

COVERINGS OF CUBIC GRAPHS AND 3-EDGE COLORABILITY

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Abstract

Let $h: \tilde{G} \rightarrow G$ be a finite covering of 2-connected cubic (multi)graphs where G is 3-edge uncolorable. In this paper, we describe conditions under which \tilde{G} is 3-edge uncolorable. As particular cases, we have constructed regular and irregular 5-fold coverings $f: \tilde{G} \rightarrow G$ of uncolorable cyclically 4-edge connected cubic graphs and an irregular 5-fold covering $g: \tilde{H} \rightarrow H$ of uncolorable cyclically 6-edge connected cubic graphs.

In [13], Steffen introduced the resistance of a subcubic graph, a characteristic that measures how far is this graph from being 3-edge colorable. In this paper, we also study the relation between the resistance of the base cubic graph and the covering cubic graph.

Keywords: uncolorable cubic graph, covering of graphs, voltage permutation graph, resistance, nowhere-zero 4-flow.

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1. INTRODUCTION

1.1. Motivation and statement of results

We will consider proper edge colorings of G . Let $\Delta(G)$ be the maximum degree of vertices in a graph G . Denote by $\chi'(G)$ the minimum number of colors needed for (proper) edge coloring of G and call it the chromatic index of G . Recall that according to Vizing's theorem (see [14]), either $\chi'(G) = \Delta(G)$, or $\chi'(G) = \Delta(G) + 1$.

In this paper, we shall consider only subcubic graphs i.e., graphs G such that $\Delta(G) \leq 3$. The subcubic graph G is called colorable if $\chi'(G) \leq 3$, otherwise it is

called uncolorable. An uncolorable cubic graph is called a snark if it is cyclically 4-edge connected and its girth is at least five.

There are known constructions that allow to produce new snarks starting from small cubic graphs and applying to them some operations (for example, via the dot product, the vertex and edge superpositions [7], Loupekin construction [3], gluing multipoles *etc.*).

The motivation of this paper is an attempt to understand whether the uncolorable cubic graphs (in particular, snarks) can be obtained via covering maps i.e., starting from an uncolorable cubic graph and lifting it via a covering map, and what are the conditions under which such lifting is successful (see Subsection 1.2 for the definition of covering graphs). Intuitively, a covering map of graphs is a "regular" homomorphism of them, so the question seems to be natural. Covering of graphs are usually described via voltage graph or permutation voltage graph constructions [5].

The structure of this paper is the following. In Introduction, we define some notions and concepts from topological graph theory, such as coverings of graphs, voltage graph, voltage permutation graph, i.e., graphs enhanced with an additional structure which allow to describe coverings. For details see also [5].

In Section 2, we describe general conditions under which, for a given covering of cubic (multi)graphs, the covering graph is to be uncolorable (Theorem 5). Theorem 5 relies basically on a standard procedure of gluing several copies of the same multipole in some consistent way (in particular, in a cyclic order) and allows to restate many other results on multiplying snarks in terms of topological graph theory. On the other hand, we provide nonstandard procedure for obtaining uncolorable graphs by using 5-fold regular and irregular coverings of cubic graphs. Under certain conditions, this allows to produce a big class of cyclically 4-edge connected and 6-edge connected uncolorable cubic graphs.

In Section 3, we study coverings of cubic graphs G with respect to resistance $r(G)$, a parameter of uncolorable cubic graphs that measures how far is a given cubic graph from being 3-edge colorable. More precisely, the resistance of a cubic graph G is the minimum number of edges such that after removing all them from G the remaining graph is 3-edge colorable.

Another interesting measure of non-colorability of a bridgeless cubic graph G is its oddness, denoted by $\omega(G)$. This is the minimum number of odd cycles that are in G after removing in this graph an 1-factor. By definitions, we have obviously $r(G) \leq \omega(G)$. The parameters $r(G)$ and $\omega(G)$ were introduced and studied by Steffen in [13]. The main problem was to construct for each natural number n a cubic graph of minimum order such that $r(G) = n$ ($\omega(G) = n$, respectively). In an equivalent form, the problem is to construct 2-connected cubic graphs G with the maximum ratio $\rho(G) = r(G)/|G|$ ($\mu(G) = \omega(G)/|G|$, respectively) or estimate these parameters asymptotically. In [6], Hägglund has

improved previous results of Steffen. The best known estimates of ratios $\rho(G) = r(G)/|G|$ and $\mu(G) = \omega(G)/|G|$ were given by Lukot'ka, Máčajová, Mazák and Škoviera in [8]. A good survey on measures of non-colorability of cubic graphs is the recent paper [2] where an improvement of the previous known results is also given.

In Section 3, we show that under certain conditions the resistance of a cubic graph increases when passing from the base graph G to the covering graph \tilde{G} (Theorem 12). We supply our general consideration with particular examples.

1.2. Coverings and voltage permutation graphs

Finite coverings of cubic graphs were the powerful tool in proving the Heawood conjecture on the chromatic number of a closed surface. By using them, one can construct triangular embeddings of complete graphs K_n (in regular cases) or the complete graphs with a few edges removed into closed surfaces of corresponding genus. The combinatorial schemes of such triangulations were described by means of current and voltage graphs that are modeled over cubic graphs with the assignment in a finite group H .

Definition. A surjective (continuous) map $p: \tilde{S} \rightarrow S$ of topological spaces \tilde{S} and S is called a *covering map* (*covering*) if for each $x \in S$ there exists a neighbourhood $U(x)$ such that $p^{-1}(U(x))$ is decomposed into disjoint sum $\bigsqcup_{i \in I} U_i$ of sets U_i such that for each $i \in I$, where I is a countable set, the restriction $p|_{U_i}: U_i \rightarrow U(x)$ is a homeomorphism. Then \tilde{S} is called the *covering space* and S the *base space* (or simply the base) of the covering p .

Moreover, restricted to graphs, we also require that the covering $p: \tilde{G} \rightarrow G$ is a graph map. In the case when \tilde{G} and G are both connected, the cardinal number $n = |p^{-1}(x)|$ does not depend on the choice of $x \in S$. In the following, we require also that both the cover graph \tilde{G} and the base graph G of the covering $p: \tilde{G} \rightarrow G$ are finite and connected.

Definition. Let $p: \tilde{G} \rightarrow G$ be a finite covering with connected graphs \tilde{G} and G and $n = |p^{-1}(x)|$. Then p is called *n-fold covering*.

A covering map p is called *regular* if the deck transformation group X acts on \tilde{S} transitively [5]. Otherwise it is called *irregular*.

Definition. Let $G = (V, E)$ be a connected graph. We can replace each edge $e \in E$ with the two arcs, e' and e'' , joining the same pair of vertices, but with opposite directions. As a result, we shall obtain a directed graph G' with the set of arcs E' . Let A be a finite group and let $\alpha: E' \rightarrow A$ be a map which satisfies the following condition: for any $e \in E$, if $\alpha(e') = h \in A$, then $\alpha(e'') = h^{-1} \in A$. The pair (G, α) is called then a *voltage graph* and the mapping α a *voltage assignment* on G .

Let G be a graph. By taking an orientation of edges of G we obtain an orgraph \vec{G} . It is clear that the voltage assignment α on G is uniquely determined by its values on the arcs of \vec{G} . For this reason, when defining a voltage assignment $\alpha: E' \rightarrow A$, we indicate only the values of α on arcs from \vec{E} .

Definition. The *derived graph* G^α is defined in the following way: $V(G^\alpha) = V \times A$ and $E(G^\alpha) = E \times A$. More precisely, if $e = (u, v)$ is an arc from u to v in \vec{E} , then the edge (e, g) of G^α joins the vertices (u, g) and $(v, g \cdot \alpha(e))$.

Let \vec{G} be an orgraph obtained from G as before and the number n is fixed. Denote by Σ_n the symmetric group on n -element set $[n] = \{1, \dots, n\}$.

Definition. A *permutation voltage assignment* on G with values in Σ_n is a function $\beta: E(\vec{G}) \rightarrow \Sigma_n$, which assigns to each arc of $e \in E(\vec{G})$ a permutation $\beta(e) \in \Sigma_n$. The pair (G, β) is called a *permutation voltage graph*. As in the case of the voltage assignment, we assume that the function β can be extended to the whole set of arcs of the directed graph G' , so that the following condition is satisfied: if $e \in \vec{G}$ and $\beta(e) = \omega \in \Sigma_n$, then $\beta(e^{-1}) = \omega^{-1}$. The derived graph associated with a permutation voltage graph (G, β) is denoted by G^β .

