2'\}$, $L(v_m) = \{c_3, c_3'\}$. Since $L(v_i) \nsubseteq (L(v_{m-1}) \cup L(v_m))$, there exist three colors, say c_1, c_2 and c_3, such that $c_1 \in L(v_i)$ and $c_1 \notin L(v_{m-1}) \cup L(v_m)$, $c_2 \in L(v_{m-1})$ and $c_2 \notin \{c_1, c_1'\}$, and $c_3 \in L(v_m)$ and $c_3 \notin \{c_1, c_1'\}$. Let c_4, c_4' be two distinct non-listed-colors. We construct the desired irlist-edge-colorings f of C by the following four rules.

For (1), $m - 1 \equiv 0 \pmod 3$. If $c_2 \in \{c_3, c_3'\}$ and $c_3 \in \{c_2, c_2'\}$, let f be the following: $f(v_{m-1}v_m) = c_1$, $f(v_mv_1) = c_4$, and for $i = 1, 2, \ldots, m-2$, $f(v_iv_{i+1}) = c_2$ when $i \equiv 1 \pmod 3$, $f(v_iv_{i+1}) = c_4'$ when $i \equiv 2 \pmod 3$ and $f(v_iv_{i+1}) = c_4$ when $i \equiv 0 \pmod 3$ (Rule ($\star$1)). Clearly, under f, any two edges within distance two receive distinct colors. Note that $c_1 \notin L(v_{m-1}) \cup L(v_m)$ and $\{c_2, c_4, c_4'\} \cap \{c_1, c_1'\} = \emptyset$. Therefore, f is a strong irlist-edge-coloring of C using two non-listed-colors c_4, c_4'. If $c_2 \notin \{c_3, c_3'\}$ (or $c_3 \notin \{c_2, c_2'\}$), then $c_2 \neq c_3$. Let f be the following: $f(v_{m-1}v_m) = c_1$, $f(v_mv_1) = c_2$ (or c_4), and for $i = 1, 2, \ldots, m-2$, $f(v_iv_{i+1}) = c_3$ (or c_2) when $i \equiv 1 \pmod 3$, $f(v_iv_{i+1}) = c_4$ (or c_3) when $i \equiv 2 \pmod 3$ and $f(v_iv_{i+1}) = c_2$ (or c_4) when $i \equiv 0 \pmod 3$ (Rule ($\star$2)). Additionally, under f, any two edges within distance two receive distinct colors. Since $\{c_2, c_3\} \cap \{c_1, c_1'\} = \emptyset$ and $c_1 \notin L(v_{m-1}) \cup L(v_m)$, it holds that f is a strong irlist-edge-coloring of C using one non-listed-color c_4.

For (2), $m - 2 \equiv 0 \pmod 3$. If $c_2 = c_3$, let f be $f(v_{m-1}v_m) = c_1$, $f(v_mv_1) = c_4$, and for $i = 1, 2, \ldots, m-2$, $f(v_iv_{i+1}) = c_2$ when $i \equiv 1 \pmod 3$, $f(v_iv_{i+1}) = c_4'$

when $i \equiv 2 \pmod 3$ and $f(v_iv_{i+1}) = c_4$ when $i \equiv 0 \pmod 3$ (Rule ($\star$3)). By the definition of f, it has that $f(e) \neq f(e')$ for any $e, e' \in (E(C) \setminus \{v_{m-2}v_{m-1}, v_mv_1\})$ such that the distance between them is at most two. Additionally, $c_1 \notin L(v_{m-1}) \cup L(v_m)$ and $\{c_2, c_4, c_4'\} \cap \{c_1, c_1'\} = \emptyset$. Therefore, f is the desired irlist-edge-coloring of C using two non-listed-colors c_4, c_4'.

