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GRAPHS WITH LARGE SEMIPAIRED DOMINATION NUMBER

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Abstract

Let G be a graph with vertex set V and no isolated vertices. A subset $S \subseteq V$ is a semipaired dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex in S and S can be partitioned into two element subsets such that the vertices in each subset are at most distance two apart. The semipaired domination number $\gamma_{\rm pr2}(G)$ is the minimum cardinality of a semipaired dominating set of G. We show that if G is a connected graph G of order $n \geq 3$, then $\gamma_{\rm pr2}(G) \leq \frac{2}{3}n$, and we characterize the extremal graphs achieving equality in the bound.

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1. Introduction

Paired domination was introduced in [6, 7] and a relaxed version of paired domination, called semipaired domination, was defined in [5]. Specifically, a set S of vertices in a graph G is a *dominating set* of G if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. Further, the set S is a *total dominating set* of G if every vertex of $V(G)$ is adjacent to a vertex in S. A dominating set S is a *paired dominating set* of G if the subgraph induced by S, denoted $G[S]$, contains a perfect matching. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G and the *paired domination number* $\gamma_{\rm pr}(G)$ is the minimum cardinality of a paired dominating set of G.

A relaxed form of total domination called semitotal domination was introduced by Goddard, Henning and McPillan [2], and studied further in [9, 10, 11, 12] and elsewhere. A set S of vertices in a graph G with no isolated vertices is a *semitotal dominating set* of G if S is a dominating set of G and every vertex in S is within distance 2 of another vertex of S.

We introduced a similar relaxation of paired domination in [5]. A set S of vertices in a graph G with no isolated vertices is a *semipaired dominating set*, abbreviated SPD-set, of G if S is a dominating set of G and every vertex in S is paired with exactly one other vertex in S that is within distance 2 from it. In other words, the vertices in the dominating set S can be partitioned into 2-sets such that if $\{u, v\}$ is a 2-set, then $uv \in E(G)$ or the distance between u and v is 2. We say that u and v are *paired*. We call such a pairing a *semi-matching*. The *semipaired domination number*, denoted by $\gamma_{\rm pr2}(G)$, is the minimum cardinality of a SPD-set of G. We call a semipaired dominating set of cardinality $\gamma_{\rm pr2}(G)$ a $\gamma_{\rm pr2}$*-set* of G. Note that both the paired domination number and the semipaired domination number are even integers. For more thorough treatment of domination, see the books [3, 4]. For a survey of paired domination, see [1].

1.1. Terminology and notation

For notation and graph theory terminology, we in general follow [13]. Specifically, let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n(G) = |V|$ and edge set $E = E(G)$ of size $m(G) = |E|$, and let v be a vertex in V. We denote the *degree* of v in G by $d_G(v)$. The minimum degree among the vertices of G is denoted by $\delta(G)$. The *open neighborhood* of v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, the graph obtained from G by deleting the vertices in S and all edges incident with S is denoted by $G - S$. If the graph G is clear from the context, we omit it in the above expressions. For example, we write n, m, $d(u)$, $N(v)$ and $N[v]$ rather than $n(G)$, $m(G)$, $d_G(u)$, $N_G(v)$ and $N_G[v]$, respectively.

A *leaf* of G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex with at least two leaf-neighbors. A *star* is a tree with at most one vertex that is not a leaf. The *double star* $S_{r,s}$ is the tree with exactly two adjacent non-leaf vertices, one of which is adjacent to r leaves and the other to s leaves. A *cycle* and *path* on n vertices are denoted by C_n and P_n, respectively.

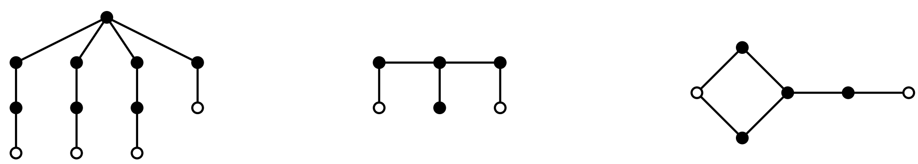
A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. We denote all the children of a vertex v by $C(v)$. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. Thus, every child of v is a descendant of v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

The *distance* between two vertices u and v in a connected graph G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. A subset S of vertices in a graph G is a *packing* if the closed neighborhoods of vertices in S are pairwise disjoint. An *isolate-free* graph is a graph with no isolated vertex.

We use the standard notation $[k] = \{1, \ldots, k\}$.

