

## ON ACCURATE DOMINATION IN GRAPHS

JOANNA CYMAN

*Faculty of Applied Physics and Mathematics*  
*Gdańsk University of Technology, 80–233 Gdańsk, Poland*

**e-mail:** joanna.cyman@pg.edu.pl

MICHAEL A. HENNING

*Department of Pure and Applied Mathematics*  
*University of Johannesburg*  
*Auckland Park 2006, South Africa*

**e-mail:** mahenning@uj.ac.za

AND

JERZY TOPP

*Faculty of Mathematics, Physics and Informatics*  
*University of Gdańsk, 80–952 Gdańsk, Poland*

**e-mail:** jtopp@inf.ug.edu.pl

### Abstract

A dominating set of a graph  $G$  is a subset  $D \subseteq V_G$  such that every vertex not in  $D$  is adjacent to at least one vertex in  $D$ . The cardinality of a smallest dominating set of  $G$ , denoted by  $\gamma(G)$ , is the domination number of  $G$ . The accurate domination number of  $G$ , denoted by  $\gamma_a(G)$ , is the cardinality of a smallest set  $D$  that is a dominating set of  $G$  and no  $|D|$ -element subset of  $V_G \setminus D$  is a dominating set of  $G$ . We study graphs for which the accurate domination number is equal to the domination number. In particular, all trees  $G$  for which  $\gamma_a(G) = \gamma(G)$  are characterized. Furthermore, we compare the accurate domination number with the domination number of different coronas of a graph.

**Keywords:** domination number, accurate domination number, tree, corona.

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## 1. INTRODUCTION AND NOTATION

We generally follow the notation and terminology of [1] and [9]. Let  $G = (V_G, E_G)$  be a graph with vertex set  $V_G$  of order  $n(G) = |V_G|$  and edge set  $E_G$  of size  $m(G) = |E_G|$ . If  $v$  is a vertex of  $G$ , then the *open neighborhood* of  $v$  is the set  $N_G(v) = \{u \in V_G : uv \in E_G\}$ , while the *closed neighborhood* of  $v$  is the set  $N_G[v] = N_G(v) \cup \{v\}$ . For a subset  $X$  of  $V_G$  and a vertex  $x$  in  $X$ , the set  $\text{pn}_G(x, X) = \{v \in V_G : N_G[v] \cap X = \{x\}\}$  is called the  *$X$ -private neighborhood* of the vertex  $x$ , and it consists of those vertices of  $N_G[x]$  which are not adjacent to any vertex in  $X \setminus \{x\}$ ; that is,  $\text{pn}_G(x, X) = N_G[x] \setminus N_G[X \setminus \{x\}]$ . The *degree*  $d_G(v)$  of a vertex  $v$  in  $G$  is the number of vertices in  $N_G(v)$ . A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. The set of leaves of a graph  $G$  is denoted by  $L_G$ , while the set of support vertices by  $S_G$ . For a set  $S \subseteq V_G$ , the subgraph induced by  $S$  is denoted by  $G[S]$ , while the subgraph induced by  $V_G \setminus S$  is denoted by  $G - S$ . Thus the graph  $G - S$  is obtained from  $G$  by deleting the vertices in  $S$  and all edges incident with  $S$ . Let  $\kappa(G)$  denote the number of components of  $G$ .

A *dominating set* of a graph  $G$  is a subset  $D$  of  $V_G$  such that every vertex not in  $D$  is adjacent to at least one vertex in  $D$ , that is,  $N_G(x) \cap D \neq \emptyset$  for every  $x \in V_G \setminus D$ . The *domination number* of  $G$ , denoted by  $\gamma(G)$ , is the cardinality of a smallest dominating set of  $G$ . An *accurate dominating set* of  $G$  is a dominating set  $D$  of  $G$  such that no  $|D|$ -element subset of  $V_G \setminus D$  is a dominating set of  $G$ . The *accurate domination number* of  $G$ , denoted by  $\gamma_a(G)$ , is the cardinality of a smallest accurate dominating set of  $G$ . We call a dominating set of  $G$  of cardinality  $\gamma(G)$  a  $\gamma$ -*set* of  $G$ , and an accurate dominating set of  $G$  of cardinality  $\gamma_a(G)$  a  $\gamma_a$ -*set* of  $G$ . Since every accurate dominating set of  $G$  is a dominating set of  $G$ , we note that  $\gamma(G) \leq \gamma_a(G)$ . The accurate domination in graphs was introduced by Kulli and Kattimani [11], and further studied in a number of papers (see, for example, [3, 6, 7, 10, 12–14, 16, 17]). A comprehensive survey of concepts and results on domination in graphs can be found in [9].

We denote the path and cycle on  $n$  vertices by  $P_n$  and  $C_n$ , respectively. We denote by  $K_n$  the *complete graph* on  $n$  vertices, and by  $K_{m,n}$  the *complete bipartite graph* with partite sets of size  $m$  and  $n$ . The accurate domination numbers of some common graphs are given by the following formulas.

**Observation 1.** *The following holds.*

- (a) For  $n \geq 1$ ,  $\gamma_a(K_n) = \lfloor \frac{n}{2} \rfloor + 1$  and  $\gamma_a(K_{n,n}) = n + 1$ .
- (b) For  $n > m \geq 1$ ,  $\gamma_a(K_{m,n}) = m$ .
- (c) For  $n \geq 3$ ,  $\gamma_a(C_n) = \lfloor \frac{n}{3} \rfloor - \lfloor \frac{3}{n} \rfloor + 2$ .
- (d) For  $n \geq 1$ ,  $\gamma_a(P_n) = \lceil \frac{n}{3} \rceil$  unless  $n \in \{2, 4\}$  when  $\gamma_a(P_n) = \lceil \frac{n}{3} \rceil + 1$  (see Corollary 6).