If $c_2 \neq c_3$, let f be the following: $f(v_{m-1}v_m) = c_1$, $f(v_mv_1) = c_4$, and for $i = 1, 2, \ldots, m-2$, $f(v_iv_{i+1}) = c_2$ when $i \equiv 1 \pmod 3$, $f(v_iv_{i+1}) = c_3$ when $i \equiv 2 \pmod 3$ and $f(v_iv_{i+1}) = c_4$ when $i \equiv 0 \pmod 3$ (Rule ($\star$4)). Analogously, f is the desired irlist-edge-coloring of C using one non-listed-colors c_4. ■

Theorem 5. *Let ℓ be the greatest common divisor of n and k. When $\ell \geq 3$, with the exception of $\ell = 3$, $k \neq \ell$, and $\frac{n}{3} \equiv 1 \pmod 3$, $P(n, k)$ has a star 5-edge-coloring.*

Proof. Let $i_j = i + (j-1)k$ for $j = 1, 2, \ldots, p = \frac{n}{\ell}$. Then, by the definition, the subgraph of $P(n, k)$ induced by $\{v_0, v_1, \ldots, v_{n-1}\}$ is the union of ℓ vertex-disjoint p-cycles, denoted by $C_i = v_{i_1}v_{i_2}\cdots v_{i_p}v_{i_1}$, $i = 0, 1, \ldots, \ell-1$. Let $C = u_0u_1\cdots u_{n-1}u_0$.

We first partition C into five edge-disjoint paths as follows.

Path-A. $u_0u_1u_2, \ldots, u_{n-2k-1}u_{n-2k}$.

Path-B. $u_{n-2k}u_{n-2k+1}u_{n-2k+2}\cdots u_{n-2k+\ell-1}u_{n-2k+\ell}$.

Path-C. $u_{n-2k+\ell}u_{n-2k+\ell+1}u_{n-2k+\ell+2}\cdots u_{n-k-1}u_{n-k}$.

Path-D. $u_{n-k}u_{n-k+1}u_{n-k+2}\cdots u_{n-k+\ell-1}u_{n-k+\ell}$.

Path-E. $u_{n-k+\ell}u_{n-k+\ell+1}u_{n-k+\ell+2}\cdots u_{n-1}u_0$.

Note that the length of each path defined above is a multiple of ℓ. Both Path-B and Path-D contain exactly ℓ edges, and when $k = \ell$, Path-C and Path-E are empty.

We now color edges of C by coloring edges of Paths-A, C, E, B and D, respectively, according to the coloring rules indicated in Table 1. We distinguish 11 cases (each row denotes one case) based on values of p and ℓ. Each column contains 11 coloring rules of the corresponding paths (for example, the second column corresponds to Path-A, Path-C and Path-E). Each rule is a cyclic coloring of ℓ colors. When we use the rule to color the edges of the corresponding path, say $P = u_xu_{x+1}\cdots u_{x+m}$, we first partition the path into q small paths of length $\ell(\geq 3)$, $P_1, P_2, \ldots, P_q$, where $P_1 = u_xu_{x+1}\cdots u_{x+\ell}, P_2 = u_{x+\ell+1}u_{x+\ell+2}\cdots u_{x+2\ell+1}, \ldots, P_q = u_{x+m-\ell}u_{x+m-\ell+1}\cdots u_{x+m}$; then, for each P_i, we color it from the first edge to the last edge one by one consecutively, according to the rule. For example, in the case of $p \equiv 1 \pmod 3$ and $\ell \equiv 1 \pmod 3$, if $P \in \{$Path-A, Path-C, Path-E$\}$, then we color P_i (P_i is a subgraph of P) with 1, 2, 3, 1, 2, 3, and 4 when $|E(P_i)| = 7$ and with 1, 2, 3, and 4 when $|E(P_i)| = 4$;

Table 1. Coloring rules of edges of C.