Let $c = (e_1, \dots, e_{k-1}, e_k)$ be an oriented path in the voltage permutation graph (G, β) . We define the permutation $\beta(c) \in \Sigma_n$ as follows: $\beta(c) = \beta(e_1) \cdot \beta(e_2) \cdots \beta(e_k)$. If c is an oriented cycle in G , the element $\beta(c)$ of the group Σ_n is defined up to conjugation.

The derived graphs G^α with the voltage assignment in a group H of order n describe regular n -fold coverings of the graph G as follows.

Proposition 1 [4]. *Every regular n -fold covering map $f: \tilde{G} \rightarrow G$ of graphs with the finite deck transformation group H where G is connected and $|H| = n$ is realized by a voltage graph (G, α) with a voltage assignment in H .*

In general, coverings of graphs are described by the following.

Proposition 2 [4]. *Every n -fold covering map $f: \tilde{G} \rightarrow G$ of graphs is realized by a permutation voltage graph with an assignment in the symmetric group Σ_n .*

1.3. Coloring of multipoles and nowhere-zero flows

In the following, we also consider graphs with semi-edges (see also [8]).

Definition. A *multipole* is a triple $M = (V; E; S)$ where $V = V(M)$ is the vertex set, $E = E(M)$ is the edge set and $S = S(M)$ is the set of semi-edges. Each semi-edge is incident to exactly one vertex v of M and is denoted by (v) (the second end of semi-edge contains no vertex of G).

It follows from the definition that no loop cannot serve as a semi-edge of M . Semi-edges are usually grouped into pairwise disjoint connectors [7, 9]. A multipole with k semi-edges is called k -pole. If $S(M) = \emptyset$, then M is simply a graph. A multipole is called *cubic* if each its vertex is of degree three. We say that the graph M' is obtained from the $2k$ -multipole M by identifying the pairs (v_i) and (u_i) of semi-edges where $i = 1, \dots, k$, if each such pair (v_i) and (u_i) is replaced with an edge $\{v_i, u_i\}$ in M' .

Let M be a multipole and let $[k] = \{1, 2, \dots, k\}$ be a set of colors. Let $f: M \rightarrow [k]$ be a mapping that assigns to each $e \in E \cup S$ a color from $[k]$ in such a way that for every vertex v in M the ends incident with v (edges or semi-edges) have pairwise distinct colors. Then f is called a k -edge coloring of M . Therefore if M is a cubic multipole that has a k -edge coloring, then $k \geq 3$. Moreover, if M is a loopless cubic multipole, then there exists an m -edge coloring of M with $m \leq 4$. If there is a 3-edge coloring of M , we say that M is colorable, otherwise it is uncolorable. In the following, we shall consider only cubic multipoles.

Sometimes it is convenient to consider the colors 1, 2, and 3 as nonzero elements of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ and redefine a 3-edge coloring of a graph or a multipole in terms of nowhere-zero flows. For convenience of the reader, below we provide some relevant information on this subject.

Let G be a (multi)graph, \vec{G} an orientation of G and H be an abelian group. Under an H -flow on G we shall mean a nowhere-zero circulation $f: \vec{E} \rightarrow H$ [1]. The term "nowhere-zero" means that for each $e \in \vec{E}$ we have $f(e) \neq 0$, where 0 denotes the neutral element of the abelian group H . Moreover, under a k -flow on G we shall mean a nowhere-zero circulation f with values in the cyclic group \mathbf{Z}_k . We shall say that the (multi)graph G has a k -flow if such k -flow exists for some orgraph (oriented multigraph) \vec{G} with the underlying (multi)graph G .

Nowhere-zero k -flows on a multipole are defined in the same way as for cubic (multi)graphs. The only difference is that any k -flow on an l -multipole M has nontrivial sources (sinks) just at the semi-edges of M . We consider nowhere-zero flows on graphs and multipoles G with values in $\mathbf{Z}_2 \times \mathbf{Z}_2$. In this case, the orientation of edges (semi-edges) of G is irrelevant.

Theorem 3 [1, 8]. *For cubic (multi)graphs and multipoles G the following conditions are equivalent.*

- (a) G has a 4-flow;
- (b) G is 3-edge colorable.

Note that if $M = (V, E, S)$ is a cubic multipole and $\varphi: E \cup S \rightarrow \mathbf{Z}_2 \times \mathbf{Z}_2$ is a (nowhere zero) 4-flow on M , then $\sum_{e \in S} \varphi(e) = 0$ [7].

A simple graph or a multigraph that does not have a 4-flow is called *4-snark*. Cyclically 4-edge connected uncolorable cubic graphs with girth at least 5 are called *snarks*.

Below we provide an example of uncolorable graph G and its 3-fold covering graph \tilde{G} which is colorable.

Example 1. In Figure 1, it is shown a 16-pole G' embedded into the rectangle R . Gluing together the pair of vertical sides and the pair of horizontal sides of R , we obtain a torus T . The corresponding six pairs of "vertical" semi-edges (e_1 and e_2 , a_1 and a_2 , d_1 and d_2 , b_1 and b_2 , f_1 and f_2 , c_1 and c_2) and the pairs of "horizontal" semi-edges (s_1 and s_2 , t_1 and t_2) in G' are also to be identified. As a result, we shall obtain a graph G embedded in the torus T (in which each pair of corresponding semi-edges of G' is replaced with a unique edge of G).

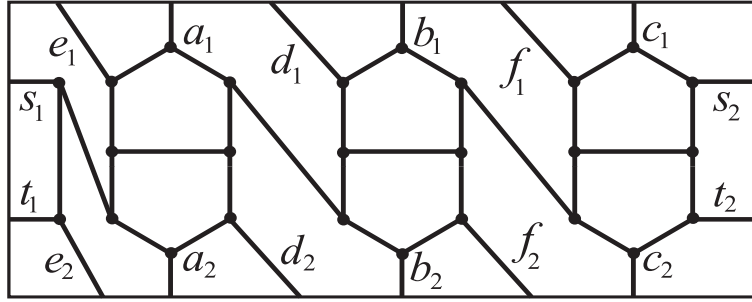


Figure 1. The 16-pole G' .

The snark G is one of the third powers of the Petersen graph P (via the dot product), so we simply write $G = P^3$ (see [11]).

Take the orientation of the six "vertical" edges of P^3 (i.e., a, b, c, d, e and f) from bottom to the top and an arbitrary orientation of the remaining edges. Cutting the graph P^3 along the six "vertical" edges, we shall obtain a 12-pole H which has a natural embedding in a cylinder.

Fix a natural number $n \geq 2$. Define the voltage assignment $\alpha: E(P^3) \rightarrow \mathbf{Z}_n$ as follows: $\alpha(h) = 1$ if h is one of six "vertical" arcs a, b, c, d, e, f and $\alpha(h) = 0$ in the remaining cases. The voltage graph (P^3, α) defines a derived cubic graph $\widetilde{P^3}$. The corresponding n -fold covering map $p: \widetilde{P^3} \rightarrow P^3$ of graphs is cyclic. The covering map of graphs can be extended to a cyclic n -fold covering $f: \widetilde{T} \rightarrow T$ of tori in a natural way. For $n = 3$ the 3-fold covering graph $\widetilde{P^3}$ embedded in the torus \widetilde{T} is pictured in Figure 2 (here we identify the corresponding semi-edges in the pairs).

Note that the multipole H has a 3-edge coloring in which all six bottom semi-edges receive a color x and all six top semi-edges receive a color y where $x \neq y$. It follows that for any choice $n \geq 2$ the covering cubic graph $\widetilde{P^3}$ is colorable. The details of the proof are left to the reader as an easy exercise.

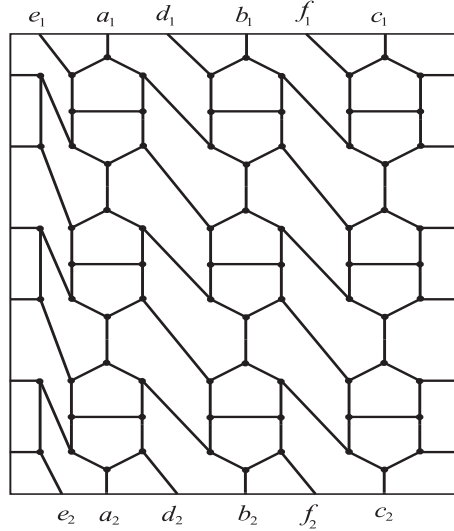


Figure 2. The 3-fold covering graph \tilde{P}^3 embedded in the torus \tilde{T} .

2. COVERINGS OF UNCOLORABLE CUBIC GRAPHS

2.1. General results

The following is an immediate consequence of definitions of n -flow and a covering map.