In this paper we study graphs for which the accurate domination number is equal to the domination number. In particular, all trees  $G$  for which  $\gamma_a(G) = \gamma(G)$  are characterized. Furthermore, we compare the accurate domination number with the domination number of different coronas of a graph. Throughout the paper, we use the symbol  $\mathcal{A}_\gamma(G)$  (respectively,  $\mathcal{A}_{\gamma_a}(G)$ ) to denote the set of all minimum dominating sets (respectively, minimum accurate dominating sets) of  $G$ .

## 2. GRAPHS WITH $\gamma_a$ EQUAL TO $\gamma$

We are interested in determining the structure of graphs for which the accurate domination number is equal to the domination number. The question about such graphs has been stated in [12]. We begin with the following general property of the graphs  $G$  for which  $\gamma_a(G) = \gamma(G)$ .

**Lemma 2.** *Let  $G$  be a graph. Then  $\gamma_a(G) = \gamma(G)$  if and only if there exists a set  $D \in \mathcal{A}_\gamma(G)$  such that  $D \cap D' \neq \emptyset$  for every set  $D' \in \mathcal{A}_\gamma(G)$ .*

**Proof.** First assume that  $\gamma_a(G) = \gamma(G)$ , and let  $D$  be a minimum accurate dominating set of  $G$ . Since  $D$  is a dominating set of  $G$  and  $|D| = \gamma_a(G) = \gamma(G)$ , we note that  $D \in \mathcal{A}_\gamma(G)$ . Now let  $D'$  be an arbitrary minimum dominating set of  $G$ . If  $D \cap D' = \emptyset$ , then  $D' \subseteq V_G \setminus D$ , implying that  $D'$  would be a  $|D|$ -element dominating set of  $G$ , contradicting the fact that  $D$  is an accurate dominating set of  $G$ . Hence,  $D \cap D' \neq \emptyset$ .

Now assume that there exists a set  $D \in \mathcal{A}_\gamma(G)$  such that  $D \cap D' \neq \emptyset$  for every set  $D' \in \mathcal{A}_\gamma(G)$ . Then,  $D$  is an accurate dominating set of  $G$ , implying that  $\gamma_a(G) \leq |D| = \gamma(G) \leq \gamma_a(G)$ . Consequently, we must have equality throughout this inequality chain, and so  $\gamma_a(G) = \gamma(G)$ . ■

It follows from Lemma 2 that if  $G$  is a disconnected graph, then  $\gamma_a(G) = \gamma(G)$  if and only if  $\gamma_a(H) = \gamma(H)$  for at least one component  $H$  of  $G$ . In particular, if  $G$  has an isolated vertex, then  $\gamma_a(G) = \gamma(G)$ . It also follows from Lemma 2 that for a graph  $G$ ,  $\gamma_a(G) = \gamma(G)$  if  $G$  has one of the following properties: (1)  $G$  has a unique minimum dominating set (see, for example, [4] or [8] for some characterizations of such graphs); (2)  $G$  has a vertex which belongs to every minimum dominating set of  $G$  (see [15]); (3)  $G$  has a vertex adjacent to at least two leaves. Consequently, there is no forbidden subgraph characterization for the class of graphs  $G$  for which  $\gamma_a(G) = \gamma(G)$ , as for any graph  $H$ , we can add an isolated vertex (or two leaves to one vertex of  $H$ ), and in this way form a graph  $H'$  for which  $\gamma_a(H') = \gamma(H')$ .

The *corona*  $F \circ K_1$  of a graph  $F$  is the graph formed from  $F$  by adding a new vertex  $v'$  and edge  $vv'$  for each vertex  $v \in V(F)$ . A graph  $G$  is said to be

a *corona graph* if  $G = F \circ K_1$  for some connected graph  $F$ . We note that each vertex of a corona graph  $G$  is a leaf or it is adjacent to exactly one leaf of  $G$ . Recall that we denote the set of all leaves in a graph  $G$  by  $L_G$ , and set of support vertices in  $G$  by  $S_G$ .

**Lemma 3.** *If  $G$  is a corona graph, then  $\gamma_a(G) > \gamma(G)$ .*

**Proof.** Assume that  $G$  is a corona graph. If  $G = K_1 \circ K_1$ , then  $G = K_2$  and  $\gamma_a(G) = 2$  and  $\gamma(G) = 1$ . Hence, we may assume that  $G = F \circ K_1$  for some connected graph  $F$  of order  $n(F) \geq 2$ . If  $v \in V_G \setminus L_G$ , then let  $\bar{v}$  denote the unique leaf-neighbor of  $v$  in  $G$ . Now let  $D$  be an arbitrary minimum dominating set of  $G$ , and so  $D \in \mathcal{A}_\gamma(G)$ . Then,  $|D \cap \{v, \bar{v}\}| = 1$  for every  $v \in V_G \setminus L_G$ . Consequently,  $D$  and its complement  $V_G \setminus D$  are minimum dominating sets of  $G$ . Thus,  $D$  is not an accurate dominating set of  $G$ . This is true for every minimum dominating set of  $G$ , implying that  $\gamma_a(G) > \gamma(G)$ . ■

**Lemma 4.** *If  $T$  is a tree of order at least three, then there exists a set  $D \in \mathcal{A}_\gamma(T)$  such that the following hold.*

- (a)  $S_T \subseteq D$ .
- (b)  $N_T(v) \subseteq V_T \setminus D$  or  $|\text{pn}_T(v, D)| \geq 2$  for every  $v \in D \setminus S_T$ .

**Proof.** Let  $T$  be a tree of order  $n(T) \geq 3$ . Among all minimum dominating sets of  $T$ , let  $D \in \mathcal{A}_\gamma(T)$  be chosen that

- (1)  $D$  contains as many support vertices as possible.
- (2) Subject to (1), the number of components  $\kappa(T[D])$  is as large as possible.