1.2. Special graphs and families

The *corona* $G \circ P_1$ of a graph G, also denoted $\mathrm{cor}(G)$ in the literature, is the graph obtained from G by adding a pendant edge to each vertex of G. The 2-*corona* $G \circ P_2$ of a graph G is the graph of order $3|V(G)|$ obtained from G by attaching a path of length 2 to each vertex of G so that the resulting paths are vertex-disjoint. The 2-corona $K_{1,3} \circ P_2$ of a star $K_{1,3}$ and the corona $P_3 \circ P_1$ of a path P_3 are illustrated in Figure 1(a) and 1(b), respectively, where the darkened vertices represent a minimum semipaired dominating set. The graph illustrated in Figure 1(c) that is obtained from a cycle C_4 by attaching a path of length 2 to one of its vertices is called the *stingray*, or just SR for short.



(a) $K_{1,3} \circ P_2$ (b) $P_3 \circ P_1$ (c) SR

Figure 1. Special graphs.

1.3. Known results

Every paired dominating set of a graph G is a SPD-set and every SPD-set is a dominating set. Hence, we have the following observation, where it is observed in [7] that $\gamma_{\rm pr}(G) \le 2\gamma(G)$ for every graph G with no isolated vertices.

Observation 1. *If G is an isolate-free graph, then $\gamma(G) \le \gamma_{\rm pr2}(G) \le \gamma_{\rm pr}(G) \le 2\gamma(G)$.*

The following sharp upper bound on the paired-domination number of a connected graph of order at least 3 was given in [7].

Theorem 2 [7]**.** *If G is a connected graph of order $n \ge 3$, then $\gamma_{\rm pr}(G) \le n-1$ with equality if and only if G is C_3, C_5 or a subdivided star.*

If minimum degree is at least 2 and the order at least 6, then the upper bound in Theorem 2 on the paired-domination number can be improved from one less than its order to two-thirds its order.

Theorem 3 [7, 14]**.** *If G is a connected graph of order $n \ge 6$ and minimum degree at least 2, then $\gamma_{\rm pr}(G) \le \frac{2}{3}n$.*

The graphs achieving equality in Theorem 3 are characterized in [8]. As a consequence of this result, if G is a connected graph of order $n \ge 10$ with minimum degree at least 2, then $\gamma_{\rm pr}(G) \le \frac{2}{3}(n-1)$, and this bound is tight.

1.4. Main results

Our aim in this paper is to show that the tight upper bound of $n-1$ on $\gamma_{\rm pr}(G)$ given in Theorem 2 can be significantly improved for the semipaired domination number. More precisely, we prove that the upper bound of $2n/3$ on $\gamma_{\rm pr}(G)$ given in Theorem 3 holds for $\gamma_{\rm pr2}(G)$ if we relax the minimum degree two condition. A proof of Theorem 4 is given in Section 2.

Theorem 4. *If T is a tree of order $n \ge 3$, then $\gamma_{\rm pr2}(G) \le \frac{2}{3}n$, with equality if and only if T is the corona, $P_3 \circ P_1$, of a path P_3 or T is the 2-corona of a tree.*

More generally, we prove the following result. A proof of Theorem 5 is given in Section 3.

Theorem 5. *If G is a connected graph of order $n \ge 3$, then $\gamma_{\rm pr2}(G) \le \frac{2}{3}n$, with equality if and only if one of the following hold.*

(a) *G is a cycle C_3 or a cycle C_6.*
(b) *G is the corona $P_3 \circ P_1$ of a path P_3.*
(c) *G is the corona $C_3 \circ P_1$ of a cycle C_3.*
(d) *G is the stingray SR.*
(e) *G is the 2-corona of a connected graph.*

2. Proof of Theorem 4

In this section, we prove Theorem 4. We first prove two preliminary lemmas.

Lemma 6. *If T is a tree of order at least 2, then there exists a minimum SPD-set of T that contains all the support vertices of T.*

Proof. Let T be a tree of order at least 2, and let S be a minimum SPD-set of T that contains as many support vertices of T as possible. Suppose, to the contrary, that there is a support vertex v of T that does not belong to S. Let u be a leaf neighbor of v. In order to dominate u, we note that $u \in S$. Let u' be the vertex of S that is paired with u. Since $v \notin S$, we note that u' is a neighbor of v distinct from u. Replacing u in S with the vertex v, produces a minimum SPD-set, S', of T where v is paired with u' and where all other pairings of vertices remain the same as the original pairings in S. Since S' is a minimum SPD-set of T that contains more support vertices than does S, we contradict our choice of the set S. Hence, every support vertex of T belongs to S. ∎

We prove next that the semipaired domination number of the 2-corona of a tree is exactly two-thirds its order.

