

ON SOME PROPERTIES OF ANTIPODAL PARTIAL CUBES

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Abstract

We prove that an antipodal bipartite graph is a partial cube if and only if it is interval monotone. Several characterizations of the principal cycles of an antipodal partial cube are given. We also prove that an antipodal partial cube G is a prism over an even cycle if and only if its order is equal to $4(\text{diam}(G) - 1)$, and that the girth of an antipodal partial cube is less than its diameter whenever it is not a cycle and its diameter is at least equal to 6.

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1. INTRODUCTION

If x, y are two vertices of a connected graph G , then y is said to be a *relative antipode* of x if $d_G(x, y) \geq d_G(x, z)$ for every neighbor z of y , where d_G denotes the usual distance in G ; and it is said to be an *absolute antipode* of x if $d_G(x, y) = \text{diam}(G)$ (the diameter of G). The graph G is said to be *antipodal* if every vertex x of G has exactly one relative antipode.

Bipartite antipodal graphs were introduced by Kotzig [14] under the name of S -graphs. Later Glivjak, Kotzig and Plesník [9] proved in particular that *a graph G is antipodal if and only if for any $x \in V(G)$ there is an $\bar{x} \in V(G)$ such that*

$$(1) \quad d_G(x, y) + d_G(y, \bar{x}) = d_G(x, \bar{x}), \quad \text{for all } y \in V(G).$$

By (1), the vertex \bar{x} , which is clearly the unique relative antipode of x , is obviously an absolute antipode. The *antipodal map* $x \mapsto \bar{x}$, $x \in V(G)$, is an automorphism

of G , i.e., $\overline{xy} \in E(G)$ whenever $xy \in E(G)$. The definition of antipodal graph was extended to the non-bipartite case by Kotzig and Laufer [15]. Several papers followed.

Partial cubes, i.e., isometric subgraphs of hypercubes, which were introduced by Firsov [7] and characterized by Djoković [6] and Winkler [23], have been extensively studied, see [4, 12, 13, 16, 20] for recent papers. In [12, 13, 20], antipodal partial cubes and very closed concepts, such as diametrical and harmonic partial cubes, play a very important role.

Recall that partial cubes are interval monotone bipartite graphs, i.e., bipartite graphs all of whose intervals are convex, but that there exist interval monotone bipartite graphs that are not partial cubes (see [2]). In Section 4, we show that the condition of interval monotony is sufficient for an antipodal bipartite graph to be a partial cube.

In the subsequent sections, besides some characterizations of the principal cycles of an antipodal partial cube (see Section 5), we study two properties of partial cubes that are specific to these particular graphs.

Göbel and Veldman [10, Theorem 18] proved that *if G is an antipodal graph of order n and diameter d which is not a cycle, then $n \geq 4d - 4$* . They noticed that the equality is attained for prisms over a cycle. Actually, these are not the only examples of bipartite graphs for which this equality is attained. We will see in Section 4 that the bipartite graph $K_{4,4}$ minus a perfect matching is antipodal but is not a partial cube; this graph has diameter $d = 3$ and order $n = 8$, and thus satisfies the equality $n = 4d - 4$. In Section 6, we show that the prisms over an even cycle are the only antipodal partial cubes that satisfy the above equality.

In Section 7, we are concerned with the relation between the girth and the diameter of an antipodal partial cube G . By a result of Kotzig (see [1]), $\gamma(G) \leq \text{diam}(G) + 1$, where $\gamma(G)$ denotes the girth of the graph G , whenever G is neither K_2 nor an even cycle. In [21, Corollary 4.5 and Theorem 4.6] Sabidussi improved this result by showing that $\gamma(G) \leq \text{diam}(G)$ whenever G is not a cycle and its diameter is greater than 6. For a partial cube we improve this last result by extending it to partial cubes of diameter 6, and by showing that the equality is impossible.

To prove several of those results we use the concept of expansion of a graph that we recall in Section 3.

2. PRELIMINARIES

The graphs we consider are undirected, without loops or multiple edges, and are finite and connected. If $x \in V(G)$, the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ is the *neighborhood* of x in G . For a set S of vertices of a graph G we denote by

$G[S]$ the subgraph of G induced by S , and we put $G - S = G[V(G) \setminus S]$. We also denote by $\partial_G(S)$ the *edge-boundary* of S in G , that is, the set of all edges of G having exactly one endvertex in S . A *path* $P = \langle x_0, \dots, x_n \rangle$ is a graph with $V(P) = \{x_0, \dots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{x_i x_{i+1} : 0 \leq i < n\}$. If x and y are two vertices of a path P , then we denote by $P[x, y]$ the subpath of P whose endvertices are x and y . A *cycle* C with $V(C) = \{x_1, \dots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(C) = \{x_i x_{i+1} : 1 \leq i < n\} \cup \{x_n x_1\}$, is denoted by $\langle x_1, \dots, x_n, x_1 \rangle$.

The usual *distance* between two vertices x and y of a graph G , that is, the length of any (x, y) -*geodesic* (shortest (x, y) -path) in G , is denoted by $d_G(x, y)$. A connected subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H . The *(geodesic) interval* $I_G(x, y)$ between two vertices x and y of G consists of the vertices of all (x, y) -geodesics in G .