If the set  $D$  contains a leaf  $v$  of  $T$ , then we can simply replace  $v$  in  $D$  with the support vertex adjacent to  $v$  to produce a new minimum dominating set with more support vertices than  $D$ , a contradiction. Hence, the set  $D$  contains no leaves, implying that  $S_T \subseteq D$ . Suppose, next, that there exists a vertex  $v$  in  $D$  that is not a support vertex of  $T$  and such that  $N_T(v) \not\subseteq V_T \setminus D$ . Thus,  $v$  has at least one neighbor in  $D$ ; that is,  $N_T(v) \cap D \neq \emptyset$ . By the minimality of the set  $D$ , we therefore note that  $\text{pn}_T(v, D) \neq \emptyset$ . If  $|\text{pn}_T(v, D)| = 1$ , say  $\text{pn}_T(v, D) = \{u\}$ , then letting  $D' = (D \setminus \{v\}) \cup \{u\}$ , the set  $D' \in \mathcal{A}_\gamma(T)$  and satisfies  $S_T \subseteq D \setminus \{v\} \subset D'$  and  $\kappa(T[D']) > \kappa(T[D])$ , which contradicts the choice of  $D$ . Hence, if  $v \in D$  is not a support vertex of  $T$  and  $N_T(v) \not\subseteq V_T \setminus D$ , then  $|\text{pn}_T(v, D)| \geq 2$ . ■

We are now in a position to present the following equivalent characterizations of trees for which the accurate domination number is equal to the domination number.

**Theorem 5.** *If  $T$  is a tree of order at least two, then the following statements are equivalent.*

- (1)  $T$  is not a corona graph.
- (2) There exists a set  $D \in \mathcal{A}_\gamma(T)$  such that  $\kappa(T - D) > |D|$ .
- (3)  $\gamma_a(T) = \gamma(T)$ .
- (4) There exists a set  $D \in \mathcal{A}_\gamma(T)$  such that  $D \cap D' \neq \emptyset$  for every  $D' \in \mathcal{A}_\gamma(T)$ .

**Proof.** The statements (3) and (4) are equivalent by Lemma 2. The implication (3)  $\Rightarrow$  (1) follows from Lemma 3. To prove the implication (2)  $\Rightarrow$  (3), let us assume that  $D \in \mathcal{A}_\gamma(T)$  and  $\kappa(T - D) > |D|$ . Thus,  $\gamma(T - D) \geq \kappa(T - D) > |D| = \gamma(T)$ . This proves that  $D$  is an accurate dominating set of  $T$ , and therefore  $\gamma_a(T) = \gamma(T)$ .

Thus it suffices to prove that (1) implies (2). The proof is by induction on the order of a tree. The implication (1)  $\Rightarrow$  (2) is obvious for trees of order two, three, and four. Thus assume that  $T$  is a tree of order at least five and  $T$  is not a corona graph. Let  $D \in \mathcal{A}_\gamma(T)$  and assume that  $S_T \subseteq D$ . Since  $T$  is not a corona graph, the tree  $T$  has a vertex which is neither a leaf nor adjacent to exactly one leaf. We consider two cases, depending on whether  $d_T(v) \geq 3$  for some vertex  $v \in S_T$  or  $d_T(v) = 2$  for every vertex  $v \in S_T$ .

*Case 1.*  $d_T(v) \geq 3$  for some  $v \in S_T$ . Let  $v'$  be a leaf of  $T$  adjacent to  $v$ . Let  $T'$  be a component of  $T - \{v, v'\}$ . Now let  $T_1$  and  $T_2$  be the subtrees of  $T$  induced on the vertex sets  $V_{T'} \cup \{v, v'\}$  and  $V_T \setminus V_{T'}$ , respectively. We note that both trees  $T_1$  and  $T_2$  have order strictly less than  $n(T)$ . Further,  $V(T_1) \cap V(T_2) = \{v, v'\}$ ,  $E(T_1) \cap E(T_2) = \{vv'\}$ , and at least one of  $T_1$  and  $T_2$ , say  $T_1$ , is not a corona graph. Applying the induction hypothesis to  $T_1$ , there exists a set  $D_1 \in \mathcal{A}_\gamma(T_1)$  such that  $\kappa(T_1 - D_1) > |D_1|$ . If  $T_2$  is a corona graph, then choosing  $D_2$  to be the set of support vertices in  $T_2$  we note that  $D_2 \in \mathcal{A}_\gamma(T_2)$  and  $\kappa(T_2 - D_2) = |D_2|$ . If  $T_2$  is not a corona graph, then applying the induction hypothesis to  $T_2$ , there exists a set  $D_2 \in \mathcal{A}_\gamma(T_2)$  such that  $\kappa(T_2 - D_2) > |D_2|$ . In both cases, there exists a set  $D_2 \in \mathcal{A}_\gamma(T_2)$  such that  $\kappa(T_2 - D_2) \geq |D_2|$ . We may assume that all support vertices of  $T_1$  and  $T_2$  are in  $D_1$  and  $D_2$ , respectively. Thus,  $v \in D_1 \cap D_2$ , the union  $D_1 \cup D_2$  is a  $\gamma$ -set of  $T$ , and  $\kappa(T - (D_1 \cup D_2)) = \kappa(T_1 - D_1) + \kappa(T_2 - D_2) - 1 > |D_1| + |D_2| - 1 = |D_1 \cup D_2|$ .

*Case 2.*  $d_T(v) = 2$  for every  $v \in S_T$ . We distinguish two subcases, depending on whether  $D \setminus S_T \neq \emptyset$  or  $D \setminus S_T = \emptyset$ .