Lemma 7. *If T is the 2-corona of a tree and T has order n, then $\gamma_{\rm pr2}(T) = \frac{2}{3}n$.*

Proof. Let T be the 2-corona of a tree T', and so $T = T' \circ P_2$. Let T' have order n', and so T has order $n = 3n'$. If $n' = 1$, then $T = P_3$, $n = 3$, and $\gamma_{\rm pr2}(T) = 2 = 2n/3$. If $n' = 2$, then $T' = P_2$, $T = P_6$, $n = 6$, and $\gamma_{\rm pr2}(T) = 4 = 2n/3$. Hence, we may assume that $n' \ge 3$, and so $n \ge 9$. Let X be the set of support vertices in T, and so $|X| = n/3$. We note that $\gamma(T) = |X| = n/3$ and the set X is the unique minimum dominating set of T. By Observation 1, $\gamma_{\rm pr2}(T) \le 2\gamma(T) = 2|X| = 2n/3$. We show next that $\gamma_{\rm pr2}(T) \ge 2n/3$. By Lemma 6, there exists a minimum SPD-set, S, of T that contains all the support vertices of T. Thus, $X \subseteq S$. Since the set X is a packing in T, no two vertices of X are paired together in S, implying that each vertex in X is paired with a vertex in $V(T) \setminus X$. Thus, $\gamma_{\rm pr2}(T) = |S| \ge 2|X| = 2n/3$. Consequently, $\gamma_{\rm pr2}(T) = 2n/3$. ∎

We are now in a position to prove Theorem 4. Recall its statement.

Theorem 4. *If T is a tree of order $n \ge 3$, then $\gamma_{\rm pr2}(G) \le \frac{2}{3}n$, with equality if and only if T is the corona, $P_3 \circ P_1$, of a path P_3 or T is the 2-corona of a tree.*

Proof. We proceed by induction of the order $n \ge 3$ of a tree T to prove that $\gamma_{\rm pr2}(T) \le 2n/3$ and that if equality holds, then $T = P_3 \circ P_1$ or T is the 2-corona of a tree. If $n = 3$, then $T = P_3$ and $\gamma_{\rm pr2}(T) = 2 = 2n/3$. Further in this case we note that $T = K_1 \circ P_2$ is the 2-corona of a trivial tree K_1. This establishes the base case. Suppose that $n \ge 4$ and that for every tree T' of order n', where

$3 \le n' < n$, $\gamma_{\rm pr2}(T') \le 2n'/3$, and that if equality holds, then $T' = P_3 \circ P_1$ or T' is the 2-corona of a tree. Let T be a tree of order n.

Suppose that T has a strong support vertex z. Let u and v be two leaf neighbors of z, and consider the tree $T' = T - v$ of order $n' = n - 1 \ge 3$. By Lemma 6, there exists a minimum SPD-set, S', of T' that contains all the support vertices of T'. In particular, the set S' contains the support vertex z of T', implying that S' is a SPD-set of T. Applying our inductive hypothesis to the tree T', we have $\gamma_{\rm pr2}(T) \le |S'| = \gamma_{\rm pr2}(T') \le 2n'/3 = 2(n-1)/3 < 2n/3$. Hence, we may assume that T has no strong support vertex, for otherwise the desired result holds. Thus, every support vertex of T has exactly one leaf neighbor. Since T has order $n \ge 4$ and T has no strong support vertex, we note that ${\rm diam}(T) \ge 3$. If ${\rm diam}(T) = 3$, then T is a path P_4, and so $n = 4$ and $\gamma_{\rm pr2}(T) = 2 < 2n/3$. Hence, ${\rm diam}(T) \ge 4$. We proceed further with the following claim.

Claim 8. *If* ${\rm diam}(T) = 4$*, then* $\gamma_{\rm pr2}(T) \le \frac{2}{3}n$*, with equality if and only if* $T = P_3 \circ P_1$*.*

Proof. Suppose that ${\rm diam}(T) = 4$. Since T has no strong support vertex, either T is obtained from a star $K_{1,k}$ where $k \ge 2$ by subdividing every edge of T exactly once or T is obtained from a star $K_{1,k+1}$ where $k \ge 2$ by subdividing k edges of T exactly once.

Suppose firstly that T is obtained from a star $K_{1,k}$ where $k \ge 2$ by subdividing every edge of T exactly once. In this case, $n = 2k + 1$. Let w denote the central vertex of T and let $v_1, v_2, \ldots, v_k$ denote the neighbors of w. If $k \ge 2$ is even, then the set $N(w)$ is a SPD-set of T, with v_{2i-1} paired with v_{2i} for $i \in \left[\frac{k}{2}\right]$, and so $\gamma_{\rm pr2}(T) \le k = (n-1)/2$. If $k \ge 3$ is odd, then the set $N[w]$ is a SPD-set of T, with v_{2i-1} paired with v_{2i} for $i \in \left[\frac{k-1}{2}\right]$ and with w paired with v_k, and so $\gamma_{\rm pr2}(T) \le k + 1 = (n+1)/2$. In both cases, $\gamma_{\rm pr2}(T) \le (n+1)/2 < 2n/3$.