values of p and ℓ	Path-A, Path-C, Path-E	Path-B	Path-D
$p \equiv 0 \pmod 3$ and $\ell \equiv 0 \pmod 3$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3}}_{\ell\ \ elements}$	
$p \equiv 0 \pmod 3$ and $\ell \equiv 1 \pmod 3$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3},\underline{1,2,3,4}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3},\underline{1,2,3,4}}_{\ell\ \ elements}$	
$p \equiv 0 \pmod 3$ and $\ell \equiv 2 \pmod 3$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3},\underline{1,2,3,4,5}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3},\underline{1,2,3,4,5}}_{\ell\ \ elements}$	
$p \equiv 1 \pmod 3$ and $\ell \equiv 0 \pmod 3$ (1) $\ell = k$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{4,1,3},\dots,\underline{4,1,3}}_{\ell\ \ elements}$	by $\underbrace{\underline{4,5,3},\dots,\underline{4,5,3}}_{\ell\ \ elements}$
$p \equiv 1 \pmod 3$ and $\ell \equiv 0 \pmod 3$ (2) $\ell \neq k$ and $\ell \geq 6$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{4,1,3},\dots,\underline{4,1,3},\underline{2,4,3}}_{\ell\ \ elements}$	
$p \equiv 1 \pmod 3$ and $\ell \equiv 1 \pmod 3$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3},\underline{1,2,3,4}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{2,3,1},\dots,\underline{2,3,1},\underline{2,3,5,4}}_{\ell\ \ elements}$	
$p \equiv 1 \pmod 3$, $\ell \equiv 2 \pmod 3$ (1) $\ell \geq 8$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3},\underline{1,2,3,4,5}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{3,2,4},\dots,\underline{3,2,4},\underline{1,3,4,2,5}}_{\ell\ \ elements}$	
$p \equiv 1 \pmod 3$, $\ell \equiv 2 \pmod 3$ (2) $\ell = 5$	by $\underbrace{1,2,3,4,5}$, repeatedly	by 1,3,4,5,3	by 1,4,5,2,3
$p \equiv 2 \pmod 3$ and $\ell \equiv 0 \pmod 3$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{4,5,3},\dots,\underline{4,5,3}}_{\ell\ \ elements}$	
$p \equiv 2 \pmod 3$ and $\ell \equiv 1 \pmod 3$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3},\underline{1,2,3,4}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{2,3,5},\dots,\underline{2,3,5},\underline{2,3,5,4}}_{\ell\ \ elements}$	
$p \equiv 2 \pmod 3$ and $\ell \equiv 2 \pmod 3$	by $\underbrace{\underline{1,2,3},\dots,\underline{1,2,3},\underline{1,2,3,4,5}}_{\ell\ \ elements}$, repeatedly	by $\underbrace{\underline{2,4,1},\dots,\underline{2,4,1},\underline{2,4,1,3,5}}_{\ell\ \ elements}$	

if $P \in \{\text{Path-B, Path-D}\}$, then we color P_i with 2, 3, 1, 2, 3, 5, and 4 when $|E(P_i)| = 7$ and with 2, 3, 5, and 4 when $|E(P_i)| = 4$.

The resulting coloring of C is denoted by f. One can readily check that f is a strong edge-coloring. We now assign list L to C_i for $i = 0, 1, \ldots, \ell - 1$. Let

$$L(v_i) = \{f(u_i u_{i+1}), f(u_i u_{i-1})\},\ i = 0, 1, \ldots, n-1.$$

Then, we obtain ℓ listed-cycles C_i of length $p = \frac{n}{\ell}$, $i = 0, 1, \ldots, \ell - 1$.

Case 1. When $p \equiv 0 \pmod 3$, then $|L(V(C_i))| = 2$ (since k is a multiple of ℓ) for each $i \in \{0, 1, \ldots, \ell - 1\}$. Observe that $|V(C_i)| = p \equiv 0 \pmod 3$. Hence, by Lemma 3, C_i has a strong irlist-edge-coloring with $\{1, 2, 3, 4, 5\} \setminus \{x, y\}$, where x, y are the two listed-colors of C_i.

Case 2. When $p \equiv 1 \pmod 3$, we further consider the following three subcases.