*Case 2.1.*  $D \setminus S_T \neq \emptyset$ . Let  $v$  be an arbitrary vertex belonging to  $D \setminus S_T$ . It follows from the second part of Lemma 4 that there are two vertices  $v_1$  and  $v_2$  belonging to  $N_T(v) \setminus D$ . Let  $R$  be the tree obtained from  $T$  by adding a new vertex  $v'$  and the edge  $vv'$ . We note that  $D$  is a minimum dominating set of  $R$  and  $S_R \subseteq D$ . Let  $R'$  be the component of  $R - \{v, v'\}$  containing  $v_1$ . Now let  $R_1$  and  $R_2$  be the subtrees of  $R$  induced by the vertex sets  $V_{R'} \cup \{v, v'\}$  and  $V_R \setminus V_{R'}$ , respectively. We note that both trees  $R_1$  and  $R_2$  have order strictly

less than  $n(T)$ . Further,  $V(R_1) \cap V(R_2) = \{v, v'\}$ ,  $E(R_1) \cap E(R_2) = \{vv'\}$ , and neither  $R_1$  nor  $R_2$  is a corona graph. By the induction hypothesis, there exists a set  $D_1 \in \mathcal{A}_\gamma(R_1)$  and a set  $D_2 \in \mathcal{A}_\gamma(R_2)$  such that  $\kappa(R_1 - D_1) > |D_1|$  and  $\kappa(R_2 - D_2) > |D_2|$ . We may assume that all support vertices of  $R_1$  and  $R_2$  are in  $D_1$  and  $D_2$ , respectively. Thus,  $v \in D_1 \cap D_2$ , the union  $D_1 \cup D_2$  is a  $\gamma$ -set of  $R$ , and

$$\begin{aligned} \kappa(T - (D_1 \cup D_2)) &= \kappa(R - (D_1 \cup D_2)) - 1 = (\kappa(R_1 - D_1) + \kappa(R_2 - D_2) - 1) - 1 \\ &= (\kappa(R_1 - D_1) - |D_1| + \kappa(R_2 - D_2) - |D_2|) - 2 + |D_1| + |D_2| \\ &\geq |D_1| + |D_2| = |D_1 \cup D_2| + 1 > |D_1 \cup D_2|. \end{aligned}$$

*Case 2.2.*  $D \setminus S_T = \emptyset$ . In this case, we note that  $D = S_T$ . Let  $v$  be an arbitrary vertex belonging to  $D$  and assume that  $N_T(v) = \{u, w\}$ , where  $u \in L_T$ . If  $w \in L_T$ , then  $T = K_{1,2}$ , contradicting the assumption that  $n(T) \geq 5$ . If  $w \in S_T$ , then  $T = P_4 = K_2 \circ K_1$ , contradicting the assumption that  $T$  is not a corona graph (and the assumption that  $n(T) \geq 5$ ). Therefore,  $w \in V_T \setminus (L_T \cup S_T)$ . Thus,  $V_T \setminus (L_T \cup S_T)$  is nonempty and  $T - D$  has  $|D|$  one-element components induced by leaves of  $T$  and at least one component induced by  $V_T \setminus (L_T \cup S_T)$ . Consequently,  $\kappa(T - D) \geq |D| + 1 > |D|$ . This completes the proof of Theorem 5.  $\blacksquare$

The equivalence of the statements (1) and (3) of Theorem 5 shows that the trees  $T$  for which  $\gamma_a(T) = \gamma(T)$  are easy to recognize. From Theorem 5 and from the well-known fact that  $\gamma(P_n) = \lceil n/3 \rceil$  for every positive integer  $n$ , we also immediately have the following corollary which provides a slight improvement on Proposition 3 in [12].

**Corollary 6.** *For  $n \geq 1$ ,  $\gamma_a(P_n) = \gamma(P_n) = \lceil n/3 \rceil$  if and only if  $n \in \mathbb{N} \setminus \{2, 4\}$ .*

### 3. DOMINATION OF GENERAL CORONAS OF A GRAPH

Let  $G$  be a graph, and let  $\mathcal{F} = \{F_v : v \in V_G\}$  be a family of nonempty graphs indexed by the vertices of  $G$ . By  $G \circ \mathcal{F}$  we denote the graph with vertex set

$$V_{G \circ \mathcal{F}} = (V_G \times \{0\}) \cup \bigcup_{v \in V_G} (\{v\} \times V_{F_v})$$

and edge set determined by open neighborhoods defined in such a way that

$$N_{G \circ \mathcal{F}}((v, 0)) = (N_G(v) \times \{0\}) \cup (\{v\} \times V_{F_v})$$

for every  $v \in V_G$ , and

$$N_{G \circ \mathcal{F}}((v, x)) = \{(v, 0)\} \cup (\{v\} \times N_{F_v}(x))$$

if  $v \in V_G$  and  $x \in V_{F_v}$ . The graph  $G \circ \mathcal{F}$  is said to be the  $\mathcal{F}$ -corona of  $G$ . Informally,  $G \circ \mathcal{F}$  is the graph obtained by taking a disjoint copy of  $G$  and all the graphs of  $\mathcal{F}$  with additional edges joining each vertex  $v$  of  $G$  to every vertex in the copy of  $F_v$ . If all the graphs of the family  $\mathcal{F}$  are isomorphic to one and the same graph  $F$  (as it was defined by Frucht and Harary [5]), then we simply write  $G \circ F$  instead of  $G \circ \mathcal{F}$ . Recall that a graph  $G$  is said to be a *corona graph* if  $G = F \circ K_1$  for some connected graph  $F$ .

The *2-subdivided graph*  $S_2(G)$  of a graph  $G$  is the graph with vertex set

$$V_{S_2(G)} = V_G \cup \bigcup_{vu \in E_G} \{(v, vu), (u, vu)\}$$

and the adjacency is defined in such a way that

$$N_{S_2(G)}(x) = \{(x, xy) : y \in N_G(x)\}$$

if  $x \in V_G \subseteq V_{S_2(G)}$ , while

$$N_{S_2(G)}((x, xy)) = \{x\} \cup \{(y, xy)\}$$

if  $(x, xy) \in \bigcup_{vu \in E_G} \{(v, vu), (u, vu)\} \subseteq V_{S_2(G)}$ . Less formally,  $S_2(G)$  is the graph obtained from  $G$  by subdividing every edge with two new vertices; that is, by replacing edges  $vu$  of  $G$  with disjoint paths  $(v, (v, vu), (u, vu), u)$ .