In the *geodesic convexity*, that is, the convexity on $V(G)$ which is induced by the *geodesic interval operator* I_G , a subset C of $V(G)$ is convex provided it contains the geodesic interval $I_G(x, y)$ for all $x, y \in C$. The *convex hull* $co_G(A)$ of a subset A of $V(G)$ is the smallest convex set which contains A . A subset H of $V(G)$ is a *half-space* if H and $V(G) \setminus H$ are convex. We denote by \mathcal{I}_G the *pre-hull operator* of the geodesic convex structure of G , i.e., the self-map of $\mathcal{P}(V(G))$ such that $\mathcal{I}_G(A) = \bigcup_{x, y \in A} I_G(x, y)$ for each $A \subseteq V(G)$. The convex hull of a set $A \subseteq V(G)$ is then $co_G(A) = \bigcup_{n \in \mathbb{N}} \mathcal{I}_G^n(A)$. Furthermore we say that a subgraph of a graph G is *convex* if its vertex set is convex, and by the *convex hull* $co_G(H)$ of a subgraph H of G we mean the smallest convex subgraph of G containing H as a subgraph, that is,

$$co_G(H) = G[co_G(V(H))].$$

A graph is said to be *interval monotone* if all its intervals are convex.

For an edge ab of a graph G , let

$$W_{ab} = \{x \in V(G) : d_G(a, x) < d_G(b, x)\}.$$

Note that the sets W_{ab} and W_{ba} are disjoint and that $V(G) = W_{ab} \cup W_{ba}$ if G is bipartite.

Two edges xy and uv are in the Djoković-Winkler relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

If G is bipartite, the edges xy and uv are in relation Θ if and only if $d_G(x, u) = d_G(y, v)$ and $d_G(x, v) = d_G(y, u)$. The relation Θ is clearly reflexive and symmetric.

We recall that, by Djoković [6, Theorem 1] and Winkler [23], *a connected bipartite graph G is a partial cube, that is, an isometric subgraph of some hypercube, if it has the following equivalent properties.*

(Conv.) For every edge ab of G , the sets W_{ab} and W_{ba} are convex.

(Trans.) The relation Θ is transitive, and thus is an equivalence relation.

It follows in particular that the half-spaces of a partial cube G are the sets W_{ab} , $ab \in E(G)$. In the following lemma we recall two well-known properties of partial cubes.

Lemma 2.1. *Let G be a partial cube. We have the following properties.*

- (i) *Let x, y be two vertices of G , P an (x, y) -geodesic and W an (x, y) -path of G . Then each edge of P is Θ -equivalent to some edge of W .*
- (ii) *A path P in G is a geodesic if and only if no two distinct edges of P are Θ -equivalent.*
- (iii) *Any shortest cycle of G is convex in G .*

Lemma 2.2. *If H is connected subgraph of a partial cube G , then any edge of $co_G(H)$ is Θ -equivalent to an edge of H .*

Proof. We recall that $co_G(H) = G [\bigcup_{n \in \mathbb{N}} \mathcal{I}_G^n(V(H))]$. We prove by induction on n that any edge of $G [\mathcal{I}_G^n(V(H))]$ is Θ -equivalent to an edge of H . This is trivial if $n = 0$. Suppose that this is true for some $n \geq 0$. Let e be an edge of $G [\mathcal{I}_G^{n+1}(V(H))]$ that is not an edge of $G [\mathcal{I}_G^n(V(H))]$. Then e is an edge of an (x, y) -geodesic P , for some $x, y \in \mathcal{I}_G^n(V(H))$. Let W be an (x, y) -path of $G [\mathcal{I}_G^n(V(H))]$, note that this graph is connected. Then e is Θ -equivalent to some edge e' of W by Lemma 2.1(i). By the induction hypothesis, e' is Θ -equivalent to an edge e'' of H . Hence e and e'' are Θ -equivalent by the transitivity of the relation Θ . ■

3. EXPANSION

In this section we recall some properties of expansions of a graph, a concept that we will need in the next section and which was introduced by Mulder [17] to characterize median graphs and which was later generalized by Chepoi [3].

Definition 3.1. A pair (V_0, V_1) of sets of vertices of a graph G is called a *proper cover* of G if it satisfies the following conditions:

- $V_0 \cap V_1 \neq \emptyset$ and $V_0 \cup V_1 = V(G)$;
- there is no edge between a vertex in $V_0 \setminus V_1$ and a vertex in $V_1 \setminus V_0$;
- $G[V_0]$ and $G[V_1]$ are isometric subgraphs of G .

Recall that the *prism over* a graph G is the Cartesian product of G and K_2 , i.e., the graph denoted by $G \square K_2$ whose vertex set is $V(G) \times V(K_2)$, and such that, for all $x, y \in V(G)$ and $i, j \in V(K_2) = \{0, 1\}$, $(x, i)(y, j) \in E(G \square K_2)$ if $xy \in E(G)$ and $i = j$, or $x = y$ and $i \neq j$.

Definition 3.2. An *expansion* of a graph G with respect to a proper cover (V_0, V_1) of G is the subgraph of the prism over G induced by the vertex set $(V_0 \times \{0\}) \cup (V_1 \times \{1\})$ (where $\{0, 1\}$ is the vertex set of K_2).