Case 2.1. $\ell \equiv 0 \pmod 3$. First, $\ell = k$. Then, C_i is a listed-cycle such that (1) $L(v_{i_j}) = \{1, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3, 4\}$; or (2) $L(v_{i_j}) = \{1, 2\}$ and $L(v_{i_{p-1}}) = \{1, 4\}, L(v_{i_p}) = \{4, 5\}$; or (3) $L(v_{i_j}) = \{2, 3\}$ and $L(v_{i_{p-1}}) = \{1, 3\}, L(v_{i_p}) = \{3, 5\}$, where $j \in \{1, 2, \ldots, p-2\}$.

Second, $\ell \neq k$ and $\ell \geq 6$. Then, C_i is a listed-cycle satisfying one of the following conditions. For $j \in \{1, 2, \ldots, p-2\}$, (1) $L(v_{i_j}) = \{1, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3, 4\}$; (2) $L(v_{i_j}) = \{1, 2\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{1, 4\}$; (3) $L(v_{i_j}) = \{2, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{1, 3\}$; (4) $L(v_{i_j}) = \{1, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2, 3\}$; (5) $L(v_{i_j}) = \{1, 2\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2, 4\}$; (6) $L(v_{i_j}) = \{2, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3, 4\}$.

Case 2.2. $\ell \equiv 1 \pmod 3$. Then, for $j \in \{1, 2, \ldots, p-2\}$, it follows that (1) $L(v_{i_j}) = \{1, 4\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2, 4\}$; or (2) $L(v_{i_j}) = \{1, 2\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2, 3\}$; or (3) $L(v_{i_j}) = \{2, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{1, 3\}$; or (4) $L(v_{i_j}) = \{1, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{1, 2\}$; or (5) $L(v_{i_j}) = \{2, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3, 5\}$; or (6) $L(v_{i_j}) = \{3, 4\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{4, 5\}$.

Case 2.3. $\ell \equiv 2 \pmod 3$. First, when $\ell \geq 8$, it has that for $j \in \{1, 2, \ldots, p-2\}$, (1) $L(v_{i_j}) = \{1, 5\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3, 5\}$; or (2) $L(v_{i_j}) = \{1, 2\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2, 3\}$; or (3) $L(v_{i_j}) = \{2, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2, 4\}$; or (4) $L(v_{i_j}) = \{1, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3, 4\}$; or (5) $L(v_{i_j}) = \{1, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{1, 4\}$; or (6) $L(v_{i_j}) = \{1, 2\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{1, 3\}$; or (7) $L(v_{i_j}) = \{2, 3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3, 4\}$; or (8) $L(v_{i_j}) = \{3, 4\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2, 4\}$; or (9) $L(v_{i_j}) = \{4, 5\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2, 5\}$.

Second, when $\ell = 5$, C_i is a listed-cycle such that (1) $L(v_{i_j}) = \{1, 5\}$ for $j \in \{1, 2, \ldots, p\} \setminus \{j', j'+1\}$, and $L(v_{i_{j'}}) = L(v_{i_{j'+1}}) = \{1, 3\}$, where $j', j'+1$ are read model p; or (2) $L(v_{i_j}) = \{1,2\}$ and $L(v_{i_{p-1}}) = \{1,3\}, L(v_{i_p}) = \{1,4\}$; or (3) $L(v_{i_j}) = \{2,3\}$ and $L(v_{i_{p-1}}) = \{3,4\}, L(v_{i_p}) = \{4,5\}$; or (4) $L(v_{i_j}) = \{3,4\}$ and $L(v_{i_{p-1}}) = \{4,5\}, L(v_{i_p}) = \{2,5\}$; or (5) $L(v_{i_j}) = \{4,5\}$ and $L(v_{i_{p-1}}) = \{3,5\}, L(v_{i_p}) = \{2,3\}$, where $j \in \{1,2,\dots,p-2\}$ in (2)–(5).

Obviously, in each of the above subcases, C_i is a quaint listed-cycle satisfying the condition of Lemma 4(1). Therefore, C_i has a strong irlist-edge-coloring using some colors in $\{1,2,3,4,5\}$ by Rules ($\star$1) and ($\star$2).