Proposition 4. *Let $p: \tilde{G} \rightarrow G$ be an m -fold covering map of graphs. If G has an n -flow (where $n \geq 2$), then \tilde{G} also has an n -flow.*

Proof. Let \vec{G} be an orgraph with the underlying graph G and let f be an n -flow on \vec{G} . Then the orientation of edges of the graph G is lifting uniquely to an orientation of edges in the covering graph \tilde{G} . Let \vec{G}' denote the resulting orgraph with the underlying graph \tilde{G} . We define the function \vec{f} on $E(\vec{G}')$ as follows. If e' is an arc of \vec{G} , we set $\vec{f}(e') = f(e')$ for each arc e' in the preimage $p^{-1}(e')$. The "lifted" function \vec{f} on arcs of \vec{G}' defines obviously an n -flow of the graph \tilde{G} . ■

In particular, if G has a 4-flow, then the covering graph \tilde{G} also has a 4-flow. Moreover, if G is an uncolorable cubic graph, then \tilde{G} is also so. A similar statement holds for multigraphs.

Question 1. *What are the conditions under which the covering graph of an uncolorable graph is an uncolorable graph?*

The class of uncolorable cubic graphs G obtained via covering maps of simple graphs and multigraphs of degree 3 includes the well known subclasses of them

such as Isaac's flowers, Goldberg snarks *etc.* Below we describe a general concept of these coverings.

Let G be a connected cubic (multi)graph and let $p: \tilde{G} \rightarrow G$ be an n -fold covering map of connected graphs that is defined via a permutation voltage assignment $\lambda: E(\tilde{G}) \rightarrow \Sigma_n$. Moreover, let $E' = \{e_1, e_2, \dots, e_r\}$ be a set of arcs in \tilde{G} (here we use the same notations e_i for arcs in \tilde{G} and corresponding edges in G). Cutting the edges e_1, \dots, e_r in interior points, we shall obtain a $2r$ -pole L' with the r "initial" semi-edges e'_1, e'_2, \dots, e'_r and the corresponding r "terminal" semi-edges, denoted by $e''_1, e''_2, \dots, e''_r$.

Definition. We say that a subset of edges $E' \subset E(G)$ satisfies the condition (i) if the multipole L' is connected. Moreover E' satisfies condition (ii) if for each oriented cycle c in $G \setminus E'$ we have $\lambda(c) = e$ where $e = (1)(2) \cdots (n)$ is the identity permutation of Σ_n .

We now associate with a subset $E' \subset E(G)$ satisfying condition (i) a hypergraph $H = H(E', p)$ as follows. Let L'_1, \dots, L'_k be connected components of the multipole $p^{-1}(L')$. In each L'_s , we identify the input semi-edges e'_i and the output semi-edges e''_j if and only if $\lambda(e)(i) = j$. Denote the resulting multipoles by L_1, \dots, L_k . The vertex set $V(H)$ of H consists of multipoles L_1, \dots, L_k .

For each pair $\{L_i, L_j\}, i \neq j$, there are $r_{i,j}$ edges in G joining the multipole L_i to the multipole L_j . Denote the corresponding edge set by $R_{i,j}$. Note that $R_{i,j} \subseteq p^{-1}(E')$. The pair $\{L_i, L_j\}, i \neq j$, is consistent if there exists a 3-edge coloring of multipoles L_i and L_j which is compatible on the pairs of semi-edges e' and e'' for each edge e from the set $R_{i,j}$. The subset $\{L_{i_1}, \dots, L_{i_m}\}$ is a hyperedge of H if and only if there is a 3-edge colorings of the multipoles $\{L_{i_1}, \dots, L_{i_k}\}$ which is consistent for each pair $\{L_{i_s}, L_{i_t}\}, s \neq t, s, t \leq k$.

Theorem 5 (the decomposition theorem). *Let G be a connected cubic (multi) graph that is a 4-snark and let $p: \tilde{G} \rightarrow G$ be an n -fold covering of connected graphs that is defined via a permutation voltage assignment $\lambda: E(\tilde{G}) \rightarrow \Sigma_n$. Assume that the set of arcs $E' = \{e_1, e_2, \dots, e_r\}$, where $E' \subset E(\tilde{G})$, satisfies conditions (i)–(ii). Then G^λ is colorable if and only if the hypergraph $H(E', p)$ associated with the set E' is a complete hypergraph on n vertices.*

Proof. Let E' be the set of arcs of \tilde{G} with the properties under assumption. Cutting in G all edges e from E' , we shall obtain a connected multipole L' .

Let L'_1, \dots, L'_k be the connected components of the multipole $p^{-1}(L')$. Since $p: \tilde{G} \rightarrow G$ is n -fold covering, the condition (i) implies that $k \leq n$. Suppose that $k < n$. Then there are a component L'_i of $p^{-1}(L')$ and an edge $f \in E(G) \setminus E'$ such that L_i contains at least two (disjoint) edges, say f_1 and f_2 , of the preimage $p^{-1}(f)$. It follows that there is a path q in $p^{-1}(L')$ which starts at the edge f_1 and ends at the edge f_2 . Without loss of generality, we can suggest that f_1 and

f_2 are unique double edges in the path q . Now, projecting the path q on G by the map p , we shall get a cycle c in $G \setminus E'$. But this contradicts the condition (ii), since $\lambda(c) = e$ and so c lifts to a cycle via the covering map p . Note that all multipoles L'_i , $i = 1, \dots, n$, are isomorphic to L' in a natural way.

Therefore, after identifying the corresponding pairs of input and output semi-edges in each L'_i , $i = 1, \dots, n$, we get exactly n disjoint multipoles L_1, \dots, L_n . It may occur that L' is uncolorable. It follows immediately that $p^{-1}(L')$ is uncolorable, so is the graph G^λ . It may also occur that L' is colorable but some multipole L_i is not uncolorable. In any case the hypergraph $H(E', p)$ obviously does not contain the n -hyperedge $h = \{L_1, \dots, L_n\}$. Assume now that each L_i is colorable.

Let $(e'_i, 1), \dots, (e'_i, n)$ be the lifts of the semi-edge e'_i under the covering map p , where $i = 1, \dots, r$. Similarly, for each $i = 1, \dots, r$ let $(e''_i, 1), \dots, (e''_i, n)$ be the lifts of the semi-edge e''_i under the covering map p . By conditions (i) and (ii), the covering graph G^λ is obtained from multipoles L_1, \dots, L_n by identifying the corresponding pairs of semi-edges, for each set $R_{i,j}$, $i \neq j$. More precisely, $r_{i,j}$ semi-edges of L_i are identified with $r_{i,j}$, $i \neq j$, semi-edges of L_j .

After identification of all pairs of semi-edges for each pair $\{L_i, L_j\}$, we shall obtain a graph G' . It is not difficult to verify that G' is isomorphic to \tilde{G} . The condition that the hypergraph $H(E', p)$ is complete means that there is the n -hyperedge h in $H(E', p)$. But the last condition implies that there is consistent 3-edge coloring of the disjoint multipoles L_1, \dots, L_n which is equivalent to the existence of 3-edge coloring of the covering graph \tilde{G} . ■

Theorem 5 describes, in particular, cyclic coverings of uncolorable cubic graphs (the deck transformation group of such coverings is cyclic and acts transitively). This allows to multiply uncolorable cubic graphs and obtain on this way a wide class of snarks (including Isaac's flowers, Goldberg snarks and many other known uncolorable graphs). However the cyclic covering method repeats more or less the other well known constructions of snarks. The real meaning of Theorem 5 is that it also provides some nonstandard tools for obtaining larger uncolorable cubic graphs starting from smaller ones. The corresponding procedures are given by special coverings of cubic graphs and described below in Examples 2.1, 2.2 and 2.3 and Propositions 7–10. They can be considered as new operations on uncolorable cubic graphs. Moreover under certain conditions, they allow to obtain new cyclically 4-connected and cyclically 6-connected cubic graphs.

2.2. Examples

Before describing examples of coverings, we will introduce some needed notions and prove auxiliary assertions.

Definition. A connected cubic graph G is called *cyclically k -edge connected* if no

set of fewer than k edges is cycle-separating in G . The edge cyclic connectivity $\zeta(G)$ of the cubic graph G is the largest integer $k \leq \beta(G)$ for which G is cyclically k -edge connected where $\beta(G)$ denotes the Betti number of G .

For any cubic connected graph G we have obviously $\kappa(G) = \lambda(G) \leq \zeta(G)$ where $\kappa(G)$ and $\lambda(G)$ denote the vertex and the edge connectivity of G , respectively. For cubic graphs G with $\zeta(G) \leq 3$ the values of vertex connectivity, edge connectivity and cyclic k -edge connectivity coincide (see [12]). Moreover, with the exception of graphs $K_{3,3}, K_4$ and θ , the conditions " G is cyclically k -vertex connected" and " G is cyclically k -edge connected" coincide for connected cubic graphs [10]. For this reason, in the following we usually use the term " G is cyclically k -connected" in both the cases. Note also that if G is cyclically 4-connected and E is an edge cut of G consisting of three edges, then there is a vertex v of G such that all these edges are incident to v . We will use this fact in the future.