An expansion of a bipartite graph (respectively, a partial cube) is a bipartite graph (respectively, a partial cube (see [3])). If G' is an expansion of a partial cube G , then we say that G is a Θ -contraction of G' , because, as we can easily see, G is obtained from G' by contracting each element of some Θ -class of edges of G' . More precisely, let G be a partial cube different from K_1 and let uv be an edge of G . Let G/uv be the quotient graph of G whose vertex set $V(G/uv)$ is the partition of $V(G)$ such that x and y belong to the same block of this partition if and only if $x = y$ or xy is an edge which is Θ -equivalent to uv . The natural surjection γ_{uv} of $V(G)$ onto $V(G/uv)$ is a contraction (weak homomorphism in [11]) of G onto G/uv , that is, an application which maps any two adjacent vertices to adjacent vertices or to a single vertex. Then clearly the graph G/uv is a partial cube and $(\gamma_{uv}(W_{uv}^G), \gamma_{uv}(W_{vu}^G))$ is a proper cover of G/uv with respect to which G is an expansion of G/uv . We will say that G/uv is the Θ -contraction of G with respect to the Θ -class of uv .

Let G' be an expansion of a graph G with respect to a proper cover (V_0, V_1) of G . We will use the following notation.

- For $i = 0, 1$ denote by $\psi_i : V_i \rightarrow V(G')$ the natural injection $\psi_i : x \mapsto (x, i)$, $x \in V_i$, and let $V'_i = \psi_i(V_i)$. Note that V'_0 and V'_1 are complementary half-spaces of G' .
- For $A \subseteq V(G)$ put

$$\psi(A) = \psi_0(A \cap V_0) \cup \psi_1(A \cap V_1).$$

The following lemma is a restatement with more precisions of [19, Lemma 4.5] (also see [18, Lemma 8.1]).

Lemma 3.3. *Let G be a connected bipartite graph and G' an expansion of G with respect to a proper cover (V_0, V_1) of G , and let $P = \langle x_0, \dots, x_n \rangle$ be a path in G . We have the following properties.*

- (i) *If $x_0, x_n \in V_i$ for some $i = 0$ or 1 , then*
 - $d_{G'}(\psi_i(x_0), \psi_i(x_n)) = d_G(x_0, x_n)$;
 - $I_{G'}(\psi_i(x_0), \psi_i(x_n)) = \psi_i(I_{G[V_i]}(x_0, x_n)) \subseteq \psi(I_G(x_0, x_n))$.
- (ii) *If $x_0 \in V_i$ and $x_1 \in V_{1-i}$ for some $i = 0$ or 1 , then*
 - $d_{G'}(\psi_i(x_0), \psi_{1-i}(x_n)) = d_G(x_0, x_n) + 1$;
 - $I_{G'}(\psi_i(x_0), \psi_{1-i}(x_n)) = \psi(I_G(x_0, x_n))$.

Now we introduce a variety of expansions that are related to antipodal partial cubes.

If A is a set of vertices of an antipodal graph G , we write

$$\bar{A} = \{\bar{x} : x \in A\}.$$

Note that, by (1), a graph G is antipodal if and only if

$$(2) \quad I_G(x, \bar{x}) = V(G), \quad \text{for all } x \in V(G).$$

Lemma 3.4 (Polat [20, Lemma 4.4]). *If G is an antipodal partial cube, then $\overline{W_{ab}} = W_{ba}$ for every edge ab of G .*

Definition 3.5. A proper cover (V_0, V_1) of an antipodal partial cube G is said to *respect the antipodality*, or to be *antipodality-respectful*, if $\overline{V_0} = V_1$.

Clearly, if $\overline{V_0} = V_1$, then $\overline{V_1} = V_0$ and $\overline{V_0 \cap V_1} = V_0 \cap V_1$. For any antipodal partial cube G , there always exists a proper cover that respects the antipodality. For example, the proper cover (V_0, V_1) such that $V_0 = V_1 = V(G)$ respects the antipodality, and the expansion of G with respect to this proper cover is the prism over G .

Definition 3.6. An expansion of an antipodal partial cube G with respect to an antipodality-respectful proper cover of G is called an *antipodality-respectful expansion* of G .

These antipodality-respectful expansions were already defined in [8] under the name of acycloidal expansions.

Lemma 3.7 (Polat [20, Lemma 4.7]). *Any antipodality-respectful expansion of an antipodal partial cube is an antipodal partial cube.*

Lemma 3.8 (Polat [20, Lemma 4.8]). *Let G' be an expansion of a partial cube G with respect to a proper cover (V_0, V_1) . If G' is antipodal, then so is G and moreover (V_0, V_1) is an antipodality-respectful proper cover of G .*

We obtain immediately.

Corollary 3.9. *Any Θ -contraction of an antipodal partial cube is an antipodal partial cube.*

Proposition 3.10 (Polat [20, Theorem 4.9]). *A finite graph is an antipodal partial cube if and only if it can be obtained from K_1 by a sequence of antipodality-respectful expansions.*

The number of iterations to obtain some antipodal partial cube G from K_1 is equal to the number of Θ -classes of $E(G)$, that is, to the *isometric dimension* of G , i.e., the least non-negative integer n such that G is an isometric subgraph of an n -cube. We denote it by $\text{idim}(G)$. Note that a result similar to the above ones is given by Knauer and Marc [13, Lemma 2.14].