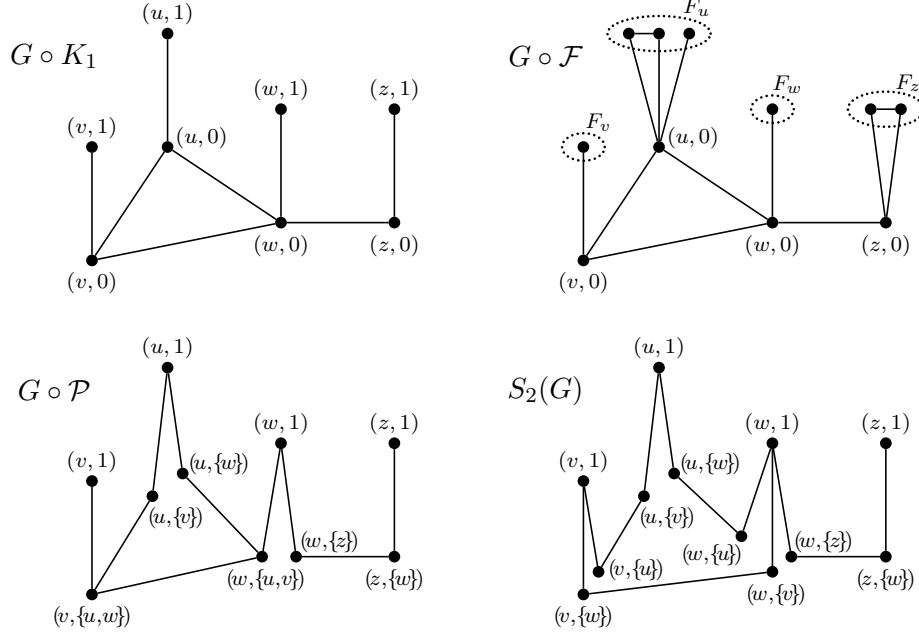
For a graph  $G$  and a family  $\mathcal{P} = \{\mathcal{P}(v) : v \in V_G\}$ , where  $\mathcal{P}(v)$  is a partition of the neighborhood  $N_G(v)$  of the vertex  $v$ , by  $G \circ \mathcal{P}$  we denote the graph with vertex set

$$V_{G \circ \mathcal{P}} = (V_G \times \{1\}) \cup \bigcup_{v \in V_G} (\{v\} \times \mathcal{P}(v))$$

and edge set

$$E_{G \circ \mathcal{P}} = \bigcup_{v \in V_G} \{(v, 1)(v, A) : A \in \mathcal{P}(v)\} \cup \bigcup_{uv \in E_G} \{(v, A)(u, B) : (u \in A) \wedge (v \in B)\}.$$

The graph  $G \circ \mathcal{P}$  is called the  $\mathcal{P}$ -corona of  $G$  and was defined by Dettlaff *et al.* in [2]. It follows from this definition that if  $\mathcal{P}(v) = \{N_G(v)\}$  for every  $v \in V_G$ , then  $G \circ \mathcal{P}$  is isomorphic to the corona  $G \circ K_1$ . On the other hand, if  $\mathcal{P}(v) = \{\{u\} : u \in N_G(v)\}$  for every  $v \in V_G$ , then  $G \circ \mathcal{P}$  is isomorphic to the 2-subdivided graph  $S_2(G)$  of  $G$ . Examples of  $G \circ K_1$ ,  $G \circ \mathcal{F}$ ,  $G \circ \mathcal{P}$ , and  $S_2(G)$  are shown in Figure 1. In this case  $G$  is the graph  $(K_2 \cup K_1) + K_1$  with vertex set  $V_G = \{v, u, w, z\}$  and edge set  $E_G = \{vu, vw, uw, wz\}$ , where the family  $\mathcal{F}$  consists of the graphs  $F_v = F_w = K_1$ ,  $F_z = K_2$ , and  $F_u = K_2 \cup K_1$ , while  $\mathcal{P} = \{\mathcal{P}(x) : x \in V_G\}$  is the family in which  $\mathcal{P}(v) = \{\{u, w\}\}$ ,  $\mathcal{P}(u) = \{\{v\}, \{w\}\}$ ,  $\mathcal{P}(w) = \{\{u, v\}, \{z\}\}$ , and  $\mathcal{P}(z) = \{\{w\}\}$ .

Figure 1. Coronas of  $G = (K_2 \cup K_1) + K_1$ .

We now study relations between the domination number and the accurate domination number of different coronas of a graph. Our first theorem specifies when these two numbers are equal for the  $\mathcal{F}$ -corona  $G \circ \mathcal{F}$  of a graph  $G$  and a family  $\mathcal{F}$  of nonempty graphs indexed by the vertices of  $G$ .

**Theorem 7.** *If  $G$  is a graph and  $\mathcal{F} = \{F_v : v \in V_G\}$  is a family of nonempty graphs indexed by the vertices of  $G$ , then the following holds.*

- (1)  $\gamma(G \circ \mathcal{F}) = |V_G|$ .
- (2)  $\gamma_a(G \circ \mathcal{F}) = \gamma(G \circ \mathcal{F})$  if and only if  $\gamma(F_v) > 1$  for some vertex  $v$  of  $G$ .
- (3)  $|V_G| \leq \gamma_a(G \circ \mathcal{F}) \leq |V_G| + \min\{|V_{F_v}| : v \in V_G\}$ .

**Proof.** (1) It is obvious that  $V_G \times \{0\}$  is a minimum dominating set of  $G \circ \mathcal{F}$  and therefore  $\gamma(G \circ \mathcal{F}) = |V_G \times \{0\}| = |V_G|$ .