Suppose secondly that T is obtained from a star $K_{1,k+1}$ where $k \ge 2$ by subdividing k edges of T exactly once. In this case, $n = 2k+2$. Once again, let w denote the central vertex of T. Further, let x denote the leaf neighbor of w and let $v_1, v_2, \ldots, v_k$ denote the non-leaf neighbors of w. If $k \ge 3$ is odd, then the set $N[w] \setminus \{x\}$ is a SPD-set of T, with v_{2i-1} paired with v_{2i} for $i \in \left[\frac{k-1}{2}\right]$ and with w paired with v_k, and so $\gamma_{\rm pr2}(T) \le k + 1 = n/2 < 2n/3$. If $k \ge 4$ is even, then the set $N[w]$ is a SPD-set of T, with v_{2i-1} paired with v_{2i} for $i \in \left[\frac{k}{2}\right]$ and with w paired with x, and so $\gamma_{\rm pr2}(T) \le k+2 = n/2+1 \le 2n/3$. If $k \ge 3$, then $n \ge 8$ and $\gamma_{\rm pr2}(T) \le n/2+1 < 2n/3$. If $k = 2$, then $T = P_3 \circ P_1$ and $\gamma_{\rm pr2}(T) = 4 = 2n/3$. $\Box$

By Claim 8, we may assume that ${\rm diam}(T) \ge 5$, for otherwise the desired result holds. This implies that $n \ge 6$. If $n = 6$, then $T = P_6$ is the 2-corona of a tree P_2. Hence, we may further assume that $n \ge 7$. Let u and r be two vertices at maximum distance apart in T. Necessarily, u and r are leaves and

$d_T(u,v) = \text{diam}(T)$. We now root the tree T at the vertex r. Let v be the parent of u, w the parent of v, x the parent of w, y the parent of x, and z the parent of y. If $\text{diam}(T) = 5$, we note that $r = z$.

By our choice of u, every child of v is a leaf of T. Since T has no strong support vertex, $d_T(v) = 2$ and so $N_T(v) = \{u, w\}$. Furthermore, every child of w is either a leaf or a support vertex of degree 2, and w has at most one leaf neighbor. We consider two cases depending on the degree of w in T. Let $T' = T - T_w$ and let T' have order n'. Recall that $n \geq 7$. Since $\text{diam}(T) \geq 5$, we note that $\{x, y, z\} \subseteq V(T')$, and so $n' \geq 3$. With our earlier assumptions, we prove next the following two claims.

Claim 9. *If $d_T(w) = 2$, then $\gamma_{\text{pr2}}(T) \leq \frac{2}{3}n$, with equality if and only if T is the 2-corona of the tree.*

Proof. Suppose that $d_T(w) = 2$. In this case, $n' = n - 3 \geq 4$. By the inductive hypothesis, $\gamma_{\text{pr2}}(T') \leq 2n'/3$, and if equality holds, then $T' = P_3 \circ P_1$ or T' is the 2-corona of a tree. Every γ_{pr2}-set of T' can be extended to a SPD-set of T by adding to it the pair of vertices v and w, and so $\gamma_{\text{pr2}}(T) \leq \gamma_{\text{pr2}}(T') + 2 \leq 2n'/3 + 2 = 2n/3$. Suppose that $\gamma_{\text{pr2}}(T) = 2n/3$. Thus, we must have equality throughout the above inequality chain. In particular, $\gamma_{\text{pr2}}(T') = 2n'/3$, and so $T' = P_3 \circ P_1$ or $T' = H' \circ P_2$ is the 2-corona of some tree H'.

Suppose that $T' = P_3 \circ P_1$, and so T' is the tree illustrated in Figure 1(b). We note that $n' = 6$ and $n = 9$. Let $\{a, b, c\}$ be the set of support vertices of T', and let a', b' and c' be the leaf neighbors of a, b and c, respectively, where abc is a path P_3. By symmetry, we may assume renaming vertices of T' if necessary, that $x \in \{a, a', b, b'\}$. If $x \in \{a, a'\}$, then $S = \{b, c, v, x\}$ is a SPD-set where v and x are paired and b and c are paired. If $x \in \{b, b'\}$, then $S = \{a, c, v, x\}$ is a SPD-set where a and c are paired and v and x are paired. In both cases, $\gamma_{\text{pr2}}(T) \leq |S| = 4 < 2n/3$, a contradiction.