4. ANTIPODAL BIPARTITE GRAPHS VS ANTIPODAL PARTIAL CUBES

By Tardif [22] (see [5]), an antipodal bipartite graph is a partial cube if and only if it contains no subdivision of $K_{3,3}$. It follows that the bipartite antipodal graph $K_{n,n} - M$, where $n > 4$ and M is a perfect matching of $K_{n,n}$, is not a partial cube since it clearly contains a subdivision of $K_{3,3}$. Note that this graph is not interval monotone. Indeed, let $\{u_i : 0 \leq i < n\}$ and $\{v_i : 0 \leq i < n\}$ be the natural partitions of $K_{n,n}$, and let $G = K_{n,n} - \{u_i v_i : 0 \leq i < n\}$. Then $\text{diam}(G) = 3$, and $\bar{u}_i = v_i$ for $0 \leq i < n$, and thus the interval $I_G(u_i, v_i) = V(G)$ is trivially convex. On the other hand $\text{dist}_G(u_0, u_1) = 2$, $v_2, v_3 \in I_G(u_0, u_1)$, $u_4 \in I_G(v_2, v_3)$, but $u_4 \notin I_G(u_0, u_1)$. Hence the interval $I_G(u_0, u_1)$ is not convex.

We will show that the condition of interval monotony is sufficient for an antipodal bipartite graph to be a partial cube, which is not the case if the graph is not antipodal (see [2]). We need the following lemma where we use the notation introduced in Section 3.

Lemma 4.1. *Let G' be an expansion of a connected bipartite graph with respect to a proper cover (V_0, V_1) of G . Let K' be a convex set of G' which meets both V'_0 and V'_1 . Then $K = \text{pr}(K')$ is a convex set of G , where pr denotes the projection of G' onto G .*

Proof. Let $u, v \in K$. If $u \in V_i$ and $v \in V_{1-i}$ for some $i = 0$ or 1 , then $I_G(u, v) = \text{pr}(I_{G'}(u', v'))$ by Lemma 3.3, and hence $I_G(u, v) \subseteq K$.

Now assume that $u, v \in V_i$ for some $i = 0$ or 1 , say $i = 0$. Let $P = \langle x_0, \dots, x_n \rangle$ be a (u, v) -geodesic in G with $x_0 = u$ and $x_n = v$. In general, not all of P is contained in $G[V_0]$. Let $0 = i_0 < i_1 < \dots < i_{2p+1} = n$ be subscripts such that the segments $P[x_{i_0}, x_{i_1}], P[x_{i_1}, x_{i_2}], \dots, P[x_{i_{2p}}, x_{i_{2p+1}}]$ are alternatively contained in $G[V_0]$ and $G[V_1]$. Thus $x_{i_1}, \dots, x_{i_{2p}} \in V_0 \cap V_1$. Since $G[V_0]$ is isometric in G there is an $(x_{i_{2h-1}}, x_{i_{2h}})$ -geodesic P_h in $G[V_0]$, $h = 1, \dots, p$. Replacing each $(x_{i_{2h-1}}, x_{i_{2h}})$ -segment of P by the corresponding P_h one obtains a new (u, v) -geodesic P_0 with $V(P_0) \subseteq V_0$. Hence $\psi_0(P_0)$ is a (u', v') -geodesic in G' , and therefore $V(P_0) \subseteq K$.

It follows in particular that $\psi_0(x_{i_k}) \in K' \cap V'_0$, $k = 1, \dots, 2p$. By hypothesis, there exists a vertex $w \in K' \cap V'_1$. From the construction of G' it then follows that $y_k = \psi_1(x_{i_k}) \in I_{G'}(\psi_0(x_{i_k}), w)$, and hence $y_k \in K'$. Since $G[V_1]$ is an isometric subgraph of G we deduce that $\psi_1(P[x_{i_{2k-1}}, x_{i_{2k}}])$ is a (y_{2k-1}, y_{2k}) -geodesic. Hence $V(P[x_{i_{2k-1}}, x_{i_{2k}}]) \subseteq K$, and therefore $V(P) \subseteq K$. ■

Definition 4.2. We say that a graph G is *weakly interval monotone* if every interval of G whose length is at least $\text{diam}(G) - 1$ is convex.

Theorem 4.3. *Let G be a bipartite antipodal graph. The following assertions are equivalent.*

- (i) G is a partial cube.
- (ii) G is interval monotone.
- (iii) G is weakly interval monotone.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are obvious.

(iii) \Rightarrow (i) First note that, in an antipodal graph H , an interval of length $\text{diam}(H)$ is an interval $I_H(x, \bar{x})$ for some vertex x , and thus is equal to $V(H)$, which implies that this interval is convex. Hence H is weakly interval monotone if and only if any interval of length $\text{diam}(H) - 1$ is convex, and such an interval is equal to $I_H(x, y)$ for some $x \in V(H)$ and $y \in N_H(\bar{x})$.