Let G_1 and G_2 be two 3-connected cubic graphs. Take in G_1 a pair of independent edges (e_1, e_2) , and in G_2 a pair of independent edges (f_1, f_2) . Denote the vertices of e_1 by u_1, u_2 , and the vertices of e_2 by v_1, v_2 , respectively. Similarly, let s_1, s_2 be the vertices of f_1 , and t_1, t_2 the vertices of f_2 . Remove from G_1 the edges e_1 and e_2 , and from G_2 the edges f_1 and f_2 . Add to the graph $G = (G_1 - e_1 - e_2) \sqcup (G_2 - f_1 - f_2)$ the edges joining the following pairs of vertices: u_1 and s_1 , u_2 and s_2 , v_1 and t_1 , and v_2 and t_2 , respectively. Denote the resulting connected graph $G_1 * G_2$ and call it a *double connected sum* of G_1 and G_2 (see Figure 3).

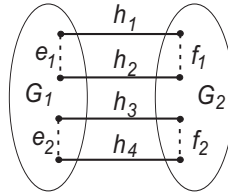


Figure 3. Double connected sum of graphs G_1 and G_2 .

Lemma 6. *Let G_1 and G_2 be two cubic graphs, e_1, e_2 a pair of independent edges in G_1 and f_1, f_2 a pair of independent edges in G_2 . If G_1 and G_2 are cyclically 4-connected, then $G_1 * G_2$ is also cyclically 4-connected.*

Proof. Let E be the minimal edge cut in $G_1 * G_2$ and let $G_1 * G_2 \setminus E = H_1 \sqcup H_2$ be the decomposition of $G_1 * G_2 \setminus E$ into two components H_1 and H_2 , where the component H_1 contains a cycle C_1 and the component H_2 contains a cycle C_2 . Now we have to consider several cases describing possible positions of cycles C_1 and C_2 in $G_1 * G_2 \setminus E$.

Case 1. C_1 and C_2 are both contained in G_1 or G_2 . Assume that C_1 and C_2 are contained in G_1 . Note that E contains at least two edges of G_1 since $\zeta(G) \geq 4$. Put $E' = E \cap (G_1 - e_1 - e_2)$. E' is a cut set of $G_1 - e_1 - e_2$ that separates C_1 from C_2 . Suppose that $|E'| = 2$. Then the vertices u_1 and u_2 belong to different connected component of $(G_1 - e_1 - e_2) \setminus E'$, the same as the vertices v_1 and v_2 . For instance, let $u_1, v_1 \in V(H_1)$ and $u_2, v_2 \in V(H_2)$. Put $W = (G_2 - f_1 - f_2) \cup \{h_1, h_2, h_3, h_4\}$. Now, to separate in W the pair of vertices u_1, v_1 from the pair u_2, v_2 , we need to remove from W at least two bridge edges or two edges of the graph $G_2 - f_1 - f_2$. Therefore $|E| \geq 4$, which contradicts the assumption.

Suppose that $|E'| = 3$. Then the vertices u_1 and u_2 belong to different connected components of $(G_1 - e_1 - e_2) \setminus E'$ or the vertices v_1 and v_2 have such a property. Without loss of generality, suppose that the first possibility occurs. However in order to separate u_1 from u_2 in the graph W we need to remove at least one bridge edge or two edges of the graph $G_2 - f_1 - f_2$. Therefore E contains at least one extra edge and so $|E| \geq 4$.

Case 2. C_1 is contained in $G_1 - e_1 - e_2$ and C_2 is contained in $G_2 - f_1 - f_2$. If E contains all four bridge edges, then $|E| \geq 4$ and we are done. Assume that there is a component H_i containing a bridge edge of $G_1 * G_2$, say $h_1 = (u_1, s_1)$. If H_{3-i} also contains a bridge edge of $G_1 * G_2$, we need at least four edges to separate H_1 from H_2 in $G_1 * G_2$, two in G_1 and two in G_2 . It follows that $|E| \geq 4$.

Now consider the remaining subcase, i.e. H_{3-i} is contained in G_1 or in G_2 . Without loss of generality, we may suppose that H_{3-i} is contained in G_1 . Assume that $|E| \leq 3$. Note that E must contain at least two edges from $G_1 - e_1 - e_2$, in order to separate the subgraph $H_i \cap G_1$ from the subgraph H_{3-i} . If $|E| = 2$, then the pair of vertices u_1 and u_2 would belong to different components H_i and H_{3-i} , the same as the pair of vertices v_1, v_2 . It follows that at least one bridge edge h_i connects H_i with H_{3-i} in $G_1 * G_2 \setminus E$, which is impossible. Consider now the subcase $|E| = 3$.

Suppose that E consists of edges from G_1 . In this case, we have that the ends of the edge e_1 (i.e. the vertices u_1 and u_2) or the ends of the edge e_2 belong to distinct components H_1 and H_2 . Then there is at least one bridge edge joining H_i to H_{3-i} in $G_1 * G_2 \setminus E$, which is impossible. Therefore E contains a unique bridge edge h_j of $G_1 * G_2 \setminus E$. It follows that the remaining three bridge edges $h_k, k \neq j$, belong to H_i . Therefore the end vertices of the edge e_1 or the end vertices of the edge e_2 belong to H_i . But the latter implies that G_1 can be divided into two components by a cut consisting of three independent edges, which contradicts the condition that $\zeta(G_1) \geq 4$.

Case 3. The circle C_1 contains the bridge edges h_1 and h_2 (so the vertices u_1 and u_2) and the circle C_2 contains the bridge edges h_3 and h_4 (so the vertices v_1 and v_2). Now, to separate $H_1 \cap (G_2 - f_1 - f_2)$ from $H_2 \cap (G_2 - f_1 - f_2)$ we need

at least two edges from $G_2 - f_1 - f_2$. Similarly, to separate $H_1 \cap (G_1 - e_1 - e_2)$ from $H_2 \cap (G_1 - e_1 - e_2)$ we need at least two edges from $G_1 - e_1 - e_2$. Totally to separate H_1 from H_2 in $G_1 * G_2$ we need at least four edges, so $|E| \leq 4$.

Case 4. The circle C_1 contains the bridge edges h_1 and h_3 and the circle C_2 contains the bridge edges h_2 and h_4 . This case can be done just in the same way as Case 3.

Case 5. C_1 contains the only bridge edges h_1 and h_2 and C_2 is contained either in $G_2 - f_1 - f_2$ or $G_1 - e_1 - e_2$. For instance, suppose that $C_2 \subset G_2 - f_1 - f_2$. The cycle C_1 can be represented as follows: $C_1 = (h_1, a_1, h_2, a_2)$ where a_1 is a path in $G_1 - e_1 - e_2$ joining u_1 to u_2 and a_2 is a path in $G_2 - f_1 - f_2$ joining s_1 to s_2 . The path $p = (h_1, a_1, h_2)$ can be considered as a subdivision of the removed edge f_1 in G_2 . It follows that in order to separate C_2 from $C_1 \cap (G_2 - f_1 - f_2)$ in $G_2 - f_1 - f_2$ we need to remove at least three edges from $G_2 - f_1 - f_2$. Let E' be the corresponding cut set in $G_2 - f_1 - f_2$ where $E' \subset E$. Let $U_1 \sqcup U_2$ be the decomposition of $(G_2 - f_1 - f_2) \setminus E'$ into two connected components where $U_1 \subset H_1$ and $U_2 \subset H_2$. If $|E'| \geq 4$, then $|E| \geq 4$ and we are done. If $|E'| = 3$, then the end vertices of the edge f_2 should be at different components U_1 and U_2 . It follows that in order to separate H_1 from H_2 in $G_1 * G_2$, we should remove from the graph $(G_2 - f_1 - f_2) + h_3 + h_4$ at least one extra edge h_i . But this contradicts the last assumption.

Case 6. C_1 contains the only bridge edges h_1 and h_3 and C_2 is contained either in $G_1 - e_1 - e_2$ or $G_2 - f_1 - f_2$. For instance, suppose first that i.e., that $C_2 \subset G_1 - e_1 - e_2$. Put $E' = E \cap (G_1 - e_1 - e_2)$. E' is a cut set of $G_1 - e_1 - e_2$. Assume first that H_1 contains both the vertices u_2 and v_2 . Then E' contains at least four edges, so we have $|E| \geq 4$. Suppose that H_2 contains both the vertices u_2 and v_2 . Then E' contains at least two edges. Moreover, in order to separate H_1 from H_2 we have to remove from $G_1 * G_2$ two extra edges of the subgraph $(G_2 - f_1 - f_2) + h_2 + h_4$. It follows that totally E must contain at least four edges. Now suppose that u_2 is a vertex of H_1 and v_2 is a vertex of H_2 . Then E' contains at least three edges. Moreover, in order to separate H_1 from H_2 we have to remove from $G_1 * G_2$ one extra edge of the subgraph $(G_2 - f_1 - f_2) + h_4$. It follows that E consists of at least four edges.