Hence, $T' = H' \circ P_2$ is the 2-corona of some tree H'. Since $n' \geq 4$, we note that $n(H') \geq 2$. Let X' be the set of support vertices of T', and let $S' = X' \cup V(H')$. We note that S' is a SPD-set of T' of size $2n'/3$, and is therefore a γ_{pr2}-set of T'. If x is leaf in T', then noting that $n(H') \geq 2$, the set $(S' \setminus \{y, z\}) \cup \{x, v\}$ is a SPD-set of T, implying that $\gamma_{\text{pr2}}(T) \leq |S'| = \gamma_{\text{pr2}}(T') < 2n/3$, a contradiction. Suppose that x is a support vertex in T'. Since $n(H') \geq 2$, we note that in this case the vertex y is the neighbor of x that belongs to $V(H')$. The set $(S' \setminus \{y\}) \cup \{v\}$ is a SPD-set of T, and so $\gamma_{\text{pr2}}(T) \leq |S'| = \gamma_{\text{pr2}}(T') < 2n/3$, a contradiction. Hence, $x \in V(H')$. Let H be the tree obtained from H' by adding to it the vertex w and the edge wx. We note that $H = T[V(H') \cup \{w\}]$ and that T is the 2-corona of the tree H; that is, $T = H \circ P_2$. Thus, if $\gamma_{\text{pr2}}(T) = 2n/3$, then T is the 2-corona of the tree. This completes the proof of Claim 9. □

Claim 10. *If $d_T(w) \geq 3$, then $\gamma_{\rm pr2}(T) < \frac{2}{3}n$.*

Proof. Suppose that $d_T(w) \geq 3$. We note that the maximal subtree, T_w, of T at w is either obtained from a star $K_{1,k}$ where $k \geq 2$ by subdividing every edge of T exactly once or is obtained from a star $K_{1,k+1}$ where $k \geq 2$ by subdividing k edges of T exactly once. Let $n_w = n(T_w)$. An identical proof as in the proof of Claim 8 shows that either $\gamma_{\rm pr2}(T_w) < \frac{2}{3}n_w$ or $T_w = P_3 \circ P_1$ and $\gamma_{\rm pr2}(T_w) = \frac{2}{3}n_w$. Every minimum SPD-set of T' can be extended to a SPD-set of T by adding to it a minimum SPD-set of T_w, where the pairing of the vertices is preserved. Thus,

$$\gamma_{\rm pr2}(T) \leq \gamma_{\rm pr2}(T') + \gamma_{\rm pr2}(T_w) \leq \frac{2}{3}n' + \frac{2}{3}n_w = \frac{2}{3}n. \tag{1}$$

We show that $\gamma_{\rm pr2}(T) < \frac{2}{3}n$. Suppose, to the contrary, that $\gamma_{\rm pr2}(T) = \frac{2}{3}n$. Then we must have equality throughout the above inequality chain (1). In particular, $\gamma_{\rm pr2}(T') = \frac{2}{3}n'$, implying that $T' = P_3 \circ P_1$ or $T' = H' \circ P_2$ is the 2-corona of some tree H', and $\gamma_{\rm pr2}(T_w) = \frac{2}{3}n_w$, implying that $T_w = P_3 \circ P_1$. Let v' be the leaf neighbor of w in T_w, and let v_1 and v_2 be the two children of w that are support vertices. We note that either $v = v_1$ or $v = v_2$. We can choose the set S_w to consist of w and its three children, where w is paired with v' and where v_1 and v_2 are paired.

Suppose that $T' = P_3 \circ P_1$. Thus, $n' = 6$ and $n = 12$. If x is a leaf in T', then the set $(S_w \setminus \{v'\}) \cup \{x\}$ can be extended to a SPD-set of T by adding to it the two support vertices of T' that are not adjacent to x. If x is a support vertex of T, then the set $(S_w \setminus \{v'\}) \cup \{x\}$ can be extended to a SPD-set of T by adding to it the two support vertices of T' different from x. In both cases, the vertices w and x are paired and the two support vertices of T' different from x are paired. Thus, $\gamma_{\rm pr2}(T) \leq 4 < \frac{2}{3}n$, a contradiction.

Hence, $T' = H' \circ P_2$ is the 2-corona of some tree H'. Let X' be the set of support vertices of T', and let $S' = X' \cup V(H')$. We note that S' is a SPD-set of T' of size $2n'/3$, and is therefore a $\gamma_{\rm pr2}$-set of T'. Suppose that x is a leaf in T'. In this case, the set $(S' \setminus \{z\}) \cup (S_w \setminus \{v'\})$ is a SPD-set of T with w and y paired and where all other pairings of vertices remain the same as the original pairings. Suppose that x is a support vertex in T'. We note that in this case, the vertex y is the neighbor of x that belongs to H'. The set $(S' \setminus \{y\}) \cup (S_w \setminus \{v'\})$ is a SPD-set of T with w and x paired and where all other pairings of vertices remain the same as the original pairings. Suppose that x belongs to $V(H')$. Let x'' be the neighbor of x in T' that does not belong to H'. In this case, the set $(S' \setminus \{x\}) \cup (S_w \setminus \{v'\})$ is a SPD-set of T with w and x'' paired and where all other pairings of vertices remain the same as the original pairings. In all three cases, $\gamma_{\rm pr2}(T) \leq |S'| + |S_w| - 2 = \gamma_{\rm pr2}(T') + \gamma_{\rm pr2}(T_w) - 2 = \frac{2}{3}n - 2$, a contradiction. Therefore, $\gamma_{\rm pr2}(T) < \frac{2}{3}n$, as claimed. This completes the proof of Claim 10. □