(2) If  $\gamma(F_v) > 1$  for some vertex  $v$  of  $G$ , then

$$\gamma(G \circ \mathcal{F} - (V_G \times \{0\})) = \sum_{v \in V_G} \gamma((G \circ \mathcal{F})[\{v\} \times V_{F_v}]) = \sum_{v \in V_G} \gamma(F_v) > |V_G| = |V_G \times \{0\}|$$

and this proves that no subset of  $V_{G \circ \mathcal{F}} \setminus (V_G \times \{0\})$  of cardinality  $|V_G \times \{0\}|$  is a dominating set of  $G \circ \mathcal{F}$ . Consequently  $V_G \times \{0\}$  is a minimum accurate dominating set of  $G \circ \mathcal{F}$  and therefore  $\gamma_a(G \circ \mathcal{F}) = \gamma(G \circ \mathcal{F})$ .



Assume now that  $G$  and  $\mathcal{F}$  are such that  $\gamma_a(G \circ \mathcal{F}) = \gamma(G \circ \mathcal{F})$ . We claim that  $\gamma(F_v) > 1$  for some vertex  $v$  of  $G$ . Suppose, contrary to our claim, that  $\gamma(F_v) = 1$  for every vertex  $v$  of  $G$ . Then the set  $U_v = \{x \in V_{F_v} : N_{F_v}[x] = V_{F_v}\}$ , the set of universal vertices of  $F_v$ , is nonempty for every  $v \in V_G$ . Now, let  $D$  be any minimum dominating set of  $G \circ \mathcal{F}$ . Then,  $|D| = \gamma(G \circ \mathcal{F}) = |V_G \times \{0\}| = |V_G|$ ,  $|D \cap (\{(v, 0)\} \cup (\{v\} \times U_v))| = 1$ , and the set  $(\{(v, 0)\} \cup (\{v\} \times U_v)) \setminus D$  is nonempty for every  $v \in V_G$ . Now, if  $\bar{D}$  is a system of representatives of the family  $\{(\{(v, 0)\} \cup (\{v\} \times U_v)) \setminus D : v \in V_G\}$ , then  $\bar{D}$  is a minimum dominating set of  $G \circ \mathcal{F}$ . Since  $\bar{D}$  and  $D$  are disjoint,  $D$  is not an accurate dominating set of  $G \circ \mathcal{F}$ . Consequently, no minimum dominating set of  $G \circ \mathcal{F}$  is an accurate dominating set and therefore  $\gamma(G \circ \mathcal{F}) < \gamma_a(G \circ \mathcal{F})$ , a contradiction.

(3) The lower bound is obvious as  $|V_G| = \gamma(G \circ \mathcal{F}) \leq \gamma_a(G \circ \mathcal{F})$ . Since  $(V_G \times \{0\}) \cup (\{v\} \times V_{F_v})$  is an accurate dominating set of  $G \circ \mathcal{F}$  (for every  $v \in V_G$ ), we also have the inequality  $\gamma_a(G \circ \mathcal{F}) \leq |V_G| + \min\{|V_{F_v}| : v \in V_G\}$ . This completes the proof of Theorem 7. ■

As a consequence of Theorem 7, we have the following result.

**Corollary 8.** *If  $G$  is a graph, then  $\gamma_a(G \circ K_1) = \gamma(G \circ K_1) + 1 = |V_G| + 1$ .*

**Proof.** Since  $\gamma(K_1) = 1$ , it follows from Theorem 7 that  $\gamma_a(G \circ K_1) \geq \gamma(G \circ K_1) + 1 = |V_G| + 1$ . On the other hand, the set  $(V_G \times \{0\}) \cup \{(v, 1)\}$  is an accurate dominating set of  $G \circ K_1$  and therefore  $\gamma_a(G \circ K_1) \leq |(V_G \times \{0\}) \cup \{(v, 1)\}| = |V_G| + 1$ . Consequently,  $\gamma_a(G \circ K_1) = \gamma(G \circ K_1) = |V_G| + 1$ . ■

From Theorem 7 we know that  $\gamma_a(G \circ \mathcal{F}) = \gamma(G \circ \mathcal{F}) = |V_G|$  if and only if the family  $\mathcal{F}$  is such that  $\gamma(F_v) > 1$  for some  $F_v \in \mathcal{F}$ , but we do not know any general formula for  $\gamma_a(G \circ \mathcal{F})$  if  $\gamma(F_v) = 1$  for every  $F_v \in \mathcal{F}$ . The following theorem shows a formula for the domination number and general bounds for the accurate domination number of a  $\mathcal{P}$ -corona of a graph.

**Theorem 9.** *If  $G$  is a graph and  $\mathcal{P} = \{\mathcal{P}(v) : v \in V_G\}$  is a family of partitions of the vertex neighborhoods of  $G$ , then the following holds.*

- (1)  $\gamma(G \circ \mathcal{P}) = |V_G|$ .
- (2)  $\gamma_a(G \circ \mathcal{P}) \geq |V_G|$ .
- (3)  $\gamma_a(G \circ \mathcal{P}) \leq |V_G| + \min\{\min\{|\mathcal{P}(v)| : v \in V_G\}, 1 + \min\{|A| : A \in \bigcup_{v \in V_G} \mathcal{P}(v)\}\}$ .

**Proof.** It follows from the definition of  $G \circ \mathcal{P}$  that  $V_G \times \{1\}$  is a dominating set of  $G \circ \mathcal{P}$ , and therefore  $\gamma(G \circ \mathcal{P}) \leq |V_G \times \{1\}| = |V_G|$ . On the other hand, let  $D \in \mathcal{A}_\gamma(G \circ \mathcal{P})$ . Then  $D \cap N_{G \circ \mathcal{P}}[(v, 1)] \neq \emptyset$  for every  $v \in V_G$ , and, since the sets  $N_{G \circ \mathcal{P}}[(v, 1)]$  form a partition of  $V_{G \circ \mathcal{P}}$ , we have

$$\gamma(G \circ \mathcal{P}) = |D| = \left| \bigcup_{v \in V_G} (D \cap N_{G \circ \mathcal{P}}[(v, 1)]) \right| = \sum_{v \in V_G} |D \cap N_{G \circ \mathcal{P}}[(v, 1)]| \geq |V_G|.$$

Consequently, we have  $|V_G| = \gamma(G \circ \mathcal{P}) \leq \gamma_a(G \circ \mathcal{P})$ , which proves (1) and (2).