We proceed by induction on $\text{diam}(G)$. This is obvious if $\text{diam}(G) = 0$ or 1 since the only antipodal graphs of diameter 0 and 1 are K_1 and K_2 , respectively, which are hypercubes. Let $n \geq 1$. Suppose that every bipartite antipodal graph whose diameter is at most n and which is weakly interval monotone is a partial cube. Let G be a weakly interval monotone bipartite antipodal graph whose diameter is $n + 1$.

Because G is finite, there exists a non-trivial convex set K of G which is maximal with respect to inclusion. Let $x \in K$. Because $K \neq V(G) = I_G(x, \bar{x})$, it follows that $\bar{x} \in V(G) \setminus K$. Let $\bar{K} = \{\bar{x} : x \in K\}$. Then $\bar{K} \subseteq V(G) \setminus K$. Moreover \bar{K} is convex since so is K . Indeed, let P be an (x, y) -geodesic for some $x, y \in \bar{K}$. Then $\alpha(P)$, where α is the antipodal map of G , is an (\bar{x}, \bar{y}) -geodesic, and thus is a path in $G[K]$ since K is convex. Hence P is a path in $G[\bar{K}]$, which proves that \bar{K} is convex.

Claim 1. $\bar{K} = V(G) \setminus K$, i.e., K is a half-space.

Suppose that this is not true. Then there is a vertex $a \in V(G) \setminus (K \cup \bar{K})$ that is adjacent to some vertex $b \in K \cup \bar{K}$, say $b \in K$. Then $\bar{a} \in V(G) \setminus (K \cup \bar{K})$ is adjacent to \bar{b} which belongs to \bar{K} . Because G is antipodal, $I_G(a, \bar{a}) = V(G)$, and thus $K \subseteq I_G(a, \bar{a})$. On the other hand $K \subseteq W_{ba}$ since G is bipartite and K is convex. Hence $K \subseteq I_G(b, \bar{a})$. Because G is weakly interval monotone by assumption, it follows that $I_G(b, \bar{a})$ is a convex set that contains K but not \bar{b} , contrary to the maximality of K . Therefore $\bar{K} = V(G) \setminus K$, i.e., K is a half-space.

Because K is a half-space, it follows that all edges in $\partial_G(K)$ are pairwise in relation Θ . Let F be the graph obtained from G by identifying the endvertices of each edge between K and $V(G) \setminus K$. Clearly G is an expansion of F . More precisely, G is the expansion of F with respect to the proper cover (V_0, V_1) , where V_0 and V_1 are the projections of K and \bar{K} on $V(F)$, respectively.

Moreover F is bipartite since any cycle of G contains an even number of edges in $\partial_G(K)$, and $\text{diam}(F) = \text{diam}(G) - 1$. By Lemma 3.8, it is antipodal and (V_0, V_1) respects the antipodality.

Claim 2. F is weakly interval monotone.

Let $u, v \in V(F)$ be such that $d_G(u, v) = \text{diam}(F) - 1$. Then v is a neighbor of \bar{u} since F is antipodal. Assume that $u \in V_0$. If $u \in V_0 \setminus V_1$, then $\bar{u} \in V_1 \setminus V_0$, and thus $v \in V_1$. If $u \in V_0 \cap V_1$, then $\bar{u} \in V_0 \cap V_1$, and thus $v \in V_0 \cup V_1$. Suppose that $v \in V_i$ for some $i = 0$ or 1 . Then $u \in V_{1-i}$, $\bar{u} \in V_i$, and $I_G(\psi_{1-i}(u), \psi_i(v)) = \psi(I_F(u, v))$ by Lemma 3.3. The length of the interval $I_G(\psi_{1-i}(u), \psi_i(v))$ is equal to $d_F(u, v) + 1 = \text{diam}(F) = \text{diam}(G) - 1$. Hence this interval is a convex set, since G is weakly interval monotone, that meets K and \bar{K} . It follows that $I_F(u, v)$, which is equal to $\text{pr}(I_G(\psi_{1-i}(u), \psi_i(v)))$, is convex, by Lemma 4.1.

Therefore, by the induction hypothesis, F is a partial cube, and thus so is G , because any expansion of a partial cube is also a partial cube. ■

5. PRINCIPAL CYCLES OF AN ANTIPODAL PARTIAL CUBE

Definition 5.1. An isometric cycle C of an antipodal graph G is called *principal* if $\alpha(C) = C$, where α is the antipodal map of G .

Principal cycles always exist, indeed, any geodesic in G lies on a principal cycle. Moreover, we have the following result.

Proposition 5.2 (Glivjak, Kotzig and Plesník [9]). *An isometric cycle C of an antipodal graph G is principal if and only if $\text{diam}(C) = \text{diam}(G)$.*

We say that a subgraph H of a partial cube G is *median-stable* if, for any triple (x, y, z) of vertices of H , if (x, y, z) has a median m in G , then $m \in V(H)$. Note that, if H is isometric in G , then m is the median of (x, y, z) in H . The smallest median-stable subgraph F of a partial cube G which contains a subgraph H of G is called the *median-closure* of H . Such a subgraph F always exists.