The proof of Theorem 4 now follows from Claim 9 and Claim 10. ■

3. Proof of Theorem 5

In this section, we prove Theorem 5. Recall its statement.

Theorem 5. *If G is a connected graph of order $n \geq 3$, then $\gamma_{\rm pr2}(G) \leq \frac{2}{3}n$, with equality if and only if $G \in \{C_3, C_6, P_3 \circ P_1, C_3 \circ P_1\}$ or G is the stingray SR or G is the 2-corona of a connected graph.*

Proof. Let G be a connected graph of order $n \geq 3$ and let T be an arbitrary spanning tree of G. Since deleting a cycle edge from a graph cannot decrease the semipaired domination number, as an immediate consequence of Theorem 4, we have that $\gamma_{\rm pr2}(G) \leq \gamma_{\rm pr2}(T) \leq \frac{2}{3}n$.

Suppose that $\gamma_{\rm pr2}(G) = \frac{2}{3}n$. Thus, we must have equality throughout the above inequality chain, implying that $\gamma_{\rm pr2}(G) = \gamma_{\rm pr2}(T)$ and $\gamma_{\rm pr2}(T) = \frac{2}{3}n$ for every spanning tree T of G. In particular, by Theorem 4, the spanning tree T is either the corona $P_3 \circ P_1$ or is the 2-corona of a tree. We proceed further with the following claims.

Claim 11. *If $T = P_3 \circ P_1$, then $G = P_3 \circ P_1$ or $G = C_3 \circ P_1$ or G is the stingray SR.*

Proof. Suppose that $T = P_3 \circ P_1$. Let a, b and c be the three support vertices of T, with leaf neighbors a', b' and c', respectively, and where abc is a path P_3. If $T = G$, then $G = P_3 \circ P_1$, as desired. Hence we may assume that $T \neq G$. Let $e \in E(G) \setminus E(T)$. If $e = ab'$, let $S = \{a, c\}$. If $e = ac'$, let $S = \{a, b\}$. If $e = ba'$, let $S = \{b, c\}$. If $e = a'c'$, let $S = \{a', b\}$. In all cases, S is a SPD-set, and so $\gamma_{\rm pr2}(G) = 2 < \gamma_{\rm pr2}(T)$, a contradiction. Hence, $e \notin \{ab', ac', ba'\}$. By symmetry, $e \notin \{cb', ca', bc'\}$. Hence, $e \in \{ac, a'b', b'c'\}$. Thus, $E(G) \setminus E(T) \subseteq \{ac, a'b', b'c'\}$. If $\{a'b', b'c'\} \subseteq E(G)$, let $D = \{b, b'\}$. If $\{ac, a'b'\} \subseteq E(G)$, let $D = \{a', c\}$. If $\{ac, b'c'\} \subseteq E(G)$, let $D = \{a, b'\}$. In all three cases, D is a SPD-set, and so $\gamma_{\rm pr2}(G) = 2 < \gamma_{\rm pr2}(T)$, a contradiction. Hence, $E(G) \setminus E(T) = \{e\}$; that is, e is the only edge in G that is not in T, implying that either $e = ac$, in which case $G = C_3 \circ P_1$, or $e \in \{a'b', b'c'\}$, in which case G is the stingray SR. □

By Claim 11, we may assume that $T \neq P_3 \circ P_1$, for otherwise the desired result holds. Hence, T is the 2-corona of a tree, say T'. Let A be the set of leaves of T, let B be the set of support vertices of T, and let $C = V(T')$. Thus, (A, B, C) is a partition of $V(T)$. We note that $|C| = n(T')$.