From the definition of  $G \circ \mathcal{P}$  it also follows that each of the sets  $(V_G \times \{1\}) \cup N_{G \circ \mathcal{P}}[(v, 1)]$  (for every  $v \in V_G$ ) and  $(V_G \times \{1\}) \cup N_{G \circ \mathcal{P}}[(v, A)]$  (for every  $v \in V_G$  and  $A \in \mathcal{P}(v)$ ) is an accurate dominating set of  $G \circ \mathcal{P}$ . Hence,

$$\begin{aligned} |(V_G \times \{1\}) \cup N_{G \circ \mathcal{P}}[(v, 1)]| &= |V_G \times \{1\}| + |N_{G \circ \mathcal{P}}((v, 1))| = |V_G| + |\mathcal{P}(v)| \\ &\geq |V_G| + \min\{|\mathcal{P}(v)| : v \in V_G\} \geq \gamma_a(G \circ \mathcal{P}), \end{aligned}$$

and similarly

$$\begin{aligned} |(V_G \times \{1\}) \cup N_{G \circ \mathcal{P}}[(v, A)]| &= |(V_G \times \{1\}) \cup \{(v, 1)\} \cup N_{G \circ \mathcal{P}}((v, A))| \\ &= |V_G| + 1 + |A| \\ &\geq |V_G| + 1 + \min\{|A| : A \in \bigcup_{v \in V_G} \mathcal{P}(v)\}. \end{aligned}$$

Therefore,

$$\gamma_a(G \circ \mathcal{P}) \leq |V_G| + \min \left\{ \min\{|\mathcal{P}(v)| : v \in V_G\}, 1 + \min \left\{ |A| : A \in \bigcup_{v \in V_G} \mathcal{P}(v) \right\} \right\}.$$

This completes the proof of Theorem 9.  $\blacksquare$

We do not know all the pairs  $(G, \mathcal{P})$  achieving equality in the upper bound for the accurate domination number of a  $\mathcal{P}$ -corona of a graph, but Theorem 10 and Corollaries 11 and 12 show that the bounds in Theorem 9 are best possible. The next theorem also shows that the domination number and the accurate domination number of a 2-subdivided graph are easy to compute.

**Theorem 10.** *If  $G$  is a connected graph, then the following holds.*

- (1)  $\gamma(S_2(G)) = |V_G|$ .
- (2)  $|V_G| \leq \gamma_a(S_2(G)) \leq |V_G| + 2$ .
- (3)  $\gamma_a(S_2(G)) = \begin{cases} |V_G| + 2, & \text{if } G \text{ is a cycle,} \\ |V_G| + 1, & \text{if } G = K_2, \\ |V_G|, & \text{otherwise.} \end{cases}$

**Proof.** The statement (1) follows from Theorem 9(1).

(2) The inequalities  $|V_G| \leq \gamma_a(S_2(G)) \leq |V_G| + 2$  are obvious if  $G = K_1$ . Thus assume that  $G$  is a connected graph of order at least two. Let  $u$  and  $v$  be adjacent vertices of  $G$ . Then,  $V_G \cup \{(v, vu), (u, vu)\}$  is an accurate dominating set of  $S_2(G)$  and we have  $|V_G| = \gamma(S_2(G)) \leq \gamma_a(S_2(G)) \leq |V_G \cup \{(v, vu), (u, vu)\}| = |V_G| + 2$ .

(3) The connectivity of  $G$  implies that there are three cases to consider.

*Case 1.*  $|E_G| > |V_G|$ . In this case  $S_2(G) - V_G$  has  $|E_G|$  components and therefore no  $|V_G|$ -element subset of  $V_{S_2(G)} \setminus V_G$  dominates  $S_2(G)$ . Hence,  $V_G$  is an accurate dominating set of  $S_2(G)$  and  $\gamma_a(S_2(G)) = |V_G|$ .

*Case 2.*  $|E_G| = |V_G|$ . In this case,  $G$  is a unicyclic graph. First, if  $G$  is a cycle, say  $G = C_n$ , then  $S_2(G) = C_{3n}$  and  $\gamma_a(S_2(G)) = \gamma_a(C_{3n}) = n + 2 = |V_G| + 2$  (see Proposition 3 in [12]). Thus assume that  $G$  is a unicyclic graph which is not a cycle. Then  $G$  has a leaf, say  $v$ . Now, if  $u$  is the only neighbor of  $v$ , then  $(V_G \setminus \{v\}) \cup \{(v, vu)\}$  is a minimum dominating set of  $S_2(G)$ . Since  $S_2(G) - ((V_G \setminus \{v\}) \cup \{(v, vu)\})$  has  $|V_G| + 1$  components,  $(V_G \setminus \{v\}) \cup \{(v, vu)\}$  is a minimum accurate dominating set of  $S_2(G)$  and  $\gamma_a(S_2(G)) = |(V_G \setminus \{v\}) \cup \{(v, vu)\}| = |V_G|$ .

*Case 3.*  $|E_G| = |V_G| - 1$ . In this case,  $G$  is a tree. Now, if  $G = K_1$ , then  $S_2(G) = K_1$  and  $\gamma_a(S_2(G)) = \gamma_a(K_1) = 1 = |V_G|$ . If  $G = K_2$ , then  $S_2(G) = P_4$  and  $\gamma_a(S_2(G)) = \gamma_a(P_4) = 3 = 2 + 1 = |V_G| + 1$ . Finally, if  $G$  is a tree of order at least three, then the tree  $S_2(G)$  is not a corona graph and by (1) and Theorem 5 we have  $\gamma_a(S_2(G)) = \gamma(S_2(G)) = |V_G|$ . ■

As a consequence of Theorem 10, we have the following results.