Case 7. C_1 contains all bridge edges h_1, h_2, h_3 and h_4 , and C_2 is contained in $G_1 - e_1 - e_2$. Put $E' = E \cap (G_1 - e_1 - e_2)$. E' is a cut set of $G_1 - e_1 - e_2$ as before. Therefore all the vertices u_1, u_2, v_1 and v_2 are contained in the component H_1 . Since G_1 is cyclically 4-connected, the cut set E' consists of at least four edges, so $|E| \geq 4$.

We have counted all possible cases of positions the cycles C_1 and C_2 in the graph $G_1 * G_2$ and have seen that in any case the minimum set cut E consists of at least four edges. ■

Example 2. Let G be a connected uncolorable cubic graph and let e, f be two independent edges of G such that $G - e - f$ is connected. Cutting the edges e and f in G , we shall obtain a connected 4-pole L with two pairs of semi-edges, e' and e'' , and f' and f'' , respectively. Then either L does not have any 4-flow or L admits a nowhere-zero flow ξ in $\mathbf{Z}_2 \times \mathbf{Z}_2$ with the following property.

(*) ξ has the only nontrivial sources at four semi-edges, i.e., $\xi(e') = x, \xi(e'') = y$ and $\xi(f') = x, \xi(f'') = y$ or $\xi(e') = x, \xi(e'') = y$ and $\xi(f'') = x, \xi(f') = y$ where $x, y \neq 0$ and $x \neq y$.

Note that the condition (*) simply means that $r(G) = 2$.

Take an orientation of the edges of the graph G and denote the resulting orgraph by \vec{G} . Let $\beta: \vec{G} \rightarrow \Sigma_5$ be a permutation voltage assignment defined in the following way: $\beta(e) = (12345)$ and $\beta(f) = (13524)$ and $\beta(h) = (1)(2)(3)(4)(5)$ for any other arc h of the \vec{G} . The voltage permutation graph (G, β) defines the 5-fold covering map $p: G^\beta \rightarrow G$ with the covering graph G^β being connected.

Proposition 7. *The covering $p: G^\beta \rightarrow G$ is regular and the cubic graph G^β is uncolorable. Moreover if G is cyclically 4-connected, then G^β is also cyclically 4-connected.*

Proof. First note that the set of arcs $E' = \{e, f\}$ satisfies the conditions (i) and (ii) of Theorem 5. It follows that the 20-multipole $p^{-1}(L)$ is decomposed into 5 disjoint copies L_i of the 4-pole L .

Let e_1, \dots, e_5 be the lifts of the edge e and f_1, \dots, f_5 the lifts of the edge f via the covering map p . Moreover let e'_1, \dots, e'_5 and e''_1, \dots, e''_5 be the lifts of semi-edges e' and e'' , respectively, and f'_1, \dots, f'_5 and f''_1, \dots, f''_5 be the lifts of semi-edges f' and f'' , respectively. The covering graph G^β can be obtained in the following way. Take the five copies L_1, L_2, \dots, L_5 of the multipole L . Then identify the 5 pairs of semi-edges e'_i and e''_j according to the permutation $\beta(e) = (12345)$ and the 5 pairs of semi-edges f'_k and f''_t according to the permutation $\beta(f) = (13524)$ (see Figure 4). Identifying the first five pairs of semi-edges results in the edges e_1, e_2, \dots, e_5 and the second five pairs of semi-edges results in the edges f_1, \dots, f_5 of the graph G^β .

The deck transformation group of the covering $p: G^\beta \rightarrow G$ is \mathbf{Z}_5 which acts on G^β transitively. More precisely, the generator 1 of \mathbf{Z}_5 shifts the edge e_i to the edge e_{i+1} and the edge f_j to the edge f_{j+1} for each $i, j = 1, \dots, 5$. Moreover 1 permutes the components L_i of $p^{-1}(L)$ cyclically. It follows that p is a regular 5-fold covering of connected topological graphs.

If L does not have any 4-flow it follows immediately that G^β is uncolorable. If L admits a nowhere zero $\mathbf{Z}_2 \times \mathbf{Z}_2$ -flow, one can directly check that the associated hypergraph $H(E', p)$ does not contain the hyperedge $\{L_1, L_2, L_3, L_4, L_5\}$. In the other words, there is no consistent 3-coloring of the 4-poles L_i , $i = 1, \dots, 5$,

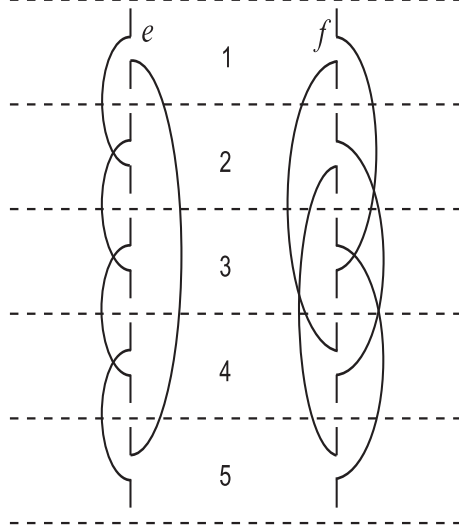


Figure 4. The graph G^β obtained by gluing the five copies of the 4-pole L .

with the set of colors $\{(1, 0), (0, 1), (1, 1)\}$. The last fact depends strongly on the property $(*)$ of the 4-pole L .

The second statement of the proposition can be proved by induction with using Lemma 6. Let us consider on the graph G the following permutation assignment $\gamma: \vec{G} \rightarrow \Sigma_4$, $\gamma(e) = (1234)$ and $\gamma(f) = (1324)$ and $\gamma(g) = (1)(2)(3)(4)$ for any other arc g of the \vec{G} . The voltage permutation graph (G, γ) defines the 4-fold covering $r: G^\gamma \rightarrow G$ with the connected cubic graph G^γ . It is not difficult to see that G^β can be represented as a double connected sum of G^γ and a copy of the graph G , where the distinguished edges in G are e and f and the distinguished edges in G^γ are lifts e_i and f_j of the edges e and f via the covering map r . Since both the permutations $\gamma(e) = (1234)$ and $\gamma(f) = (1324)$ are cyclic we can continue this process and decompose G^γ into double connected sum of four copies of the the graph G . By inductive assumption, G^γ is cyclically 4-edge connected. Since $G^\beta = G^\gamma * G$, by Lemma 6, G^β is also cyclically 4-edge connected. ■

Below we describe examples of irregular 5-fold coverings of connected uncolorable cubic graphs.

Example 3. Let G be a connected uncolorable cubic graph and let $E' = \{e, f, h\}$ be a set of independent edges of G such that $G - \{e, f, h\}$ is connected. Cutting the edges e, f and h in G , we shall obtain a connected 6-pole L with two pairs of semi-edges, e' and e'' , f' and f'' , and h' and h'' , respectively.

Take an orientation of the edges of the graph G and denote the resulting orgraph by \vec{G} . Let $\alpha: \vec{G} \rightarrow \Sigma_5$ be a permutation voltage assignment defined in

the following way: $\alpha(e) = (123)(4)(5)$, $\alpha(f) = (1)(2)(345)$, $\alpha(h) = (1245)(3)$, and $\alpha(g) = (1)(2)(3)(4)(5)$ for any other arc g of the \vec{G} . The voltage permutation graph (G, α) defines the 5-fold covering map $q: G^\alpha \rightarrow G$ with the covering graph G^α being connected.

