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GENERALIZED SUM LIST COLORINGS OF GRAPHS

Arnfried Kemnitz

Massimiliano Marangio

Computational Mathematics
Technical University Braunschweig
Universitätsplatz 2, 38 106 Braunschweig, Germany

e-mail: a.kemnitz@tu-bs.de
m.marangio@tu-bs.de

and

Margit Voigt

Faculty of Information Technology and Mathematics
University of Applied Sciences
Friedrich-List-Platz 1, 01 069 Dresden, Germany

e-mail: mvoigt@informatik.htw-dresden.de

Abstract

A (graph) property $\mathcal{P}$ is a class of simple finite graphs closed under isomorphisms. In this paper we consider generalizations of sum list colorings of graphs with respect to properties $\mathcal{P}$.

If to each vertex v of a graph G a list $L(v)$ of colors is assigned, then in an $(L, \mathcal{P})$-coloring of G every vertex obtains a color from its list and the subgraphs of G induced by vertices of the same color are always in $\mathcal{P}$. The $\mathcal{P}$-sum choice number $\chi_{sc}^{\mathcal{P}}(G)$ of G is the minimum of the sum of all list sizes such that, for any assignment L of lists of colors with the given sizes, there is always an $(L, \mathcal{P})$-coloring of G.

We state some basic results on monotonicity, give upper bounds on the $\mathcal{P}$-sum choice number of arbitrary graphs for several properties, and determine the $\mathcal{P}$-sum choice number of specific classes of graphs, namely, of all complete graphs, stars, paths, cycles, and all graphs of order at most 4.

Keywords: sum list coloring, sum choice number, generalized sum list coloring, additive hereditary graph property.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$, and for every vertex $v \in V$ let $L(v)$ be a set (list) of available colors. The graph G is called *L-colorable* if there is a proper coloring c of the vertices with $c(v) \in L(v)$ for all $v \in V$. A function f from the vertex set V of G to the positive integers is called a *choice function* of G if G is L-colorable for every list assignment L with $|L(v)| = f(v)$ for all $v \in V$. If the list length of all vertices coincide then this is the ordinary list colorability. The *sum choice number* $\chi_{sc}(G)$ denotes the minimum of $\sum_{v\in V} f(v)$ over all choice functions f of G. Since the considered colorings are proper, vertices of the same color induce an edgeless graph.

Sum list colorings were introduced by Isaak in 2002 [7]. Results on the sum choice number can be found, e.g., in [1, 2, 6–9, 11].

In this paper we examine a generalization of this concept. We consider vertex colorings such that the graphs induced by the vertices of the same color belong to some specific given class of graphs (and not necessarily to the class of edgeless graphs).

A *(graph) property* $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$, where $\mathcal{I}$ denotes the class of all finite simple graphs (see [3]). We assume in the entire paper that $K_1 \in \mathcal{P}$ for the considered properties $\mathcal{P}$. A property $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$ are disjoint where G and H are two graphs of $\mathcal{I}$. A property $\mathcal{P}$ is called *hereditary* (*induced hereditary*) if $G \in \mathcal{P}$ and $H \subseteq G$ ($H \leq G$) implies $H \in \mathcal{P}$, where $H \subseteq G$ ($H \leq G$) means that H is a subgraph (an induced subgraph) of G. Therefore, every hereditary property is also induced hereditary. Obviously, $K_1 \in \mathcal{P}$ for any (induced) hereditary property $\mathcal{P}$.

The graph G is called *$(L, \mathcal{P})$-colorable* if there exists a mapping (coloring) $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for each vertex $v \in V(G)$ and, for each $i \in \mathbb{N}$, the graph induced in G by the vertices colored i belongs to $\mathcal{P}$. Such a mapping is called an *$(L, \mathcal{P})$-coloring* or a *$\mathcal{P}$-list coloring* of G.

Let $f : V(G) \to \mathbb{N}$ be a function which assigns *list sizes* to the vertices of G. The graph G is *$(f, \mathcal{P})$-choosable* and f is a *$\mathcal{P}$-choice function* of G if for every list assignment L with list sizes specified by f, that is, $|L(v)| = f(v)$ for each $v \in V(G)$, the graph G is $(L, \mathcal{P})$-colorable. The *$\mathcal{P}$-sum choice number* $\chi_{sc}^{\mathcal{P}}(G)$ of a graph G is the minimum of the sum of list sizes in f taken over all $\mathcal{P}$-choice functions f of G. Thus

$$\chi_{sc}^{\mathcal{P}}(G) = \min\left\{ \sum_{v\in V(G)} f(v) : f \text{ is a } \mathcal{P}\text{-choice function of } G \right\}.$$

We use the following standard notation for specific graph properties.

$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$,
$\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of G has at most $k+1$ vertices$\}$,
$\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\}$,
$\mathcal{D}_k = \{G \in \mathcal{I} :$ each subgraph of G contains a vertex of degree $\leq k\}$,
$\mathcal{O}^k = \{G \in \mathcal{I} : \chi(G) \leq k\}$,
$\mathcal{J}_k = \{G \in \mathcal{I} : \chi'(G) \leq k\}$,
$\mathcal{I}_k = \{G \in \mathcal{I} : G$ does not contain $K_{k+2}\}$.

All these properties are additive induced hereditary properties.

Note that $\mathcal{O}_k \subseteq \mathcal{S}_k \subset \mathcal{D}_k \subset \mathcal{O}^{k+1} \subset \mathcal{I}_k$ for $k \geq 1$ (see [3]).

The *completeness* $c(\mathcal{P})$ of an induced hereditary property $\mathcal{P}$ is defined as $c(\mathcal{P}) = \max\{k : K_{k+1} \in \mathcal{P}\}$; we write $c(\mathcal{P}) = \infty$ if the maximum does not exist. For example, $c(\mathcal{P}) = 0$ if and only if $\mathcal{P} \subseteq \mathcal{O}$, $c(\mathcal{I}) = \infty$, and $c(\mathcal{O}_k) = c(\mathcal{S}_k) = c(\mathcal{D}_k) = c(\mathcal{O}^{k+1}) = c(\mathcal{I}_k) = k$, as well as $c(\mathcal{J}_k) = k$ if k is odd and $c(\mathcal{J}_k) = k-1$ if k is even. Moreover, if $\mathcal{P}$ is an additive hereditary property with $c(\mathcal{P}) = k$, then $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{I}_k$ (see [3]).