Lemma 5.3. *Let C be a principal cycle of an antipodal partial cube G . For each edge uv of G , if ab is an edge of C which is Θ -equivalent to uv , then $u \in I_G(a, \bar{b})$ and $v \in I_G(b, \bar{a})$.*

Proof. Let uv be an edge of G , and ab an edge of C which is Θ -equivalent to uv . Note that the antipodal edge $\bar{b}\bar{a}$ of ab in C is Θ -equivalent to ab , and thus to uv . Then $u \in I_G(a, \bar{a})$ since G is antipodal. On the other hand, because uv is Θ -equivalent to $\bar{b}\bar{a}$, it follows that $\bar{b} \in I_G(u, \bar{a})$. Hence $u \in I_G(a, \bar{b})$, and likewise $v \in I_G(b, \bar{a})$. ■

By Lemma 2.1(ii) we clearly have

$$\text{diam}(G) \leq \text{idim}(G).$$

We need the following lemma which is an immediate consequence of Desharnais [5, Lemme 1.6.9] (also see [20, Lemma 3.2] for an alternative proof, [12, Proposition

3.1] for a weaker similar result). We recall that a graph G is *diametrical* if every vertex of G has exactly one absolute antipode. Antipodal graphs are then diametrical.

Lemma 5.4. *Let G be a diametrical partial cube. Then G is antipodal if and only if $\text{diam}(G) = \text{idim}(G)$.*

Theorem 5.5. *Let C be an isometric cycle of an antipodal partial cube G . The following assertions are equivalent.*

- (i) C is a principal cycle of G .
- (ii) $\text{diam}(C) = \text{diam}(G)$.
- (iii) $\text{idim}(C) = \text{idim}(G)$.
- (iv) $\mathcal{I}_G(C) = G$.
- (v) The convex hull of C is G .
- (vi) The median-closure of C is G .

Proof. The equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) are consequences of Proposition 5.2 and Lemma 5.4.

(i) \Rightarrow (iv) Assume that C is a principal cycle of G , and let a be a vertex of C . Then $\bar{a} \in V(C)$, and thus $V(G) = I_G(a, \bar{a})$ since G is antipodal. Hence, a fortiori, $\mathcal{I}_G(C) = G$.

(iv) \Rightarrow (v) is obvious, and (v) \Rightarrow (iii) is a consequence of Lemma 2.2.

(iii) \Rightarrow (vi) Assume that $\text{dim}(C) = \text{dim}(G)$. Let F be the median-closure of C . Suppose that $F \neq G$. Because G is connected, there exists a vertex u of $G - F$ which is adjacent to some vertex v of F . Then, by (iii), the edge uv is Θ -equivalent to some edge ab of C , and thus to the edge $\bar{b}\bar{a}$. By Lemma 5.3, $u \in I_G(a, \bar{b})$ and $v \in I_G(b, \bar{a})$. It follows that u is the median of the triple (v, a, \bar{b}) , and thus $u \in V(F)$, contrary to the hypothesis. Therefore $G = F$.

(vi) \Rightarrow (v) is obvious. ■

We immediately obtain the following.

Corollary 5.6. *Let C be a principal cycle of some antipodal partial cube. The following assertions are equivalent.*

- (i) C is a retract of G .
- (ii) C is median-stable.
- (iii) $C = G$.

6. ORDER AND DIAMETER

Clearly even cycles and hypercubes are characterized by the relations $n = 2d$ and $n = 2^d$, respectively. It would be interesting to find other classes of antipodal

partial cubes which are characterized by a relation between their order and diameter. Such a characterization is not obvious even for other classes of regular antipodal partial cubes such as Cartesian products of families of even cycles and K_2 , except, as we will see below, for the class of prisms over an even cycle.

Theorem 6.1. *Let G be an antipodal partial cube of order n and diameter d . Then $n = 4d - 4$ if and only if G is either a 4-cycle or a prism over an even cycle.*

Proof. We only have to prove the necessity, the sufficiency being obvious. Let G be an antipodal partial cube such that $n = 4d - 4$. If G is a cycle, then $n = 2d$, and thus G is a 4-cycle. Assume that G is not a cycle.

Claim. *If the isometric dimension of G is at least 3, then there exists an edge e of G such that the Θ -class of e has at least four elements.*

Proof. Assume that the isometric dimension of G is at least 3. Let C be a principal cycle of G , and c a vertex of C . Because G is not a cycle, there exists a vertex x which does not lie on C . Note that \bar{x} is also not a vertex of C . Then there exists a principal cycle C' of G which passes through c and x , and thus through \bar{c} and \bar{x} . Let e be an edge of C' which is incident to x , and let e' be the edge of C' which is Θ -equivalent to e . Then e' is incident to \bar{x} , and thus is not an edge of C . On the other hand, by Theorem 5.5, $\text{idim}(C) = \text{idim}(G) = \text{idim}(C')$, and thus C has two edges that are Θ -equivalent to e . Hence the Θ -class of e has at least four elements. \square

Assume now that $n = 4d - 4$. Then G is not a cycle. Recall that the isometric dimension and the diameter of an antipodal partial cube are equal by Lemma 5.4. We prove by induction on the isometric dimension of G that G is a prism over an even cycle. This is clear if the isometric dimension of G is 3, because G is then a 3-cube. Suppose that the theorem holds for any antipodal partial cube that is not a cycle and whose isometric dimension is d for some $d \geq 3$. Let G be an antipodal partial cube that is not a cycle, whose isometric dimension is $d + 1$, and such that $n = 4d - 4$.