Proposition 8. *The covering $q: G^\alpha \rightarrow G$ is irregular and the cubic graph G^α is uncolorable.*

Proof. First note that the set of arcs $E' = \{e, f, h\}$ satisfies the conditions (i) and (ii) of Theorem 5 since e, f, h are the unique arcs of \vec{G} where the voltage assignment is a non-identical permutation. It follows that the 30-pole $p^{-1}(L)$ is decomposed into 5 disjoint copies L'_i of the 6-pole L . Let e_1, \dots, e_5 be the lifts of the edge e , f_1, \dots, f_5 the lifts of the edge f and h_1, \dots, h_5 the lifts of the edge h via the covering map p . Moreover let e'_1, \dots, e'_5 and e''_1, \dots, e''_5 be the lifts of semi-edges e' and e'' , respectively, and f'_1, \dots, f'_5 and f''_1, \dots, f''_5 be the lifts of semi-edges f' and f'' , h'_1, \dots, h'_5 and h''_1, \dots, h''_5 be the lifts of semi-edges h' and h'' , respectively. Identifying in $p^{-1}(L)$ the pair of semi-edges e'_4 and e''_4 , e'_5 and e''_5 , f'_1 and f''_1 , f'_2 and f''_2 , h'_3 and h''_3 , we shall obtain the multipole L'' consisting of five connected components, the 4-poles L_1, L_2, L_3, L_4 , and L_5 . Each component L_i has the property (*). Moreover the covering graph G^α can be obtained as follows. We identify in L'' three pairs of semi-edges e'_i and e''_j according to the permutation $\alpha(e) = (123)(4)(5)$, three pairs of semi-edges f'_k and f''_t according to the permutation $\alpha(f) = (1)(2)(345)$ and four pairs of semi-edges h'_k and h''_l according to the permutation $\alpha(h) = (1245)(3)$ (see Figure 5).

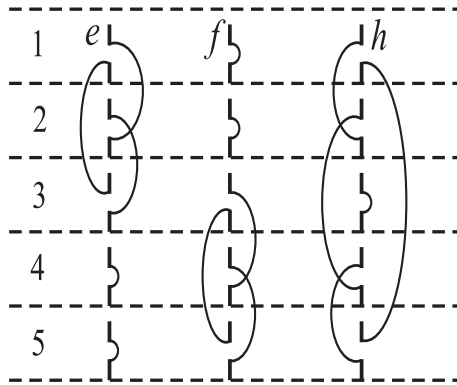


Figure 5. The graph G^α obtained by gluing the multipoles L_1, L_2, L_3, L_4 and L_5 .

The deck transformation group of the covering map $q: G^\alpha \rightarrow G$ is trivial, so q is irregular.

If L does not have any 4-flow it follows immediately that G^α is uncolorable. Assume that L admits a nowhere zero $\mathbf{Z}_2 \times \mathbf{Z}_2$ -flow. Since all 4-poles

$L_i, i = 1, \dots, 5$, have the property (*), one can easily check that the hypergraph $H(E', q)$ associated with the set of edges E' does not contain the hyperedge $\{L_1, L_2, L_3, L_4, L_5\}$. It follows that there is no consistent way to color the 20-pole L'' with the colors $\{(1, 0), (0, 1), (1, 1)\}$. ■

Example 4. Let G be a connected uncolorable cubic graph with $r(G) \geq 3$ and let e, f, h be a set of independent edges in G such that the graph $G - \{e, f, h\}$ is connected. Cut the edges e, f and h of G in internal points. As a result, we obtain a connected 6-pole M with corresponding pairs of semi-edges, e' and e'' , f' and f'' , h' and h'' , respectively. Since $r(G) \geq 3$, cutting any two of the edges e, f and h in G results in an uncolorable 4-pole. Then either (1) M does not admit 3-edge coloring, or (2) M has a 3-edge coloring ξ with the following combination of nontrivial colors $x, y, z \in \mathbf{Z}_2 \times \mathbf{Z}_2, x, y, z \neq 0$ at three pairs of semi-edges.

(**) $\xi(e') = x, \xi(e'') = y$ and $\xi(f') = y, \xi(f'') = z$ and $\xi(h') = z, \xi(h'') = x$,

or any other combination obtained from the given one by permutation of colors in corresponding pairs of semi-edges.

Note that in Case 2 the condition (**) means that $r(G) = 3$. For example, the following distribution of colors is admissible in this case: $\xi(e') = x, \xi(e'') = y$ and $\xi(f'') = z, \xi(f') = y$ and $\xi(h') = z, \xi(h'') = x$.

Let G' be an orientation of the graph G . Now consider the 5-fold covering of the graph G given by the permutation voltage assignment $\varphi: E(G') \rightarrow \Sigma_5$ as follows (see Figure 6). Put $\varphi(e) = (12345), \varphi(f) = (153)(24), \varphi(h) = (142)(35)$. For the remaining edges g put $\varphi(g) = (1)(2)(3)(4)(5)$.

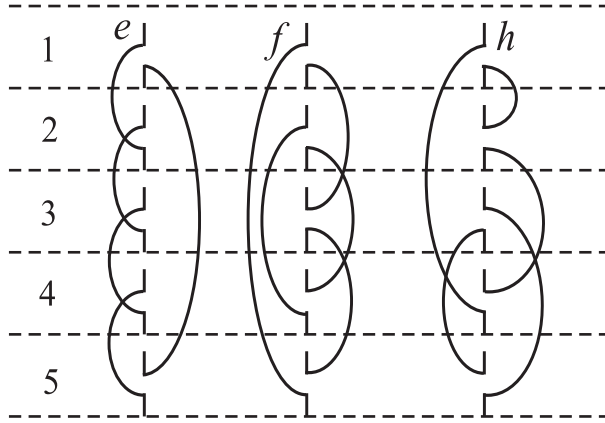


Figure 6. Obtaining the graph G^φ by gluing the five copies of the 6-pole M .

Proposition 9. *The covering $p: G^\varphi \rightarrow G$ is irregular and the cubic graph G^φ is uncolorable.*

Proof. Since $G - \{e, f, h\}$ is connected, the set of edges $E' = \{e, f, h\}$ satisfies the condition (i). Since e, f and h are the only edges with nontrivial voltage permutation assignment, E' also satisfies the condition (ii). It follows that the multipole $p^{-1}(M)$ is decomposed into 5 disjoint components M_i each of which is isomorphic to the 6-pole M .

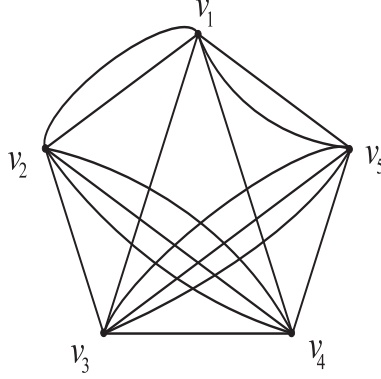
Let e_1, \dots, e_5 be the lifts of the edge e , f_1, \dots, f_5 be the lifts of the edge f and h_1, \dots, h_5 be the lifts of the edge h via the covering map p . Moreover let e'_1, \dots, e'_5 and e''_1, \dots, e''_5 the lifts of semi-edges e' and e'' , respectively, f'_1, \dots, f'_5 and f''_1, \dots, f''_5 the lifts of semi-edges f' and f'' , respectively, and h'_1, \dots, h'_5 and h''_1, \dots, h''_5 the lifts of semi-edges h' and h'' , respectively. The covering graph G^φ can be obtained as follows. Take the five copies M_1, M_2, \dots, M_5 of the multipole M . Then identify the 5 pairs of semi-edges e'_i and e''_j according to the permutation $\varphi(e) = (12345)$, the 5 pairs of semi-edges f'_k and f''_t according to the permutation $\varphi(f) = (153)(24)$ and the 5 pairs of semi-edges h'_k and h''_t according to the permutation $\varphi(h) = (142)(35)$ (see Figure 5). Identifying the first five pairs of semi-edges results in the edges e_1, e_2, \dots, e_5 , the second five pairs of semi-edges results in the edges f_1, \dots, f_5 and the third five pairs of semi-edges results in the edges h_1, \dots, h_5 of the graph G^φ . It is easy to see that there is a unique homeomorphism of G^φ which preserves the fibers of p , the identical map. Therefore the deck transformation group of the covering $p: G^\varphi \rightarrow G$ is trivial, so p is irregular.

If M does not have any 4-flow, then it follows immediately that G^φ is an uncolorable graph. If M has a $\mathbf{Z}_2 \times \mathbf{Z}_2$ -flow, then each 6-pole M_i also has a $\mathbf{Z}_2 \times \mathbf{Z}_2$ -flow and satisfies the property (**). Now, by counting all possible subcases, it is not difficult to check that there is no consistent 3-coloring of the multipole $p^{-1}(M) = \bigsqcup_{i=1}^5 M_i$ with the colors $(1,0), (0,1)$ and $(1,1)$. It follows that the associated hypergraph $H(E', p)$ does not contain the hyperedge $\{M_1, M_2, M_3, M_4, M_5\}$. The property (**) of the multipoles M_i , $i = 1, \dots, 5$, is essential here. ■

Proposition 10. *Let $p: G^\varphi \rightarrow G$ be the covering map defined in Example 4. If the graph G is cyclically 6-connected, and the edges e, f and h are contained in three disjoint circles of G , then G^φ is also cyclically 6-connected cubic graph.*

Proof. We start by considering an auxiliary multigraph H which describes connections between different multipoles M_i in the graph G^φ . For this, we constrict in G^φ all vertices and edges of each multipole M_i into one vertex v_i and remain only edges which connect different multipoles, i.e., the lifts of the edges e, f and h . The resulting multigraph H is regular of degree 6 and of order 5 (see Figure 7).