The $\mathcal{P}$-sum choice number is a generalization of the usual sum choice number since $\chi_{sc}^{\mathcal{O}}(G) = \chi_{sc}(G)$ for all graphs G. This concept was introduced in [4]. In [4,5] the $\mathcal{P}$-sum choice number for induced hereditary properties $\mathcal{P}$ was studied, especially for $\mathcal{P} = \mathcal{D}_1$, that is, for the class of acyclic graphs. In [10] lower and upper bounds on $\chi_{sc}^{\mathcal{P}}(G)$ are given for arbitrary induced hereditary properties $\mathcal{P}$ where G is the union of two graphs with exactly one vertex in common.

This paper is organized as follows. In Section 2 we collect some basic results, most of them from the literature. In Section 3 we present upper bounds on the $\mathcal{P}$-sum choice number for arbitrary graphs and specific additive induced hereditary properties $\mathcal{P}$, namely $\mathcal{D}_k$, $\mathcal{I}_k$, $\mathcal{J}_k$, $\mathcal{O}_k$, $\mathcal{O}^k$, and $\mathcal{S}_k$. Moreover, Theorem 10 contains a general upper bound for all additive hereditary properties and Theorem 14 for all additive properties. In Section 4 we determine the $\mathcal{P}$-sum choice number of some known classes of graphs including complete graphs, stars, paths, cycles, and all graphs of order at most 4 for arbitrary additive induced hereditary properties $\mathcal{P}$.

2. Preliminaries

In this section we state some basic results.

Proposition 1. *Let $\mathcal{P}, \mathcal{Q}$ be arbitrary properties. If $\mathcal{P} \subseteq \mathcal{Q}$, then $\chi_{sc}^{\mathcal{Q}}(G) \leq \chi_{sc}^{\mathcal{P}}(G)$.*

Proof. Each $(L, \mathcal{P})$-coloring of G is also an $(L, \mathcal{Q})$-coloring of G since each graph in $\mathcal{P}$ is contained in $\mathcal{Q}$. This implies that each $\mathcal{P}$-choice function of G is a $\mathcal{Q}$-choice function of G, hence $\chi_{sc}^{\mathcal{Q}}(G) \leq \chi_{sc}^{\mathcal{P}}(G)$. ■

The following proposition collects some bounds that can be found in [4] or deduced from some results there.

Proposition 2 [4]**.** *Let $\mathcal{P}$ be a hereditary (an induced hereditary) property and $H \subseteq G$ ($H \leq G$). Then $\chi_{sc}^{\mathcal{P}}(G) \geq \chi_{sc}^{\mathcal{P}}(H) + \chi_{sc}^{\mathcal{P}}(G - V(H)) \geq \chi_{sc}^{\mathcal{P}}(H) + |V(G)| - |V(H)|$.*

A direct implication is the following result.

Corollary 3. *If $\mathcal{P}$ is an induced hereditary property and $V(G) = V_1 \cup \cdots \cup V_l$ is a partition of the vertex set of a graph G, then $\chi_{sc}^{\mathcal{P}}(G) \geq \sum_{i=1}^{l} \chi_{sc}^{\mathcal{P}}(G[V_i])$.*

Proof. The result follows by iterative application of Proposition 2 on the induced subgraphs $G[V_1], \ldots, G[V_{l-1}]$. ■

For $\mathcal{P} = \mathcal{O}$ (that is, for the sum choice number) the lower bound can be improved to $\chi_{sc}^{\mathcal{O}}(G) \geq \sum_{i=1}^{l} \chi_{sc}^{\mathcal{O}}(G[V_i]) + l - c(G)$ where $c(G)$ is the number of components of G (see [6, 9]).

In the proof in [9], $l - c(G)$ edges that induce bridges (that is, blocks K_2) were added to the subgraph $G[V_1] \cup \cdots \cup G[V_l]$, and each bridge increases the sum choice number by $\chi_{sc}(K_2) - 2 = 3 - 2 = 1$. Since $\chi_{sc}^{\mathcal{P}}(K_2) = 2$ for $\mathcal{P} \neq \mathcal{O}$, the larger subgraph does not increase the lower bound on the $\mathcal{P}$-sum choice number if $\mathcal{P} \neq \mathcal{O}$.

The following result is proved in [4] using some hypergraph method.

Proposition 4 [4]**.** *Let $\mathcal{P}$ be an additive induced hereditary property. If $G = F \cup H$ is the disjoint union of F and H, then $\chi_{sc}^{\mathcal{P}}(G) = \chi_{sc}^{\mathcal{P}}(F) + \chi_{sc}^{\mathcal{P}}(H)$.*

Proof. Let $f : V(G) \to \mathbb{N}$ be a function such that $f|_{V(F)}$ and $f|_{V(H)}$ are $\mathcal{P}$-choice functions of F and H, respectively. Let L be a list assignment of G with sizes determined by f. An $(L|_{V(F)}, \mathcal{P})$-coloring of F and an $(L|_{V(H)}, \mathcal{P})$-coloring of H provide an $(L, \mathcal{P})$-coloring of $G = F \cup H$ since for each color i the subgraphs of F and of H induced by vertices of color i are disjoint and contained in $\mathcal{P}$, hence their union, that is, the corresponding induced subgraph of $G = F \cup H$, is also in $\mathcal{P}$ since $\mathcal{P}$ is additive. This implies $\chi_{sc}^{\mathcal{P}}(G) \leq \chi_{sc}^{\mathcal{P}}(F) + \chi_{sc}^{\mathcal{P}}(H)$. Equality holds by Proposition 2. ■

This result implies that the $\mathcal{P}$-sum choice number of a graph is equal to the sum of the $\mathcal{P}$-sum choice numbers of its components for additive induced hereditary properties $\mathcal{P}$.

Corollary 5. *Let $\mathcal{P}$ be an additive induced hereditary property. If G has c components $H_1, \ldots, H_c$, then $\chi_{sc}^{\mathcal{P}}(G) = \chi_{sc}^{\mathcal{P}}(H_1) + \cdots + \chi_{sc}^{\mathcal{P}}(H_c)$.*

3. Upper Bounds for Specific Properties

In this section we present upper bounds on $\chi_{sc}^{\mathcal{P}}(G)$ for arbitrary graphs $G = (V, E)$ and specific additive (induced) hereditary properties $\mathcal{P}$.

The *greedy bound* $\mathrm{GB}(G) = |V| + |E|$ is an upper bound on the sum choice number $\chi_{sc}(G) = \chi_{sc}^{\mathcal{O}}(G)$, and obviously $\chi_{sc}^{\mathcal{I}}(G) = |V|$ holds (each vertex obtains a list of size 1). Since $\mathcal{O} \subseteq \mathcal{P} \subseteq \mathcal{I}$ for any additive property $\mathcal{P}$, we have $|V| \leq \chi_{sc}^{\mathcal{P}}(G) \leq |V| + |E|$ by Proposition 1.