Claim 12. *If $|C| = 1$, then $G = C_3$ or $G = K_1 \circ P_2$.*

Proof. If $|C| = 1$, then $T = P_3$, and so $G = P_3$, which is the 2-corona $K_1 \circ P_2$ of K_1, or $G = C_3$. □

Claim 13. *If $|C| = 2$, then $G = C_6$ or G is the stingray SR or $G = P_2 \circ P_2$.*

Proof. Suppose that $|C| = 2$. In this case, $T = P_6$. If $T = G$, then $G = P_2 \circ P_2$ is the 2-corona of the graph P_2, as desired. Hence we may assume that $T \neq G$. Let $e \in E(G) \setminus E(T)$. Let T be the path $v_1v_2 \cdots v_6$. If $e = v_1v_3$ or $e = v_1v_5$, let $S = \{v_3, v_5\}$. If $e = v_2v_4$ or $e = v_2v_5$ or $e = v_2v_6$, let $S = \{v_2, v_5\}$. In all cases, S is a SPD-set, and so $\gamma_{\rm pr2}(G) = 2 < \gamma_{\rm pr2}(T)$, a contradiction. Hence, $e \notin \{v_1v_3, v_1v_5, v_2v_4, v_2v_5, v_2v_6\}$. By symmetry, $e \notin \{v_4v_6, v_3v_5\}$. Hence, $e \in \{v_1v_4, v_1v_6, v_3v_6\}$. Thus, $E(G) \setminus E(T) \subseteq \{v_1v_4, v_1v_6, v_3v_6\}$. If $\{v_1v_4, v_1v_6\} \subseteq E(G)$, let $D = \{v_1, v_4\}$. If $\{v_1v_4, v_3v_6\} \subseteq E(G)$, let $D = \{v_3, v_4\}$. If $\{v_1v_6, v_3v_6\} \subseteq E(G)$, let $D = \{v_3, v_6\}$. In all three cases, D is a SPD-set, and so $\gamma_{\rm pr2}(G) = 2 < \gamma_{\rm pr2}(T)$, a contradiction. Hence, $E(G) \setminus E(T) = \{e\}$; that is, e is the only edge in G that is not in T. If $e = v_1v_6$, then $G = C_6$, while if $e = v_1v_4$ or $e = v_3v_6$, then G is the stingray SR. □

By Claims 12 and 13, we may assume that $|C| \geq 3$, for otherwise the desired result holds. We show that every edge of G that is not in T joins two vertices of C. We shall use the following notation. Let $A = \{a_1, \ldots, a_r\}$, $B = \{b_1, \ldots, b_r\}$, and $C = \{c_1, \ldots, c_r\}$, where $r = n/3$ and where $a_ib_ic_i$ is a path in T for $i \in [r]$. Let $M_{AB} = \{a_1b_1, \ldots, a_rb_r\}$ and $M_{BC} = \{b_1c_1, \ldots, b_rc_r\}$. For sets X and Y in G, let $[X, Y]$ be the set of all edges between X and Y in G. Let $S = B \cup C$. We note that S with semi-matching $M = \{\{b_i, c_i\} \mid 1 \leq i \leq r\}$ is a $\gamma_{\rm pr2}$-set of T. In particular, $\gamma_{\rm pr2}(T) = |S| = \frac{2}{3}n$.

Claim 14. *The following hold in the graph G.*

(a) *The set A is independent.*

(b) $[A, B] = M_{AB}$.

(c) $[A, C] = \emptyset$.

(d) *The set B is independent.*

(e) $[B, C] = M_{BC}$.

Proof. (a) Suppose that there is an edge e in G that joins two vertices of A. Renaming vertices if necessary, we may assume that $e = a_1a_2$. Since $|C| \geq 3$ and $T' = T[C]$ is a tree, the vertex c_1 has a neighbor in T that belongs to C and is different from c_2 or the vertex c_2 has a neighbor in T that belongs to C and is different from c_1 (for otherwise, $T' = P_2$, a contradiction to our assumption that $n(T') = |C| \geq 3$). We may assume that c_2 has a neighbor in T that belongs to C and is different from c_1. The set $D = (S \setminus \{b_2, c_1, c_2\}) \cup \{a_2\}$ with semi-matching $(M \setminus \{\{b_1, c_1\}, \{b_2, c_2\}\}) \cup \{\{a_2, b_1\}\}$ is a SPD-set of G, implying that $\gamma_{\rm pr2}(G) \leq |D| = |S| - 2 = \gamma_{\rm pr2}(T) - 2 = \frac{2}{3}n - 2$, a contradiction. Hence, A is an independent set in G.

(b) Suppose that there is an edge e in G that joins a vertex of A and a vertex of B, but does not belong to the matching M_{AB}. Renaming vertices if necessary, we may assume that $e = a_1b_2$. In this case, the set $D = (S \setminus \{c_1, c_2\})$ with semi-matching $(M \setminus \{\{b_1, c_1\}, \{b_2, c_2\}\}) \cup \{\{b_1, b_2\}\}$ is a SPD-set of G, implying that $\gamma_{\rm pr2}(G) < \gamma_{\rm pr2}(T)$, a contradiction. Hence, the only edges in $[A, B]$ are the edge in the matching M_{AB}.