Let E be a minimal edge cut of the graph G^φ so that the disconnected graph $G^\varphi \setminus E$ is decomposed into two components S and T and both of them contain

Figure 7. The multigraph H .

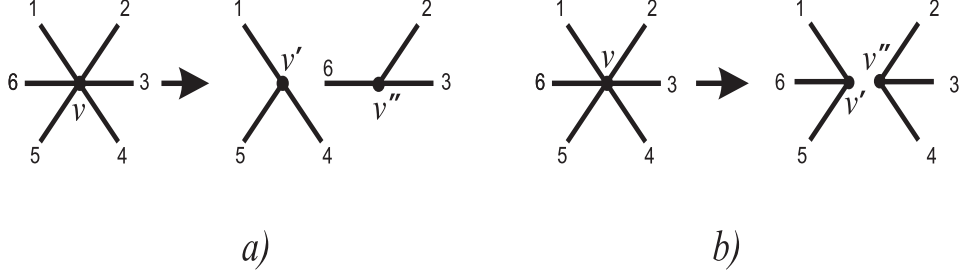
a cycle. Note that each M_i is cyclically 3-connected. Denote by M'_i the graph obtained from the multipole M_i , $i = 1, \dots, 5$, by removing all its semi-edges. Now we have to analyze the following cases.

Case 1. S and T have common vertices in two or more graphs M'_i . To separate $S \cap M'_i$ from $T \cap M'_i$ we need at least 3 edges, so totally E contains at least six edges.

Case 2. S and T have common vertices in a unique graph, say M'_j , and both S and T contain vertices of the other multipoles. By the assumption, E consists of several edges of the subgraph M'_j and lifts of the edges e, f and h only. The part of E consisting of edges of the second type is denoted by E' . The collapsing process does not influence the edges of E' , so we may identify the set E' with the set of corresponding edges of the multigraph H . Moreover, after collapsing the subgraphs M'_i , the graphs S and T will transform into multigraphs S' and T' which have a unique common vertex v_j and such that $|S'| \geq 2$ and $|T'| \geq 2$. Separating $S \cap M'_i$ from $T \cap M'_i$ leads to splitting in H the vertex v_j into two vertices v'_j and v''_j . There are $2^6 - 2$ possible splittings of a vertex of degree 6. In Figure 8 we indicate only two examples of such splittings. The resulting multigraph is denoted by H' . Since the subgraphs S and T are separated in G , the corresponding multigraphs S' and T' are also separated via E' in the multigraph H' . Now the task is to estimate the number $|E'|$.

Claim 11. *Let H' be a multigraph obtained from H by splitting a vertex v_j . Moreover, let E' be any edge cut of H' such that S' contains v'_j and T' contains v''_j , and such that $|S'| \geq 2$ and $|T'| \geq 2$. Then $|E'| \geq 3$.*

Proof. We have to consider five cases, according to a vertex of H in which splitting is performed.

Figure 8. Splitting a vertex v of the multigraph H : two types.

As an example, consider a multigraph H' obtained from H by splitting the vertex v_5 . Let E' be an edge cut in H' satisfying conditions of the claim. Suppose contrary that $|E'| < 3$. By assumption, in the multigraph H' we have the following: $v'_5 \in S'$ and $v''_5 \in T'$. Since $|S'| \geq 2$, we may suggest that $v_1 \in S'$. Assume that $v_2 \in T'$. We thus have two multiedges of E' that join v_1 to v_2 . Since $|E'| \leq 2$, v_3 and v_4 should belong to S' . However in this case we have additionally three edges from E' joining v_2 to v_4 and one edge from E' joining v_2 to v_3 , which contradicts our assumption. Therefore v_2 must be a vertex of S' . Since there are three multiedges joining v_2 and v_4 , by assumption, we have $v_4 \in S'$. Since $|T'| \geq 2$, the vertex v_3 should belong to T' . But the vertex v_3 is joining to the vertices v_2, v_1 and v_4 in H' having the color S , which also contradicts the assumption that $|E'| \leq 2$.

The remaining four cases are performing in a similar way. We omit here complete checking and remain it to the reader as an easy exercise. \square

It follows from Claim 11 that $|E'| \geq 3$ and so we have $|E| \geq 6$ in Case 2.

Case 3. S and T have common vertices in a unique subgraph, say M'_j , and T has no vertices of any other subgraph $M'_i, i \neq j$. In this case, the subgraph T is completely contained in M'_j . To continue, note that since the edges e, f and h are contained in disjoint cycles, there are disjoint paths p'_e, p'_f and p'_h in the graph $G - \{e, f, h\}$, joining the end vertices of the edges e, f and h , respectively. Let u_j, u'_j be the lifts of the end vertices of the edge e , v_j, v'_j the lifts of the end vertices of the edge f and w_j, w'_j the lifts of the end vertices of the edge h , respectively, in M'_j . Note that $\varphi(p'_e) = \varphi(p'_f) = \varphi(p'_h) = \text{id}$, since for each edge g of the subgraph $G - \{e, f, h\}$, we have $\varphi(g) = \text{id} \in \Sigma_5$. It follows that the paths p'_e, p'_f and p'_h lift to disjoint paths p_e, p_f and p_h of the graph G^φ which join the pair of vertices u_j, u'_j and v_j, v'_j and w_j, w'_j , respectively. Therefore the graph G^φ contains the subgraph G' which is homeomorphic to G (we replace here the edges e, f and h with the paths p_e, p_f and p_h , respectively). Moreover, there are disjoint paths q_e, q_f and q_h in G^φ joining the vertices u_j, u'_j , and v_j, v'_j , and w_j, w'_j , respectively, and having no other common vertices with the graph M'_j . For example, q_e looks

as follows: $(u_j, e_j, u'_{j_1}, p_e^{j_1}, u_{j_1}, e_{j_2}, p_e^{j_2}, \dots, e_{j_k}, u_{j_k}, p_e^{j_k}, u'_{j_k})$, where $j_k = j$ and each path $p_e^{j_s}$ is a lift of the path p'_e which starts at the end vertex u'_{j_s} of the edge e_{j_s} and ends at the vertex u_{j_s} . In the other words, q_e consists of the consecutive lifts of the edge e and the path p'_e .

Let $E' = E \cap M'_j$. Then $|E'| \geq 3$ as E' separates $S \cap M'_j$ from $T \cap M'_j$ and $\lambda(M'_j) \geq 3$. If $|E'| = 3$, the vertices in each pair $\{u_j, u'_j\}$, $\{v_j, v'_j\}$, and $\{w_j, w'_j\}$ belong to distinct sets S and T . Therefore to separate these pairs of vertices in the graph G^φ we need to remove at least three edges lying on the paths q_e, q_f and q_h , respectively. Then we have $|E| \geq 6$.

Assume that $|E'| = 4$. Then at least two of the pairs $\{u_j, u'_j\}$, $\{v_j, v'_j\}$, and $\{w_j, w'_j\}$ have their elements in distinct sets S and T . For instance, let $u_j, v_j \in S$, and $u'_j, v'_j \in T$. In this case, to separate these pairs of vertices in G^φ we need to remove two edges, one lying on the path q_e and the other one lying on the path q_f . Then we have $|E| \geq 6$.

Assume that $|E'| = 5$. Then at least one of the pairs $\{u_j, u'_j\}$, $\{v_j, v'_j\}$, and $\{w_j, w'_j\}$ has its elements in distinct sets S and T . For instance, let $u_j \in S$ and $u'_j \in T$. Now, to separate the vertices u_j, u'_j in G^φ we need to remove at least one edge of the path q_e . We thus also have $|E| \geq 6$ in the last case.

Case 4. S' and T' have no common vertex. It follows that E consists entirely of edges that are lifts of e, f and h . As a consequence, the edges of E and E' correspond each to other, and E' is an edge cut of the multigraph H . Let S' and T' be the two components of the multigraph $H \setminus E'$, as before. The case when $|S'| = 1$ and $|T'| = 4$ is trivial since H' is a regular multigraph of degree six, and so $|E'| = 6$ in that case. The case $|S'| = 2$ and $|T'| = 3$ is decomposing into ten subcases according to the partitions of the vertex set $V(H')$ into two sets S' and T' . For example, consider the subcase $S' = \{v_1, v_2\}$ and $T' = \{v_3, v_4, v_5\}$. We have eight edges in H' joining the sets S' and T' , so $|E'| = 8$ in that case. Checking the remaining nine cases is also immediate and shows that $|E'| \geq 6$. This completes the proof of the assertion. ■

Note that the graph G^α from Example 3 is not cyclically 6-connected (actually $\zeta(G^\alpha) \leq 4$).