The next result states an upper bound on the $\mathcal{D}_1$-sum choice number proved in [4]. We present a simple direct proof.

Theorem 6 [4]**.** $\chi_{sc}^{\mathcal{D}_1}(G) \leq |E| + c(G)$*, where* $c(G)$ *is the number of components of* G*.*

Proof. If G is connected, then order the vertices $V = \{v_1, \ldots, v_n\}$ in such a way that v_i, $i \geq 2$, is connected to v_j, $1 \leq j \leq i-1$. Set $f(v_1) = 1$ and $f(v_i) = |\{e \in E : e = v_j v_i, j < i\}| \geq 1$ for $i \geq 2$.

Then $\sum_{i=1}^{n} f(v_i) = |E| + 1$. We prove that f is a $\mathcal{D}_1$-choice function of G by a greedy $(L, \mathcal{P})$-coloring of $v_1, \ldots, v_n$ for an arbitrary list assignment L with list sizes defined by f. Obviously, v_1 can be colored. Assume that $v_1, \ldots, v_{i-1}$ are colored and consider v_i, $i \geq 2$. If all neighbors of v_i in $\{v_1, \ldots, v_{i-1}\}$ are colored distinctly, then v_i can be colored by any color from its list since neither adding an isolated vertex nor a pending edge to an acyclic graph does create a cycle. If at least two neighbors have the same color, then there is a color in $L(v_i)$ not used in any neighbor of v_i which can be used to color v_i, and we are done.

If G is not connected, then use Corollary 5 and apply the preceding result for all components of G. ■

Note that this bound can be improved (see Theorem 7), but it is also tight in some specific cases. For example, it holds that $\chi_{sc}^{\mathcal{D}_1}(C_n) = n + 1 = |E(C_n)| + 1$ (see Theorem 19).

The following result can be deduced from Corollary 9 in [4]. We give a direct proof instead without using hypergraph methods.

Theorem 7. *Let* $v_1, \ldots, v_n$ *be an arbitrary ordering of the vertices of* G *and* $G_i = G[\{v_1, \ldots, v_i\}]$*,* $i \in \{1, \ldots, n\}$*. Then*

$$\chi_{sc}^{\mathcal{D}_k}(G) \leq n + \sum_{i=1}^{n} \left\lfloor \frac{d_{G_i}(v_i)}{k+1} \right\rfloor \leq |V| + \frac{|E|}{k+1}.$$

Proof. Define $f : V \to \mathbb{N}$ by $f(v_i) = 1 + \left\lfloor \frac{d_{G_i}(v_i)}{k+1} \right\rfloor$, $i \in \{1, \ldots, n\}$. Then $\sum_{i=1}^{n} f(v_i) = n + \sum_{i=1}^{n} \left\lfloor \frac{d_{G_i}(v_i)}{k+1} \right\rfloor \leq n + \frac{1}{k+1} \sum_{i=1}^{n} d_{G_i}(v_i) = |V| + \frac{1}{k+1} |E|$.

We prove in the following that f is a $\mathcal{D}_k$-choice function of G. Let L be a list assignment with $|L(v)| = f(v)$ for every $v \in V$. Vertex v_1 can be colored with the color from its list. Assume that vertices $v_1, \ldots, v_{i-1}$ are already colored in a partial $(L, \mathcal{D}_k)$-coloring of G, and consider the next vertex v_i. If at most k neighbors of v_i are colored by a color $\alpha \in L(v_i)$, then v_i can also be colored by α since the subgraph C_α of G_i induced by vertices of color α is k-degenerate: Each subgraph of C_α without v_i has a vertex of degree at most k because of the assumed coloring, and if v_i is a vertex of the subgraph, then v_i is a vertex of degree at most k. This means that at most $\left\lfloor \frac{d_{G_i}(v_i)}{k+1} \right\rfloor$ colors cannot be used for v_i, but $L(v_i)$ has at least one color which is not forbidden. Therefore, the $(L, \mathcal{D}_k)$-coloring of G can be completed and f is a $\mathcal{D}_k$-choice function of G. ■

If $\mathcal{D}_k \subseteq \mathcal{P}$, then $\chi_{sc}^{\mathcal{P}}(G) \leq \chi_{sc}^{\mathcal{D}_k}(G)$ by Proposition 1, thus the upper bound of Theorem 7 is also an upper bound on the $\mathcal{P}$-sum choice number of G. Since $\mathcal{D}_k \subseteq \mathcal{O}^{k+1} \subseteq \mathcal{I}_k$ we obtain the following bounds.

Corollary 8. *Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of G and $G_i = G[\{v_1, \ldots, v_i\}]$, $i \in \{1, \ldots, n\}$. Then*

$$\chi_{sc}^{\mathcal{O}^k}(G) \leq n + \sum_{i=1}^{n} \left\lfloor \frac{d_{G_i}(v_i)}{k} \right\rfloor \leq |V| + \frac{|E|}{k}.$$

Note that for $\mathcal{O}^1 = \mathcal{O}$ Corollary 8 gives the greedy bound of G: $\chi_{sc}^{\mathcal{O}^1}(G) = \chi_{sc}(G) \leq \mathrm{GB}(G) = |V| + |E|$.

Corollary 9. *Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of G and $G_i = G[\{v_1, \ldots, v_i\}]$, $i \in \{1, \ldots, n\}$. Then*

$$\chi_{sc}^{\mathcal{I}_k}(G) \leq n + \sum_{i=1}^{n} \left\lfloor \frac{d_{G_i}(v_i)}{k+1} \right\rfloor \leq |V| + \frac{|E|}{k+1}.$$

Let us mention that it is possible to generalize these results and prove that $f : V \to \mathbb{N}$ with $f(v_i) = 1 + \left\lfloor \frac{d_{G_i}(v_i)}{d(\mathcal{P},G)} \right\rfloor$ is a $\mathcal{P}$-choice function of G for an appropriate divisor $d(\mathcal{P}, G)$. Note that $d(\mathcal{P}, G) = 1$ leads to a choice function f with sum of list sizes equal to the greedy bound $\mathrm{GB}(G)$ which is indeed an upper bound on the $\mathcal{P}$-choice number of G. In Corollary 9 in [4] a divisor $d(\mathcal{P}, G) = \delta(\mathcal{P})$ was used, that is, the smallest minimum degree of a minimal forbidden graph of $\mathcal{P}$ (which is a graph not contained in $\mathcal{P}$ whose proper induced subgraphs are all in $\mathcal{P}$). Obviously, it suffices to consider just subgraphs of G.

The following result provides a general upper bound on $\chi_{sc}^{\mathcal{P}}(G)$.