By the claim, there exists an edge e of G such that the Θ -class of e has at least four elements. Let $H = G/e$ be the Θ -contraction of G with respect to the Θ -class of e . By Corollary 3.9, H is an antipodal partial cube whose isometric dimension is d . Denote by n' and d' the order and the diameter of H , respectively. Then $d' = d - 1$ and $n' \leq n - 4$. It follows that

$$n' \leq n - 4 = 4(d' + 1) - 8 = 4d' - 4.$$

We have two cases. If H is a cycle, then G is either a cycle, which is impossible by assumption, or the prism over H , and thus we are done.

If H is not a cycle, then $n' \geq 4d' - 4$ by [10, Theorem 18]. It follows that

$$4d' - 4 \leq n' \leq 4d' - 4$$

by the inequality above. Hence $n' = 4d' - 4$. By the induction hypothesis, H is a prism $C \square K_2$ over an even cycle C of length $2(d' - 1)$, i.e., $2d - 4$. Because $n = 4d - 4$, $n' = 4d' - 4$ and $d = d' + 1$, it follows that $n = n' + 4$. Hence, if G is the expansion of H with respect to some antipodality-respectful proper cover (V_0, V_1) of H , then we must have $|V_0 \cap V_1| = 4$.

On the other hand, let (W_0, W_1) be some antipodality-respectful proper cover of C . Note that, either $W_0 = W_1 = V(C)$ or W_0 and W_1 are the vertex sets of two geodesics of length $d' - 1$ in C whose union is C . By the properties of antipodality-respectful proper covers, any antipodality-respectful proper cover of H is equal either to $A' = (W_0 \times \{0, 1\}, W_1 \times \{0, 1\})$ or $A'' = ((W_0 \times \{0, 1\}) \cup (W_1 \times \{0\}), (W_1 \times \{0, 1\}) \cup (W_0 \times \{1\}))$.

Therefore, because $|V_0 \cap V_1| = 4$, we infer that $(V_0, V_1) = A'$ with $W_0 \neq W_1$. It follows that G is the prism over some even cycle of length $2d'$, i.e., $2d - 2$. ■

7. GIRTH OF AN ANTIPODAL PARTIAL CUBE

In this section we will improve for partial cubes the result of Sabidussi [21, Corollary 4.5 and Theorem 4.6], $\gamma(G) \leq \text{diam}(G)$ whenever $\text{diam}(G) > 6$, where $\gamma(G)$ denotes the girth of the graph G , by extending it to partial cubes of diameter 6, and by showing that the equality is impossible.

We first show, by giving two examples, that 6 is the best lower bound. The 3-cube Q_3 is an antipodal partial cube whose diameter is 3 and girth is 4. The Desargues graph D , i.e., the bipartite double of the Petersen graph, has diameter 5 and girth 6. Moreover D is an antipodal partial cube, in fact D is an antipodality-respectful expansion of the antipodal partial cube $M_{4,1}$, i.e., the cube Q_4 from which a pair of antipodal vertices has been removed.

By a recent result [16, Theorem 2.10] of Marc *there exist no finite partial cubes of girth greater than 6 and minimum degree at least 3*. On the other hand, by [9, Theorem 7], *if the degree of some vertex of an antipodal bipartite graph G is greater than 2, then all vertices of G has degree greater than 2*. Note that, for partial cubes, this last result is an immediate consequence of Proposition 3.10, because any antipodality-respectful expansion of an even cycle C is either a cycle or the prism over C , and since any antipodality-respectful expansion of a graph all of whose vertices have degree greater than 2 is also a graph all of whose vertices have degree greater than 2. From these two results we infer the following.

Proposition 7.1. *Any antipodal partial cube of girth greater than 6 is an even cycle.*

Theorem 7.2. *If G is an antipodal partial cube of diameter at least 6 that is not a cycle, then*

$$\gamma(G) < \text{diam}(G).$$

Proof. By proposition 7.1, it remains to prove the case $\text{diam}(G) = 6$. Put $d = \text{diam}(G)$. Assume that $d = 6$. By proposition 7.1, $\gamma(G) \leq 6$. We are done if $\gamma(G) = 4$. So assume that $\gamma(G) = 6$.

Let $C = \langle c_0, \dots, c_5, c_0 \rangle$ be a cycle of G of minimal length 6. Then C is convex in G by Lemma 2.1(iii). Because G is antipodal, $\langle c_0, \dots, c_3 \rangle$ is a subpath of a principal cycle

$$P = \langle c_0, \dots, c_3, x_4, \dots, x_{11}, c_0 \rangle.$$

For all $i = 0, 1, 2$, because C and P are isometric cycles of G , the edges $c_{3+i}c_{3+i+1}$ and $x_{6+i}x_{6+i+1}$ are Θ -equivalent, since both of them are Θ -equivalent to the edge $c_{i+1}c_i$. It follows that $d_G(c_{3+i}, x_{6+i}) = d_G(c_3, x_6) = 3$ if $0 \leq i \leq 3$. For all i with $0 \leq i \leq 3$, denote by X_i a (c_{3+i}, x_{6+i}) -geodesic. We can choose X_0 and X_3 as subpaths of P . For $0 \leq i \leq 3$, put

$$X_i = \langle y_0^i, \dots, y_3^i \rangle$$

with $y_0^i = c_{3+i}$, $y_3^i = x_{6+i}$ and $y_j^0 = x_{3+j}$ for $j = 1, 2$. Because G is bipartite and since $c_{3+i}c_{3+i+1}$ and $x_{6+i}x_{6+i+1}$ are Θ -equivalent, it follows that the paths X_i and X_{i+1} are disjoint, for $0 \leq i < 3$.