3. COVERINGS OF CUBIC GRAPHS AND RESISTANCE

In [13], Steffen introduced the parameter $r(G)$ of an uncolorable cubic graph G without bridges. It measures how far G is from being 3-edge colorable and is called the *resistance* of G . More precisely, $r(G) = \min\{|F| : F \subset E(G) \text{ such that } G - F \text{ is 3-edge colorable}\}$ (here we slightly modify the original definition of the parameter $r(G)$ but in an equivalent form). This parameter is related to another

measure of non-colorability, the *oddness* $\omega(G)$ of G , which is the smallest possible number of odd circuits in 2-factors of G (see [6, 13]). In particular, $r(G) \leq \omega(G)$ for any cubic graph G .

It is not difficult to see that the number $r(G)$ is equal to the minimal number of edges in the cubic graph G , say e_1, \dots, e_k , such that cutting all them in interior points results in a $2k$ -pole which has a 4-flow (with sources in the semi-edges).

It follows directly from definitions that an analogue of Proposition 7 holds true for uncolorable cubic graphs G with $r(G) \geq 3$ and for an arbitrary choice of cyclic permutations $\beta(e)$ and $\beta(f)$ in Σ_n with $n \geq 2$.

Let us consider several examples of snarks and indicate their resistance.

Example 5. Let P be the Petersen graph, and P^3 the third power of P pictured in Figure 1. In Figure 9, it is shown the snark \mathcal{G}_{26} of order 26 embedded in a torus (see [11]). By direct computation, we have $r(P) = 2$, $r(P^3) = 2$ and $r(\mathcal{G}_{26}) = 2$.

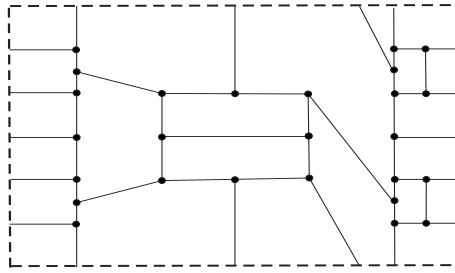


Figure 9. The snark \mathcal{G}_{26} embedded in a torus.

The following theorem allows to construct uncolored graphs G with an arbitrary value of resistance.

Theorem 12. *Let G be a connected bridgeless uncolored cubic graph with $r(G) = k$. Let (G, μ) be a permutation voltage graph with an assignment $\mu: E(\vec{G}) \rightarrow \Sigma_n$ in the symmetric group Σ_n , G^μ the corresponding covering graph and let $E = \{e_1, e_2, \dots, e_l\}$ be a subset of edges of G with $l \leq k - 1$. Assume that E satisfies the following conditions:*

- (i) *the graph $H = G - E$ is connected;*
- (ii) *for each oriented cycle c in the graph H we have $\mu(c) = e$ where e is the trivial permutation in Σ_n .*

Then the bridgeless cubic graph G^μ is uncolored. Moreover $r(G^\mu) \geq (k - l)n$.

Proof. The fact that G^μ is bridgeless follows from the fact that G is so and the covering is finite. First assume that the covering graph G^μ is connected. Let L be the $2l$ -pole obtained from G by cutting the edges e_1, \dots, e_l from E in interior

points. It follows from (i) and (ii) (see the proof of Theorem 5) that the multipole $p^{-1}(L)$ is decomposed into n disjoint (isomorphic) copies L_i of the multipole L . Moreover the covering graph G^μ can be obtained from multipoles L_1, \dots, L_n by identifying the corresponding pairs of their semi-edges in accordance with the permutation values $\mu(e_i)$ for $e_i \in E$. It follows that the graph $p^{-1}(H)$ is decomposed into n disjoint components H_1, H_2, \dots, H_n each of which is isomorphic to H . It is clear that each graph H_i is obtained from the multipole L_i by removing all its semi-edges.

Suppose that $r(G^\mu) = t < (k-l)n$. Then there are edges e'_1, \dots, e'_t of G^μ such that the graph $U = G^\mu - \{e'_1, \dots, e'_t\}$ is 3-edge colorable. Let φ be the corresponding 3-edge coloring of U . Then φ descends obviously to a proper 3-edge coloring φ_i of each subgraph $U_i = H_i - \{e'_1, \dots, e'_t\}$ where $i = 1, \dots, n$. Since U_i is 3-edge colorable and $r(G) = k \geq 2$, it follows that the graph U_i is obtained from the graph H_i by removing at least $k-l$ edges, $i = 1, \dots, n$. This means that U is obtained from G by eliminating at least $(k-l)n$ edges, i.e., $r(G^\mu) \geq (k-l)n$, contradicting our assumption.

If G^μ is disconnected, we can restrict the covering map $p: G^\mu \rightarrow G$ to each connected component of G^μ and then argue in the same way as in the first case. Now the assertion follows. ■

Note that by Proposition 4, for any n -fold covering $p: G^\mu \rightarrow G$ of the cubic graph G with $r(G) = k$, we have $r(G^\mu) \leq kn$.

Corollary 13. *Let G be a connected bridgeless uncolorable cubic graph with $r(G) = k$ and \vec{G} be an orientation of G . Moreover, let (G, μ) be a voltage graph with a voltage assignment $\mu: E(\vec{G}) \rightarrow A$ in a finite group A of order m such that μ takes the only nontrivial values at l arcs of \vec{G} with $l \leq k-1$. Then the covering cubic graph G^μ is an uncolorable graph with $r(G^\mu) \geq (k-l)m$.*

Proof. Let $E = \{e_1, e_2, \dots, e_l\}$ be the edges of G with nontrivial values of the voltage assignment μ . If the graph $G - \{e_1, e_2, \dots, e_l\}$ is connected, the assertion is a direct consequence of Theorem 12. If $G - \{e_1, e_2, \dots, e_l\}$ is disconnected, we can replace the set E with a smaller subset $E' \subset E$ such that $G - E'$ is connected and the condition (iii) of Theorem 12 is satisfied. Now the assertion follows from the proof of Theorem 12. ■

The estimation of the parameter $r(G^\mu)$ of the covering graph G^μ , given in Theorem 12, can be improved in particular cases, when the subset E of edges of the graph G is specified. We illustrate this by the following example.

Example 6. Consider the cubic graph H_2 which is depicted in Figure 10. This uncolorable graph is due to [8]. In [8], it was shown that H_2 is a unique smallest uncolorable graph with oddness 4 and with edge-cyclic connectivity 3. The order

of H_2 is equal to 28. The given graph is obtained by gluing together three copies of the 3-pole P_3 [8], where the multipole P_3 is shown in Figure 11. Note that the multipole P_3 is uncolorable.

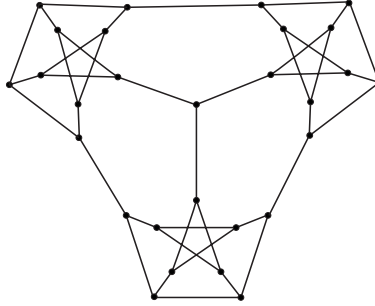


Figure 10. The cubic graph H_2 .

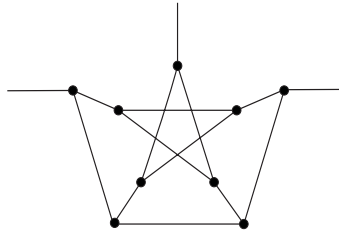


Figure 11. The multipole P_3 .

It is not difficult to show that $r(H_2) = 3$. We distinguish in H_2 three edges, e, f and g , the edges that join two different copies of the multipole P_3 in the graph H_2 . Consider the 5-fold covering map $p: H_2^\beta \rightarrow H_2$ defined via the permutation voltage assignment $\beta: E(\vec{H}_2) \rightarrow \Sigma_5$ as follows: $\beta(e) = (12345)$, $\beta(f) = (153)(24)$, $\beta(g) = (142)(35)$, and $\beta(h) = (1)(2)(3)(4)(5)$ for any other edge h of the graph H_2 . It is clear that $r(H_2^\beta) \geq 3 \cdot 5$, since in order to obtain an uncolored (subcubic) graph from H_2^β we have to remove at least one edge in each copy P_3^i , $i = 1, 2, \dots, 15$, of the 3-pole P_3 . Since the number $r(H_2^\beta)$ cannot exceed $3 \cdot 5$, it follows that $r(H_2^\beta) = 15$. Note also that $\zeta(H_2^\beta) = 3$.

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