Theorem 10. *Let $\mathcal{P}$ be an additive hereditary property, $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of G, $G_i = G[\{v_1, \ldots, v_i\}]$ for $i \in \{1, \ldots, n\}$, and $k = c(\mathcal{P})$. Then*

$$\chi_{sc}^{\mathcal{P}}(G) \le n + \sum_{i=1}^{n} \min \left\{ d_{G_i}(v_i), \left\lfloor \frac{i-1}{k+1} \right\rfloor \right\}.$$

Proof. Define $f : V(G) \to \mathbb{N}$ by $f(v_i) = 1 + \min \left\{ d_{G_i}(v_i), \left\lfloor \frac{i-1}{k+1} \right\rfloor \right\}$, $i \in \{1, \ldots, n\}$. Then $\sum_{i=1}^{n} f(v_i) = n + \sum_{i=1}^{n} \min \left\{ d_{G_i}(v_i), \left\lfloor \frac{i-1}{k+1} \right\rfloor \right\}$ as stated.

Let L be a list assignment with list sizes defined by f. Vertex v_1 can be colored with the color from its list of size $f(v_1) = 1$. Assume that vertices $v_1, \ldots, v_{i-1}$ are already colored in a partial $(L, \mathcal{P})$-coloring of G, and consider next the vertex v_i.

If $f(v_i) = 1 + d_{G_i}(v_i)$, then we can color v_i by a color distinct from the colors of all of its already colored $d_{G_i}(v_i)$ neighbors. Thus v_i belongs to none of the so far existing components induced by vertices of the same color. Let now $f(v_i) = 1 + \left\lfloor \frac{i-1}{k+1} \right\rfloor$. If at most k vertices in $v_1, \ldots, v_{i-1}$ are colored with a color $\alpha \in L(v_i)$, then v_i can also be colored by α since all subgraphs of K_{k+1} are contained in the hereditary property $\mathcal{P}$. Hence at most $\left\lfloor \frac{i-1}{k+1} \right\rfloor$ colors are forbidden for vertex v_i, but $L(v_i)$ contains at least one additional color which can be used to color v_i.

This implies that the $(L, \mathcal{P})$-coloring of G can be inductively completed, and therefore f is a $\mathcal{P}$-choice function of G. ■

Note that for $\mathcal{P} \supseteq \mathcal{D}_k$ the degree $d_{G_i}(v_i)$ can be replaced by $\left\lfloor \frac{d_{G_i}(v_i)}{k+1} \right\rfloor$. Since $d_{G_i}(v_i) \le i - 1$, we obtain the upper bound of Theorem 7.

Corollary 11. *Let $\mathcal{P}$ be an additive hereditary property, G be a graph with n vertices, and $k = c(\mathcal{P})$. Then*

$$\chi_{sc}^{\mathcal{P}}(G) \le \chi_{sc}^{\mathcal{O}_k}(G) \le n + \sum_{i=1}^{n} \left\lfloor \frac{i-1}{k+1} \right\rfloor.$$

Proof. The first inequality follows from Proposition 1 since $\mathcal{O}_k \subseteq \mathcal{P}$, the second by Theorem 10 since $c(\mathcal{O}_k) = k$. ■

For complete graphs equality holds in Theorem 10 and Corollary 11 (see Theorem 15).

The *square* G^2 of a graph G is the graph with $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if the distance between u and v in G is at most 2.

Theorem 12. *Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of G and $G_i = G[\{v_1, \ldots, v_i\}]$ for $i \in \{1, \ldots, n\}$. Then*

$$\chi_{sc}^{\mathcal{S}_k}(G) \leq n + \sum_{i=1}^{n} \left\lfloor \frac{d_{{G_i}^2}(v_i)}{k+1} \right\rfloor \leq |V(G)| + \frac{\left|E(G^2)\right|}{k+1}.$$

Proof. Define $f : V(G) \to \mathbb{N}$ by $f(v_i) = 1 + \left\lfloor \frac{d_{{G_i}^2}(v_i)}{k+1} \right\rfloor$, $i \in \{1, \ldots, n\}$. Then

$$\sum_{i=1}^{n} f(v_i) = n + \sum_{i=1}^{n} \left\lfloor \frac{d_{{G_i}^2}(v_i)}{k+1} \right\rfloor \leq n + \sum_{i=1}^{n} \left\lfloor \frac{d_{G^2[\{v_1,\ldots,v_i\}]}(v_i)}{k+1} \right\rfloor$$
$$\leq n + \frac{1}{k+1} \sum_{i=1}^{n} d_{G^2[\{v_1,\ldots,v_i\}]}(v_i) = |V(G)| + \frac{1}{k+1} \left|E(G^2)\right|.$$

The first inequality follows from $G_i = G[\{v_1, \ldots, v_i\}]$ which implies that two vertices at distance at most 2 in G_i have also distance at most 2 in G, that is, ${G_i}^2 \subseteq G^2[\{v_1, \ldots, v_i\}]$.

We prove in the following that f is an $\mathcal{S}_k$-choice function of G. Let L be a list assignment with list sizes defined by f. Vertex v_1 can be colored with the color from its list. Assume that $v_1, \ldots, v_{i-1}$ are already colored in a partial $(L, \mathcal{S}_k)$-coloring of G and consider the next vertex v_i, $i \in \{2, \ldots, n\}$. A color $\alpha \in L(v_i)$ is forbidden for v_i if either v_i is adjacent to at least $k+1$ vertices of color α in G_i, or if v_i is adjacent to a vertex v_j of color α, $j < i$, which is adjacent to at least k vertices of color α. In either case, at least $k+1$ vertices of $N_{{G_i}^2}(v_i)$ must be already colored with α in order to forbid this color for v_i. This implies that at most $\left\lfloor \frac{1}{k+1} d_{{G_i}^2}(v_i) \right\rfloor$ different colors are forbidden, hence v_i can be colored with a color from $L(v_i)$, and the $(L, \mathcal{S}_k)$-coloring of G can be inductively completed. ■

If $\mathcal{S}_k \subseteq \mathcal{P}$, then $\chi_{sc}^{\mathcal{P}}(G) \leq \chi_{sc}^{\mathcal{S}_k}(G)$ by Proposition 1, thus the upper bound of Theorem 12 is also an upper bound on the $\mathcal{P}$-sum choice number of G. This improves the upper bound on $\chi_{sc}^{\mathcal{P}}(G)$ from Corollary 11 if $\mathcal{S}_k \subseteq \mathcal{P}$ since $d_{{G_i}^2}(v_i) \leq i-1$. For example, the Theorem of Vizing states that $\chi'(G) \leq \Delta(G)+1$, that is, $\Delta(G) \leq k-1$ implies $\chi'(G) \leq k$. Therefore, $\mathcal{S}_{k-1} \subseteq \mathcal{J}_k$ and $\chi_{sc}^{\mathcal{J}_k}(G) \leq \chi_{sc}^{\mathcal{S}_{k-1}}(G)$ by Proposition 1. From Theorem 12 we obtain the following bounds.