For all i with $0 \leq i < 3$, consider the cycle

$$A_i = X_i \cup \langle x_{6+i}, x_{6+i+1} \rangle \cup X_{i+1} \cup \langle c_{3+i+1}, c_{3+i} \rangle.$$

We distinguish two cases.

Case 1. A_i is not isometric in G for some i with $0 \leq i \leq n$. Without loss of generality we suppose that A_0 is not isometric in G . Because c_3c_4 and x_6x_7 are Θ -equivalent and X_0 and X_1 are geodesics, it follows that there exist $u_0 \in V(X_0)$ and $u_1 \in V(X_1)$ such that $d_G(u_0, u_1) < d_{A_0}(u_0, u_1)$. For these same reasons, and since G contains no 4-cycles, it follows that the only possibilities are $u_0 = y_1^0 = x_4$ and $u_1 = y_2^1$, or $u_0 = y_2^0 = x_5$ and $u_1 = y_1^1$.

Suppose that $u_0 = x_4$ and $u_1 = y_2^1$. Because $d_{A_0}(x_4, y_2^1) = 4$ and since G is bipartite, it follows that $d_G(x_4, y_2^1) = 2$. Let v be the common neighbor of x_4 and y_2^1 . The cycle $\langle c_3, c_4, y_1^1, y_2^1, v, c_4 \rangle$ is isometric since G contains no 4-cycles. Hence the edge vy_2^1 is Θ -equivalent to the edge c_3c_4 , which is itself Θ -equivalent to the edge x_6x_7 . It follows that vy_2^1 and x_6x_7 are Θ -equivalent, and thus y_2^1 and x_6 must be adjacent since so are y_2^1 and x_7 , contrary to the assumption $\gamma(G) = 6$. We obtain an analogous contradiction if $u_0 = y_2^0 = x_5$ and $u_1 = y_1^1$.

Case 2. A_i is isometric in G for all i with $0 \leq i \leq n$. Because the cycles A_0 and A_1 are isometric, it follows that, for $j = 0, 1, 2$, the edges $y_{3-j-1}^0y_{3-j}^0$ and

$y_{3-j-1}^2 y_{3-j}^2$ are Θ -equivalent because both of them are Θ -equivalent to the edge $y_j^1 y_{j+1}^1$.

This entails that the paths X_0 and X_2 are disjoint. Indeed, suppose that $y_j^0 = y_k^2$ for some j, k that are as small as possible. Then $j = k$ since otherwise the edges $y_0^0 y_1^0$ and $y_0^2 y_1^2$ will not be Θ -equivalent; but, on the other hand, j cannot be equal to k since otherwise the edges $y_{j-1}^0 y_j^0$ and $y_{j-1}^2 y_j^2$ will not be Θ -equivalent.

From above we infer that $d_G(y_2^0, y_2^2) = d_G(x_6, x_8) = 2$. Clearly, the common neighbor y of y_2^0 and y_2^2 is distinct from x_{6+i} for $i = 0, 1, 2$, because G is bipartite and $\gamma(G) = 6$ by assumption. In the same way $d_G(y_1^0, y_1^2) = d_G(c_3, c_5) = 2$, and moreover, if z is the common neighbor of y_1^0 and y_1^2 , then z is distinct from x_i for $i = 3, 4, 5$.

On the other hand y_2^0 and y_1^2 , and y_1^0 and y_2^2 , are not adjacent. Indeed, suppose that y_2^0 and y_1^2 are adjacent. Then $y = y_1^2$ and $z = y_2^0$ since otherwise G would contain a 4-cycle, contrary to the assumption. It follows that the edges $c_3 c_4$ and $x_7 x_8$ are Θ -equivalent because they both are Θ -equivalent to the edge $y_2^0 y_1^2$ (recall that any 6-cycle of G is convex), and this is impossible since $c_3 c_4$ is also Θ -equivalent to $x_6 x_7$ which cannot be Θ -equivalent to $x_7 x_8$.

Because any 6-cycle is convex we have the following chain of Θ -equivalences

$$c_3 c_4 \Theta z y_1^2 \Theta y_2^0 y \Theta x_7 x_8 \Theta c_4 c_5.$$

Hence $c_3 c_4$ and $c_4 c_5$ are Θ -equivalent by the transitivity of Θ , which is impossible since these edges are adjacent.

This shows that in Case 2, as well as in Case 1, we cannot have $\gamma(G) = 6$. Hence $\gamma(G) < 6$ if $d = 6$.

Consequently we have $\gamma(G) < \text{diam}(G)$ if $\text{diam}(G) \geq 6$. ■

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