Case 3. When $p \equiv 2 \pmod 3$, there are also three subcases that need to dealt with.

Case 3.1. $\ell \equiv 0 \pmod 3$. Then, one of the following holds. For $j \in \{1,2,\dots,p-2\}$, (1) $L(v_{i_j}) = \{1,3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3,4\}$; (2) $L(v_{i_j}) = \{1,2\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{4,5\}$; (3) $L(v_{i_j}) = \{2,3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3,5\}$.

Case 3.2. $\ell \equiv 1 \pmod 3$. Then, for $j \in \{1,2,\dots,p-2\}$, it has that (1) $L(v_{i_j}) = \{1,4\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2,4\}$; or (2) $L(v_{i_j}) = \{1,2\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2,3\}$; or (3) $L(v_{i_j}) = \{2,3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3,5\}$; or (4) $L(v_{i_j}) = \{1,3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2,5\}$; or (5) $L(v_{i_j}) = \{3,4\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{4,5\}$.

Case 3.3. $\ell \equiv 2 \pmod 3$. Then, for $j \in \{1,2,\dots,p-2\}$, one of the following situations holds. (1) $L(v_{i_j}) = \{1,5\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2,5\}$; (2) $L(v_{i_j}) = \{1,2\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{2,4\}$; (3) $L(v_{i_j}) = \{2,3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{1,4\}$; (4) $L(v_{i_j}) = \{1,3\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{1,2\}$; (5) $L(v_{i_j}) = \{3,4\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{1,3\}$; (6) $L(v_{i_j}) = \{4,5\}$ and $L(v_{i_{p-1}}) = L(v_{i_p}) = \{3,5\}$.

One can readily check that in Cases 3.1–3.3, C_i is also a quaint listed-cycle. Therefore, C_i has a strong irlist-edge-coloring using colors 1, 2, 3, 4, and 5 by Rules ($\star$3) and ($\star$4) in Lemma 4(2).

Until now, we have colored edges of C_i, $i = 0,1,\dots,\ell-1$. We denote the resulting coloring of C_i by f'. Obviously, for each $i \in \{0,1,\dots,n-1\}$, it has that $|\{f(u_iu_{i+1}), f(u_iu_{i-1}), f'(v_iv_{i+1}), f'(v_iv_{i-1})\}| = 4$. We then color each u_iv_i with the unique color $\{1,2,3,4,5\} \setminus \{f(u_iu_{i+1}), f(u_iu_{i-1}), f'(v_iv_{i+1}), f'(v_iv_{i-1})\}$. This completes the edge-coloring of $P(n,k)$. We now show that such the coloring is a star 5-edge-coloring.

If not, let P be a bichromatic 4-path. Since f is a strong edge-coloring of C, and $\{f(u_iu_{i+1}), f(u_iu_{i-1})\} \cap \{f'(v_iv_{i+1}), f'(v_iv_{i-1})\} = \emptyset$ for any $i \in \{0,1,\dots,n-1\}$, P does not contain any edges of C. In addition, by Lemma 4, any three edges of C_i receive different colors under f', except $v_{i_{p-2}}v_{i_{p-1}}, v_{i_{p-1}}v_{i_p}, v_{i_p}v_{i_1}$. Therefore, $P = v_{i_{p-2}}v_{i_{p-1}}v_{i_p}v_{i_1}u_{i_1}$ or $P = u_{i_{p-2}}v_{i_{p-2}}v_{i_{p-1}}v_{i_p}v_{i_1}$. However, by Lemma 4 Rule ($\star$3) and ($\star$4), $f'(v_{i_{p-1}}v_{i_p})$ is a listed-color not in $L(v_{i_{p-1}}) \cup L(v_{i_p})$. Then, by the coloring rule of u_iv_i, $i = 0,1,\dots,n-1$, it has that $f'(v_{i_{p-1}}v_{i_p}) \neq f(v_{i_1}u_{i_1})$

and $f'(v_{i_{p-1}}v_{i_p}) \neq f(u_{i_{p-2}}v_{i_{p-2}})$. Therefore, P is not bichromatic, and it is a contradiction. ■