Corollary 13. *Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of G, $G_i = G[\{v_1, \ldots, v_i\}]$ for $i \in \{1, \ldots, n\}$, and $k \geq 1$. Then*

$$\chi_{sc}^{\mathcal{J}_k}(G) \leq n + \sum_{i=1}^{n} \left\lfloor \frac{d_{{G_i}^2}(v_i)}{k} \right\rfloor \leq |V(G)| + \frac{\left|E(G^2)\right|}{k}.$$

In [9] an upper bound on the sum choice number of G was proved that depends on a partition $V(G) = V_1 \cup \cdots \cup V_l$ of the vertex set of G and the sum choice numbers of the induced subgraphs $G[V_i]$, $i = 1, \ldots, l$. The bound can be generalized as follows.

Theorem 14. *If $\mathcal{P}$ is an additive property and $V(G) = V_1 \cup \cdots \cup V_l$ is a partition of $V(G)$, then*

$$\chi_{sc}^{\mathcal{P}}(G) \leq \sum_{i=1}^{l} \chi_{sc}^{\mathcal{P}}(G[V_i]) + |E(G)| - \sum_{i=1}^{l} |E(G[V_i])| \,.$$

Proof. For $i \in \{1, \ldots, l\}$ let $f_i : V_i \to \mathbb{N}$ be a $\mathcal{P}$-choice function of $G[V_i]$ with $\sum_{v \in V_i} f_i(v) = \chi_{sc}^{\mathcal{P}}(G[V_i])$. Define $f : V(G) \to \mathbb{N}$ as follows:

$$f(v) = f_i(v) + |N(v) \cap (V_1 \cup \cdots \cup V_{i-1})| \text{ for } v \in V_i,\ i \in \{1, \ldots, l\}.$$

Consider an arbitrary list assignment L with $|L(v)| = f(v)$ for each vertex $v \in V(G)$. Color at first the vertices of V_1 which is possible since $f|_{V_1} = f_1$ and f_1 is a $\mathcal{P}$-choice function of $G[V_1]$. Assume that all vertices of $V_1 \cup \cdots \cup V_{i-1}$ are colored by a partial $(L, \mathcal{P})$-coloring φ of G and consider next the set V_i, $i \in \{2, \ldots, l\}$.

A vertex $v \in V_i$ will be colored distinctly from the previously colored neighbors, that is, only the colors of $L_i(v) = L(v) \setminus \{\varphi(w) : w \in N(v) \cap (V_1 \cup \cdots \cup V_{i-1})\}$ will be used. Since $|L_i(v)| \geq f_i(v)$ for all $v \in V_i$ and f_i is a $\mathcal{P}$-choice function of $G[V_i]$, each vertex $v \in V_i$ can be colored with a color from $L_i(v) \subseteq L(v)$. The coloring is a partial $(L, \mathcal{P})$-coloring of G since $\mathcal{P}$ is additive.

This implies that f is a $\mathcal{P}$-choice function of G with

$$\begin{aligned} \sum_{v \in V(G)} f(v) &= \sum_{i=1}^{l} \chi_{sc}^{\mathcal{P}}(G[V_i]) + |E(G) \setminus E(G[V_1] \cup \cdots \cup G[V_l])| \\ &= \sum_{i=1}^{l} \chi_{sc}^{\mathcal{P}}(G[V_i]) + |E(G)| - \sum_{i=1}^{l} |E(G[V_i])| \,. \end{aligned}$$

■

4. SPECIFIC GRAPH CLASSES

In this section we determine the $\mathcal{P}$-sum choice number of some well-known classes of graphs for arbitrary additive induced hereditary properties $\mathcal{P}$. We begin with complete graphs whose $\mathcal{P}$-sum choice numbers only depend on the complete graphs contained in $\mathcal{P}$, that is, on the completeness $c(\mathcal{P})$ of $\mathcal{P}$. The proof is similar to the proof for the determination of the sum choice number $\chi_{sc}(K_n)$ in [8]. In fact, the following theorem is a generalization of this result.

Theorem 15. *Let* $b(n,k) = \sum_{i=1}^{n} \left(1 + \left\lfloor \frac{i-1}{k+1} \right\rfloor\right)$ *for* $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and* $\mathcal{P}$ *be an induced hereditary property. If* $c(\mathcal{P}) = k$, *then* $\chi_{sc}^{\mathcal{P}}(K_n) = b(n,k)$.

Proof. The proof of Theorem 10 implies $\chi_{sc}^{\mathcal{P}}(K_n) \le b(n,k)$ if $G = K_n$ since $d_{G_i}(v_i) = i-1$. We only need to require that $\mathcal{P}$ is induced hereditary, since the subgraphs of G induced by vertices of the same color are also complete and thus connected induced subgraphs.

Consider an arbitrary $\mathcal{P}$-choice function f of K_n and denote the vertices of K_n in increasing order with respect to f, $f(v_1) \le \cdots \le f(v_n)$. Assume that there is a vertex v_j, $1 \le j \le n$, with $f(v_j) < 1 + \left\lfloor \frac{j-1}{k+1} \right\rfloor$. Since $f(v_j) \ge 1$, $j-1 \ge k+1$. Let L be the list assignment with initial lists, $L(v_i) = \{1, \ldots, f(v_i)\}$ for each $i \in \{1, \ldots, n\}$. Then in any $(L, \mathcal{P})$-coloring the vertices in $V' = \{v_1, \ldots, v_j\}$ will be colored by at most $q = \left\lfloor \frac{j-1}{k+1} \right\rfloor \ge 1$ colors $1, \ldots, q$. By the pigeonhole principle, there is a color $\alpha \in \{1, \ldots, q\}$ used in at least $\left\lceil \frac{j}{q} \right\rceil$ vertices of V'. Let r be the integer $0 \le r \le k$ with $j - 1 = q(k+1) + r$. Then $\left\lceil \frac{j}{q} \right\rceil = \left\lceil \frac{q(k+1)+r+1}{q} \right\rceil = k + 1 + \left\lceil \frac{r+1}{q} \right\rceil > k+1$, that is, the graph induced by the vertices of color α is a complete graph with more than $k+1$ vertices, a contradiction to $c(\mathcal{P}) = k$. It follows that $f(v_i) \ge 1 + \left\lfloor \frac{i-1}{k+1} \right\rfloor$ for every $i \in \{1, \ldots, n\}$ and therefore $\chi_{sc}^{\mathcal{P}}(K_n) \ge b(n,k)$. ■

In the following proposition we compute $b(n,k)$.