**Corollary 11.** *If  $T$  is a tree and  $\mathcal{P} = \{\mathcal{P}(v) : v \in V_T\}$  is a family of partitions of the vertex neighborhoods of  $T$ , then*

$$\gamma_a(T \circ \mathcal{P}) = \begin{cases} |V_T| + 1, & \text{if } |\mathcal{P}(v)| = 1 \text{ for every } v \in V_T, \\ |V_T|, & \text{if } |\mathcal{P}(v)| > 1 \text{ for some } v \in V_T. \end{cases}$$

**Proof.** If  $|\mathcal{P}(v)| = 1$  for every  $v \in V_T$ , then  $T \circ \mathcal{P} = T \circ K_1$  and the result follows from Corollary 8. If  $|\mathcal{P}(v)| > 1$  for some  $v \in V_T$ , then the tree  $T \circ \mathcal{P}$  is not a corona and the result follows from Theorem 5 and Theorem 9 (1). ■

**Corollary 12.** *For  $n \geq 3$ , if  $\mathcal{P} = \{\mathcal{P}(v) : v \in V_{C_n}\}$  is a family of partitions of the vertex neighborhoods of  $C_n$ , then*

$$\gamma_a(C_n \circ \mathcal{P}) = \begin{cases} n + 1, & \text{if } |\mathcal{P}(v)| = 1 \text{ for every } v \in V_{C_n}, \\ n + 2, & \text{if } |\mathcal{P}(v)| = 2 \text{ for every } v \in V_{C_n}, \\ n, & \text{otherwise.} \end{cases}$$

**Proof.** If  $|\mathcal{P}(v)| = 1$  for every  $v \in V_{C_n}$ , then  $C_n \circ \mathcal{P} = C_n \circ K_1$ . Thus, by Theorem 8, we have  $\gamma_a(C_n \circ \mathcal{P}) = \gamma_a(C_n \circ K_1) = \gamma(C_n \circ K_1) = |V_{C_n}| + 1 = n + 1$ .

If  $|\mathcal{P}(v)| > 1$  (and therefore  $|\mathcal{P}(v)| = 2$ ) for every  $v \in V_{C_n}$ , then  $C_n \circ \mathcal{P} = S_2(C_n) = C_{3n}$ . Now, since  $\gamma_a(C_{3n}) = n + 2$  (as it was observed in [12]), we have  $\gamma_a(C_n \circ \mathcal{P}) = \gamma_a(C_{3n}) = n + 2$ .

Finally assume that there are vertices  $u$  and  $v$  in  $C_n$  such that  $|\mathcal{P}(v)| = 1$  and  $|\mathcal{P}(u)| = 2$ . Then the sets

$$V_{C_n}^1 = \{x \in V_{C_n} : |\mathcal{P}(x)| = 1\} \quad \text{and} \quad V_{C_n}^2 = \{y \in V_{C_n} : |\mathcal{P}(y)| = 2\}$$

form a partition of  $V_{C_n}$ . Without loss of generality we may assume that  $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_\ell, \dots, z_1, z_2, \dots, z_p, t_1, t_2, \dots, t_q$  are the consecutive vertices of  $C_n$ , where

$$x_1, x_2, \dots, x_k \in V_{C_n}^1, y_1, y_2, \dots, y_\ell \in V_{C_n}^2, \dots, z_1, z_2, \dots, z_p \in V_{C_n}^1, \\ t_1, t_2, \dots, t_q \in V_{C_n}^2,$$

and  $k + \ell + \dots + p + q = n$ . It is easy to observe that  $D = \{(x_i, N_{C_n}(x_i)) : i = 1, \dots, k\} \cup \{(y_j, 1) : j = 1, \dots, \ell\} \cup \dots \cup \{(z_i, N_{C_n}(z_i)) : i = 1, \dots, p\} \cup \{(t_j, 1) : j = 1, \dots, q\}$  is a dominating set of  $C_n \circ \mathcal{P}$ . Since the set  $D$  is of cardinality  $n = |V_{C_n}|$  and  $n = \gamma(C_n \circ \mathcal{P})$  (by Theorem 9 (1)),  $D$  is a minimum dominating set of  $C_n \circ \mathcal{P}$ . In addition, since  $C_n \circ \mathcal{P} - D$  has  $k + (2 + (\ell - 1)) + \dots + p + (2 + (q - 1)) > k + \ell + \dots + p + q = n$  components, that is, since  $\kappa(C_n \circ \mathcal{P} - D) > n$ , no  $n$ -element subset of  $V_{C_n \circ \mathcal{P}} \setminus D$  is a dominating set of  $C_n \circ \mathcal{P}$ . Thus,  $D$  is an accurate dominating set of  $C_n \circ \mathcal{P}$  and therefore  $\gamma(C_n \circ \mathcal{P}) = n$ . ■

#### 4. CLOSING OPEN PROBLEMS

We close with the following list of open problems that we have yet to settle.

**Problem 13.** Find a formula for the accurate domination number  $\gamma_a(G \circ \mathcal{F})$  of the  $\mathcal{F}$ -corona of a graph  $G$  depending only on the family  $\mathcal{F} = \{F_v : v \in V_G\}$  such that  $\gamma(F_v) = 1$  for every  $v \in V_G$ .

**Problem 14.** Characterize the graphs  $G$  and the families  $\mathcal{P} = \{\mathcal{P}(v) : v \in V_G\}$  for which  $\gamma_a(G \circ \mathcal{P}) = |V_G| + \min\{\min\{|\mathcal{P}(v)| : v \in V_G\}, 1 + \min\{|A| : A \in \bigcup_{v \in V_G} \mathcal{P}(v)\}\}$ .

**Problem 15.** It is a natural question to ask if there exists a nonnegative integer  $k$  such that  $\gamma_a(G \circ \mathcal{P}) \leq |V_G| + k$  for every graph  $G$  and every family  $\mathcal{P} = \{\mathcal{P}(v) : v \in V_G\}$  of partitions of the vertex neighborhoods of  $G$ .

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