Lemma 6. *Let $P(n,k)$ be a generalized Petersen graph such that $GCD(n,k) = 1$, $n \equiv 0 \pmod 2$ and $k \equiv 1 \pmod 2$. Then, $P(n,k)$ has a star 5-edge-coloring.*

Proof. It is sufficient to construct a star 5-edge-coloring for $P(n,k)$ in this case. Let $C = u_0u_1 \cdots u_{n-1}u_0$ be the cycle induced by $\{u_0, u_1, \ldots, u_{n-1}\}$. Since $\mathrm{GCD}(n,k) = 1$, i.e., n, k are coprime, the subgraph induced by $\{v_0, v_1, \ldots, v_{n-1}\}$ is also a cycle, denoted by C'. Since $n \equiv 0 \pmod 2$, it follows that $n \neq 5$ and by Lemma 3 both C and C' have a star 3-edge-coloring. Let f_1 and f_2 be the two star edge-colorings of C and C', respectively, using colors 1, 2, and 3. Then, we color u_iv_i with 4 when $i \equiv 0 \pmod 2$ and with 5 when $i \equiv 1 \pmod 2$, for $i = 0, 1, \ldots, n-1$. Denote by f_3 the resulting coloring, and let $f = f_1 \cup f_2 \cup f_3$. We now show that f is a star edge-coloring. On the contrary, we assume there is a bichromatic 4-path P. Let c_1 and c_2 be the two colors appearing on the edges of P. By f_1 and f_2, it is by no means that $\{c_1, c_2\} \subset \{1, 2, 3\}$. In addition, since $(n,k) = 1$, $n \equiv 0 \pmod 2$ and $k \equiv 1 \pmod 2$, $f_3(u_iv_i) \neq f_3(u_{i+1}v_{i+1})$ and $f_3(u_iv_i) \neq f_3(u_{i+k}v_{i+k})$. Therefore, together with f_3, $\{c_1, c_2\} \cap \{4, 5\} = \emptyset$. Hence, P is not bichromatic and is a contradiction. ■

Theorem 7. *$P(n, 1)$, $n \geq 5$, has a star 5-edge-coloring.*

Proof. By Lemma 6, we only need to consider the case $n \equiv 1 \pmod 2$. In this case, we can also obtain a star 5-edge-coloring by a slight change of the coloring in Lemma 6. Let $P_1 = u_{n-1}u_0u_1 \cdots u_{n-2}$ and $P_2 = v_0v_1 \cdots v_{n-1}$ be two paths. We now define a star 3-edge-coloring f_1 of P_1 as follows. First, let $f_1(u_{n-1}u_0) = 2, f_1(u_0u_1) = f_1(u_{n-3}u_{n-2}) = 3$. Then, color edges of sub-path $u_1u_2 \cdots u_{n-3}$ as follows: when $n = 5$, let $f_1(u_1u_2) = 1$; when $n \geq 7$, color edges $u_1u_2, u_2u_3, \ldots, u_{n-4}u_{n-3}$ by 1, 3, and 2, repeatedly, if $n - 4 \equiv 0 \pmod 3$; by $\underbrace{1, 3, 2, \ldots, 1, 3, 2}_{n-5 \ \text{edges}}$, 1 if $n - 4 \equiv 1 \pmod 3$; and by $\underbrace{1, 3, 2, \ldots, 1, 3, 2}_{n-6 \ \text{edges}}$, 1 and 2 if $n - 4 \equiv 2 \pmod 3$. Obviously, P_2 also has a star 3-edge-coloring, say f_2. By color permutation, we can assume $f_2(v_{n-3}v_{n-2}) = 3$, and $f_2(v_{n-2}v_{n-1}) = 2$. Based on f_1 and f_2, we color $u_{n-2}u_{n-1}$ with 4 and $v_{n-1}v_0$ with 5. And for any $i \in \{0, 1, \ldots, n-2\}$, color u_iv_i with 4 for $i \equiv 0 \pmod 2$ and with 5 for $i \equiv 1 \pmod 2$, and finally, color $u_{n-1}v_{n-1}$ with 1. Until now, we typically obtain an edge-coloring of $P(n, 1)$. One can easily see that such the coloring is a star 5-edge-coloring. ■