Proposition 16. *For* $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ *let* $n = q(k+1) + r$ *with* $q, r \in \mathbb{N}_0$, $r \le k$. *Then* $b(n,k) = \frac{1}{2}(q+1)(n+r) = \frac{1}{2}\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)\left(2n - \left\lfloor \frac{n}{k+1} \right\rfloor (k+1)\right)$.

Proof. Let $n = q(k+1) + r$ with $0 \le r \le k$. Then

$$\begin{aligned} b(n,k) &= \sum_{i=1}^{n} \left(1 + \left\lfloor \frac{i-1}{k+1} \right\rfloor\right) = n + \sum_{i=1}^{q(k+1)} \left\lfloor \frac{i-1}{k+1} \right\rfloor + \sum_{i=q(k+1)+1}^{q(k+1)+r} \left\lfloor \frac{i-1}{k+1} \right\rfloor \\ &= n + (k+1)\sum_{j=0}^{q-1} j + rq = n + \tfrac{1}{2}(k+1)(q-1)q + rq \\ &= \tfrac{1}{2}(k+1)q(q+1) + r(q+1) = \tfrac{1}{2}(q+1)(n+r) \\ &= \tfrac{1}{2}\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)\left(2n - \left\lfloor \frac{n}{k+1} \right\rfloor (k+1)\right). \end{aligned}$$

■

For example, if $\mathcal{P} = \mathcal{O}$, then $c(\mathcal{O}) = k = 0$ and therefore $\chi_{sc}^{\mathcal{O}}(K_n) = \chi_{sc}(K_n) = \frac{1}{2}(n+1)n = n + \binom{n}{2} = |V(K_n)| + |E(K_n)|$ (see [8]). For properties $\mathcal{P}$ with $c(\mathcal{P}) = k = 1$ we obtain $\chi_{sc}^{\mathcal{P}}(K_n) = \frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(2n - 2\left\lfloor \frac{n}{2} \right\rfloor\right)$, that is,

$\chi_{sc}^{\mathcal{P}}(K_n) = \frac{1}{4}n(n+2)$ if n even and $\chi_{sc}^{\mathcal{P}}(K_n) = \frac{1}{4}(n+1)^2$ if n odd. This generalizes Theorem 31 of [4] on $\chi_{sc}^{\mathcal{D}_1}(K_n)$.

In the next theorems stars, paths, and cycles are considered. Their $\mathcal{P}$-sum choice number again only depends on the connected induced subgraphs contained in $\mathcal{P}$.

Theorem 17. *Let $\mathcal{P}$ be an additive induced hereditary property. If $m \in \mathbb{N}$ and $s = \max\{k : k \leq m$ and $K_{1,k} \in \mathcal{P}\}$ for $\mathcal{P} \neq \mathcal{O}$ and $s = 0$ for $\mathcal{P} = \mathcal{O}$, then $\chi_{sc}^{\mathcal{P}}(K_{1,m}) = m + 1 + \lfloor \frac{m}{s+1} \rfloor$.*

Proof. Let $V = \{z, v_1, \ldots, v_m\}$ be the vertex set of $K_{1,m}$ such that z has degree m. Define $f : V \to \mathbb{N}$ by $f(v_i) = 1$ for $i \in \{1, \ldots, m\}$ and $f(z) = 1 + \lfloor \frac{m}{s+1} \rfloor$. We prove that f is a $\mathcal{P}$-choice function of $K_{1,m}$. Consider an arbitrary list assignment L with list sizes defined by f. Each vertex v_i must be colored with the color from its own list $L(v_i)$ for each $i \in \{1, \ldots, m\}$ which is possible since $\mathcal{P}$ is additive. Each color which is used to color at most s vertices v_i can be used to color z since $K_{1,s} \in \mathcal{P}$ and $\mathcal{P}$ is induced hereditary which implies that also all substars are in $\mathcal{P}$. Therefore, at most $\lfloor \frac{m}{s+1} \rfloor$ colors are forbidden for z which implies that z can be colored with a color from its list. Therefore, f is a $\mathcal{P}$-choice function of $K_{1,m}$ and $\chi_{sc}^{\mathcal{P}}(K_{1,m}) \leq \sum_{v \in V} f(v) = m + 1 + \lfloor \frac{m}{s+1} \rfloor$.

Consider now an arbitrary $\mathcal{P}$-choice function f of $K_{1,m}$ and assume without loss of generality that $f(v_1) = \cdots = f(v_a) = 1$ and $f(v_{a+1}), \ldots, f(v_m) \geq 2$, $a \geq 0$. Consider an arbitrary list assignment L with list sizes defined by f. As above, v_i must be colored with the color from $L(v_i)$ for $i \in \{1, \ldots, a\}$. It must hold that $f(z) \geq 1 + \lfloor \frac{a}{s+1} \rfloor$, which allows z to be colored with a color $\beta \in L(z)$. Lastly, $v_{a+1}, \ldots, v_m$ can always be colored by a color different from β since their list size is at least 2. It follows that $\sum_{v \in V} f(v) \geq a + 2(m-a) + 1 + \lfloor \frac{a}{s+1} \rfloor = m + 1 + \left\lfloor \frac{(m-a)(s+1)+a}{s+1} \right\rfloor \geq m + 1 + \lfloor \frac{m}{s+1} \rfloor$ since $s \geq 0$. Therefore, $\chi_{sc}^{\mathcal{P}}(K_{1,m}) \geq m + 1 + \lfloor \frac{m}{s+1} \rfloor$. ■

For example, if $\mathcal{P} = \mathcal{O}$, then $\chi_{sc}^{\mathcal{O}}(K_{1,m}) = \chi_{sc}(K_{1,m}) = 2m + 1$.

Theorem 18. *Let $\mathcal{P}$ be an additive induced hereditary property. If $n \in \mathbb{N}$ and $p = \max\{k : k \leq n$ and $P_k \in \mathcal{P}\}$, then $\chi_{sc}^{\mathcal{P}}(P_n) = n + \left\lfloor \frac{n-1}{p} \right\rfloor$.*

Proof. Let $P_n = (v_1, \ldots, v_n)$ and define $f : V(P_n) \to \mathbb{N}$ by $f(v_i) = 1$ if $i = 1$ or if $p \nmid (i-1)$, and $f(v_i) = 2$ otherwise. We prove that f is a $\mathcal{P}$-choice function of P_n. Consider an arbitrary list assignment L with $|L(v_i)| = f(v_i)$ for every $i \in \{1, \ldots, n\}$. We color the vertices in order, beginning with v_1. If $f(v_i) = 1$, then v_i must be colored with the single color in its list. If $f(v_i) = 2$, then v_i will be colored with a color different from the color of v_{i-1}. This implies that each graph induced by vertices of the same color consists of paths of order at most p

and therefore belongs to $\mathcal{P}$ since $\mathcal{P}$ is additive and induced hereditary. Hence the coloring is an $(L, \mathcal{P})$-coloring, and f is a $\mathcal{P}$-choice function which implies that $\chi_{sc}^{\mathcal{P}}(P_n) \leq n + \left\lfloor \frac{n-1}{p} \right\rfloor$.

Assume that there is a $\mathcal{P}$-choice function f of P_n with $\sum_{v \in V(P_n)} f(v) = n - 1 + \left\lfloor \frac{n-1}{p} \right\rfloor$. Since $\chi_{sc}^{\mathcal{P}}(P_n) \geq n$, $p \leq n - 1$, which implies $P_{p+1} \notin \mathcal{P}$. There are less than $\left\lfloor \frac{n-1}{p} \right\rfloor$ vertices with list size at least 2, all other vertices have list size 1. Therefore, we either find $p + 1$ consecutive vertices with list size 1, or $(a + 2)p + 1$ consecutive vertices $v_j, \ldots, v_{j+(a+2)p}$, $a \geq 0$, with the following list sizes: $f(v_i) = 2$ if $i = j + lp$, $l = 1, \ldots, a + 1$, and $f(v_i) = 1$ otherwise. Every sequence of consecutive vertices of list size 1 is assigned the same list, alternating between $\{1\}$ and $\{2\}$, all other vertices have initial lists $L(v) = \{1, \ldots, f(v)\}$. These lists force that any list coloring has $p + 1$ consecutive vertices of the same color, which is a contradiction to $P_{p+1} \notin \mathcal{P}$. Therefore, $\chi_{sc}^{\mathcal{P}}(P_n) \geq n + \left\lfloor \frac{n-1}{p} \right\rfloor$. ∎

For example, if $\mathcal{P} = \mathcal{O}$, then $p = 1$ and $\chi_{sc}^{\mathcal{O}}(P_n) = \chi_{sc}(P_n) = 2n - 1$.

Theorem 19. *Let $\mathcal{P}$ be an additive induced hereditary property. If $n \in \mathbb{N}$ and $p = \max\{k : k \leq n$ and $P_k \in \mathcal{P}\}$, then $\chi_{sc}^{\mathcal{P}}(C_n) = n$ if $C_n \in \mathcal{P}$, $\chi_{sc}^{\mathcal{P}}(C_n) = n + 1$ if $C_n \notin \mathcal{P}$, $p = n - 1$, and $\chi_{sc}^{\mathcal{P}}(C_n) = n + 1 + \left\lfloor \frac{n-1}{p} \right\rfloor$ otherwise.*

Proof. The result is obvious if $C_n \in \mathcal{P}$, therefore assume in the following that $C_n \notin \mathcal{P}$ which implies $\chi_{sc}^{\mathcal{P}}(C_n) \geq n + 1$.

Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of C_n, and $V = \{v_1, \ldots, v_n\}$.

If $p = n - 1$, then $f : V \to \mathbb{N}$ with $f(v_i) = 1$ for $i \in \{1, \ldots, n - 1\}$ and $f(v_n) = 2$ is a $\mathcal{P}$-choice function of C_n, since for any list assignment with list sizes determined by f the vertex v_n can be colored differently than vertex v_1, that is, the graphs induced by vertices of the same color consist of paths of order at most $p = n - 1$ which are in $\mathcal{P}$ since $\mathcal{P}$ is an additive and induced hereditary property. Hence, $\chi_{sc}^{\mathcal{P}}(C_n) = n + 1$ in this case.

If $p \neq n - 1$, then define $f : V \to \mathbb{N}$ by $f(v_i) = 1$ if $i = 1$ or if $p \nmid (i - 1)$, and $f(v_i) = 2$ otherwise (see the proof of Theorem 18) and $f' : V \to \mathbb{N}$ by $f'(v_i) = f(v_i)$ if $1 \leq i \leq n - 1$ and $f'(v_n) = f(v_n) + 1$. Consider an arbitrary list assignment L with $|L(v_i)| = f'(v_i)$ for every $i \in \{1, \ldots, n\}$. Color the vertices $v_1, \ldots, v_n$ in order as in the proof of Theorem 18, but additionally remove the color of v_1 from $L(v_n)$ which is possible since its list size was increased by 1, thus forcing v_n to be colored differently from v_1. This implies again that the coloring is an $(L, \mathcal{P})$-coloring and f' is a $\mathcal{P}$-choice function of C_n. Therefore, $\chi_{sc}^{\mathcal{P}}(C_n) \leq n + 1 + \left\lfloor \frac{n-1}{p} \right\rfloor$. Note that because of the lower bound $n + 1$, equality holds for $p \geq n$. Since $p \neq n - 1$, let $p \leq n - 2$ in the following, which implies $P_{p+1} \notin \mathcal{P}$.

Assume that there is a $\mathcal{P}$-choice function f of C_n with $\sum_{v \in V} f(v) = n + \left\lfloor \frac{n-1}{p} \right\rfloor$. If there is a vertex v_i with $f(v_i) \geq 3$, then v_i can always be colored with a color different from the colors of its neighbors. This implies that $\sum_{v \in V} f(v) \geq f(v_i) + \chi_{sc}^{\mathcal{P}}(P_{n-1}) \geq 3 + n - 1 + \left\lfloor \frac{n-2}{p} \right\rfloor \geq n + 1 + \left\lfloor \frac{n-1}{p} \right\rfloor$ by Theorem 18, a contradiction to the assumption. Hence there are exactly $a = \left\lfloor \frac{n-1}{p} \right\rfloor$ vertices with list size 2 and $n - a$ vertices with list size 1. Set $n = ap + r$ with $1 \leq r \leq p$. Since $n > ap$, by the pigeonhole principle, we either find $p + 1$ consecutive vertices with list size 1 (e.g., if $a = 1$) which leads to a list assignment with a monochromatic P_{p+1} which is not in $\mathcal{P}$, a contradiction, or we find p consecutive vertices of list size 1 bounded by two vertices of list size 2. In this case, by removing the p vertices of list size 1 and reducing the list size of the end-vertices of the resulting P_{n-p} by 1 we obtain a $\mathcal{P}$-choice function of P_{n-p} which implies $\sum_{v \in V} f(v) \geq p + 2 + \chi_{sc}^{\mathcal{P}}(P_{n-p}) = p + 2 + n - p + \left\lfloor \frac{n-p-1}{p} \right\rfloor = n + 1 + \left\lfloor \frac{n-1}{p} \right\rfloor$ by Theorem 18, a contradiction to the initial assumption. ■

For example, if $\mathcal{P} = \mathcal{O}$, then $p = 1$ and $\chi_{sc}^{\mathcal{O}}(C_n) = \chi_{sc}(C_n) = 2n$.