Note that when $n = 3$, Theorem 7 by no means hold. However, by a coloring $P(n, 3)$ with an exhausting search, we can see that $P(n, 3)$ does not contain any star 5-edge-coloring.

Lemma 8. *Let $P(n,k)$ be a generalized Petersen graph such that $(n,k)=2$. Let $C_0 = v_0v_k \cdots v_{(\frac{n}{2}-1)k}v_0$. If C_0 has a star 3-edge-coloring f such that $C_f(v_i) \neq C_f(v_{i+1})$ for any $i \in \{0,1,\ldots,n-1\}$ and $i \equiv 0 \pmod 2$, then $P(n,k)$ has a star 5-edge-coloring, where $C_f(v_i) = \{f(v_iv_{i+k}), f(v_iv_{i-k})\}$.*

Proof. Since $GCD(n,k)=2$, it has that n is an even number. Let f be a star 3-edge-coloring of C_0, such that $C_f(v_i) \neq C_f(v_{i+1})$ for any $i \equiv 0 \pmod 2$ and $i \in \{0,1,\ldots,n-1\}$. We now color $C_1 = v_1v_{1+k} \cdots v_{1+(\frac{n}{2}-1)k}v_1$ with the same pattern as C_0, that is, color each edge v_jv_{j+k} with the color $f(v_{j-1}v_{j+k-1})$, for $j \equiv 1 \pmod 2$ and $j \in \{0,1,\ldots,n-1\}$. Denote the resulting coloring of C_0 and C_1 also by f. Then, $C_f(v_i) = C_f(v_{i+1})$ for any $i = 0,2,4,\ldots,n-2$. Based on f, for any $i \in \{0,1,\ldots,n-1\}$, we color u_iu_{i+1} with the color in $\{1,2,3\} \setminus C_f(v_i)$ when $i \equiv 0 \pmod 2$, and with 4 when $i \equiv 1 \pmod 2$. Finally, color u_iv_i with 5, $i = 0,1,\ldots,n-1$. Obviously, such the coloring is a star 5-edge-coloring. ∎

Theorem 9. *$P(6m,2)$ has a star 5-edge-coloring, where $m \geq 1$ is a positive number.*

Proof. Let $n = 6m$, and $C_0 = v_0v_k \cdots v_{(\frac{n}{2}-1)k}v_0$. Obviously, C_0 has a star 3-edge-coloring f such that $C_f(v_i) \neq C_f(v_{i+1})$ for any $i \in \{0,1,\ldots,n-1\}$ and $i \equiv 0 \pmod 2$ (since $\frac{n}{2} = 3m \equiv 0 \pmod 3$, we can color edges of C_0 with 1, 2, 3, repeatedly). Therefore, by Lemma 8, $P(6m,2)$ has a star 5-edge-coloring. ∎

In this article, we determine the star chromatic index of generalized Petersen graphs $P(n,k)$ for "almost all" values of n and k. By using more involved analysis, we can also prove $P(n,k)$ has a star 5-edge-coloring for some remaining values of n and k, particularly, for the case $\ell = 3$, $k \neq \ell$, and $\frac{n}{3} \equiv 1 \pmod 3$. However, we prefer to present a short or uniform proof. In addition, we would like to stress that only one generalized Petersen graph, i.e., $P(3,1)$, is found to have the star chromatic index 6. Therefore, we conjecture that $P(n,k)$ has a star 5-edge-coloring for any $n \geq 4$.

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