The results of this section allow the computation of the $\mathcal{P}$-sum choice number of all graphs of order at most 4 with the exception of the graph isomorphic to a paw $K_{1,3} + e$ (a claw $K_{1,3}$ with an additional edge) and of $K_{1,1,2}$. Their $\mathcal{P}$-sum choice numbers will be determined in the next propositions.

Proposition 20. *Let $\mathcal{P}$ be an additive induced hereditary property. If $G \cong K_{1,3} + e$, then $\chi_{sc}^{\mathcal{O}}(G) = 8$, $\chi_{sc}^{\mathcal{P}}(G) = 4$ if $G \in \mathcal{P}$, and $\chi_{sc}^{\mathcal{P}}(G) = 5$ if $G \notin \mathcal{P} \neq \mathcal{O}$.*

Proof. If $\mathcal{P} = \mathcal{O}$, then $\chi_{sc}^{\mathcal{O}}(G) = \chi_{sc}(G) = \mathrm{GB}(G) = 8$ [1]. If $G \in \mathcal{P}$, then obviously $\chi_{sc}^{\mathcal{P}}(G) = |V(G)| = 4$. Therefore, let $G \notin \mathcal{P} \neq \mathcal{O}$. Since $G \notin \mathcal{P}$, $\chi_{sc}^{\mathcal{P}}(G) \geq |V(G)| + 1 = 5$. Denote the vertices of G such that $d_G(z) = 3$, $d_G(w) = 1$, and $d_G(v_1) = d_G(v_2) = 2$. Define $f : V(G) \to \mathbb{N}$ by $f(z) = 2$ and $f(v) = 1$ for $v \neq z$. Consider an arbitrary list assignment L with sizes defined by f. In an $(L, \mathcal{P})$-coloring of G, all vertices except z must obtain the color of their list. If v_1 and v_2 are colored by the same color α, then color z differently from α. Otherwise, if the colors of v_1 and v_2 are not equal, then color z differently from the color of w. In any case, at most two adjacent vertices share the same color, that is, f is a $\mathcal{P}$-choice function of G and $\chi_{sc}^{\mathcal{P}}(G) \leq 5$. ■

Proposition 21. *Let $\mathcal{P}$ be an additive induced hereditary property. Then it holds $\chi_{sc}^{\mathcal{O}}(K_{1,1,2}) = 9$, $\chi_{sc}^{\mathcal{O}_1}(K_{1,1,2}) = 6$, $\chi_{sc}^{\mathcal{P}}(K_{1,1,2}) = 4$ if $K_{1,1,2} \in \mathcal{P}$, and $\chi_{sc}^{\mathcal{P}}(K_{1,1,2}) = 5$ in the remaining cases.*

Proof. If $\mathcal{P} = \mathcal{O}$, then $\chi_{sc}^{\mathcal{O}}(K_{1,1,2}) = \chi_{sc}(K_{1,1,2}) = \mathrm{GB}(K_{1,1,2}) = 9$ [1]. If $K_{1,1,2} \in \mathcal{P}$, then obviously $\chi_{sc}^{\mathcal{P}}(K_{1,1,2}) = |V(K_{1,1,2})| = 4$.

If $\mathcal{P} = \mathcal{O}_1$, then each subgraph P_3, C_3 needs a list of size 2 to avoid a monochromatic P_3, $C_3 \notin \mathcal{P}$. Hence $\chi_{sc}^{\mathcal{O}_1}(K_{1,1,2}) \geq 2 \cdot 2 + 2 \cdot 1 = 6$. Denote the vertices of $K_{1,1,2}$ such that $d_{K_{1,1,2}}(v_i) = 2$ and $d_{K_{1,1,2}}(w_i) = 3$, $i = 1, 2$. Set $f(v_i) = 2$ and $f(w_i) = 1$, $i = 1, 2$. Consider a list assignment L with list sizes defined by f. Vertices w_1 and w_2 must be colored with the color from their lists. If w_1 is colored by α and w_2 by β ($\alpha = \beta$ is allowed), then color v_1 by a color $\neq \alpha$ and v_2 by a color $\neq \beta$. It follows that $\chi_{sc}^{\mathcal{O}_1}(K_{1,1,2}) \leq 6$.

In the remaining cases it holds that $K_{1,1,2} \notin \mathcal{P}$, $\mathcal{P} \neq \mathcal{O}$, and $\mathcal{P} \neq \mathcal{O}_1$. Since $K_{1,1,2} \notin \mathcal{P}$, $\chi_{sc}^{\mathcal{P}}(K_{1,1,2}) \geq |V(G)| + 1 = 5$. Set $f(w_1) = 2$ for a vertex w_1 of degree 3 and $f(v) = 1$ for $v \neq w_1$. In any list assignment L with list sized defined by f, the colors of the path $P_3 = (v_1, w_2, v_2)$ are fixed. Color then w_1 by a color different from the color of w_2. This implies that at most three vertices that induce a P_3 are colored by the same color, and $P_3 \in \mathcal{P}$. Therefore, $\chi_{sc}^{\mathcal{P}}(K_{1,1,2}) \leq 5$. ■

5. Concluding Remarks

In Section 3 we determined general upper bounds on the $\mathcal{P}$-sum choice number of arbitrary graphs for some of the most common properties $\mathcal{P}$, namely $\mathcal{O}_k$, $\mathcal{S}_k$, $\mathcal{D}_k$, $\mathcal{O}^k$, $\mathcal{J}_k$, and $\mathcal{I}_k$. It would be interesting to obtain reasonable lower bounds on the $\mathcal{P}$-sum choice number of arbitrary graphs for the same properties.

In Section 4 we determined the $\mathcal{P}$-sum choice number of complete graphs, stars, paths, cycles, and all graphs of order at most 4 for arbitrary additive induced hereditary properties $\mathcal{P}$. By the same methods and extensive case analysis we also determined the $\mathcal{P}$-sum choice number of all graphs of order 5 for arbitrary additive hereditary properties $\mathcal{P}$.

As mentioned above, we determined the $\mathcal{P}$-sum choice number of stars $K_{1,m}$. It would be an interesting task to study the $\mathcal{P}$-sum choice number of arbitrary complete bipartite graphs $K_{l,m}$. Partial results for $\mathcal{P} = \mathcal{D}_1$ can be found in [4] and for $\mathcal{P} = \mathcal{O}$ in [1, 6], for example.

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