(c) Suppose that there is an edge e in G that joins a vertex of A and a vertex of C. Suppose firstly that $e = a_ic_i$. Renaming vertices if necessary, we may assume that $e = a_1c_1$. In this case, letting c_2 be a neighbor of c_1 in T that belongs to the set C, the set $D = (S \setminus \{b_1, c_2\})$ with semi-matching $(M \setminus \{\{b_1, c_1\}, \{b_2, c_2\}\}) \cup \{\{b_2, c_1\}\}$ is a SPD-set of G, implying that $\gamma_{\rm pr2}(G) < \gamma_{\rm pr2}(T)$, a contradiction. Suppose secondly that $e = a_ic_j$ where $i \neq j$. Renaming vertices if necessary, we may assume that $e = a_1c_2$. If c_1 has a neighbor in T that belongs to C and is different from c_2, then the set $D = (S \setminus \{b_1, c_1, c_2\}) \cup \{a_1\}$ with semi-matching $(M \setminus \{\{b_1, c_1\}, \{b_2, c_2\}\}) \cup \{\{a_1, b_2\}\}$ is a SPD-set of G, implying that $\gamma_{\rm pr2}(G) < \gamma_{\rm pr2}(T)$, a contradiction. Thus, c_1 is adjacent in T to c_2 but to no other vertex of C. Since $|C| \geq 3$ and $T' = T[C]$ is a tree, the vertex c_2 has a neighbor in T, say c_3, that belongs to C and is different from c_1. Thus, the set $D = (S \setminus \{b_1, c_3\})$ with semi-matching $(M \setminus \{\{b_1, c_1\}, \{b_2, c_2\}, \{b_3, c_3\}\}) \cup \{\{b_2, c_1\}, \{b_3, c_2\}\}$ is a SPD-set of G, implying that $\gamma_{\rm pr2}(G) < \gamma_{\rm pr2}(T)$, a contradiction. Hence, $[A, C] = \emptyset$.

(d) Suppose that there is an edge e in G that joins two vertices of B. Renaming vertices if necessary, we may assume that $e = b_1b_2$. In this case, the set $D = (S \setminus \{c_1, c_2\})$ with semi-matching $(M \setminus \{\{b_1, c_1\}, \{b_2, c_2\}\}) \cup \{\{b_1, b_2\}\}$ is a SPD-set of G, implying that $\gamma_{\rm pr2}(G) < \gamma_{\rm pr2}(T)$, a contradiction. Hence, the set B is independent.

(e) Suppose that there is an edge e in G that joins a vertex of B and a vertex of C, but does not belong to the matching M_{BC}. Renaming vertices if necessary, we may assume that $e = b_1c_2$. In this case, the set $D = (S \setminus \{c_1, c_2\})$ with semi-matching $(M \setminus \{\{b_1, c_1\}, \{b_2, c_2\}\}) \cup \{\{b_1, b_2\}\}$ is a SPD-set of G, implying that $\gamma_{\rm pr2}(G) < \gamma_{\rm pr2}(T)$, a contradiction. Hence, the only edges in $[B, C]$ are the edge in the matching M_{BC}. This completes the proof of Claim 14. □

By Claim 14, if there is an edge of G that does not belong to T, then such an edge must join two vertices of C. This implies that G is the 2-corona of a connected graph G', where $G' = G[C]$. This completes the proof of Theorem 5. ■

4. Closing Comments

The concept of a semipaired dominating set can be extended to the concept of a distance paired dominating set in the natural way. For $k \geq 1$, a set S of vertices in a graph G with no isolated vertices is a *k-distance paired dominating set* of G if S is a dominating set of G and every vertex in S is paired with exactly one other vertex in S that is within distance k from it. The *k-distance paired domination number*, denoted by $\gamma_{\rm prk}(G)$, is the minimum cardinality of a k-distance paired dominating set of G. We note that a 1-distance paired dominating set is a paired dominating set, and so $\gamma_{\rm pr1}(G) = \gamma_{\rm pr}(G)$.

If G is a connected graph of order $n \ge 3$, then $\gamma_{\rm pr1}(G) \le n-1$ (see Theorem 2) and $\gamma_{\rm pr2}(G) \le \frac{2}{3}n$ (see Theorem 5), and these bounds are tight.

If G is the graph of order $n = 3\ell + 1$ obtained from a star $K_{1,\ell}$ where $\ell \ge 2$ by subdividing every edge twice, then $\gamma_{\rm pr3}(G) = 2\ell = \frac{2}{3}(n-1)$. Thus since $\gamma_{\rm pr3}(G) \le \gamma_{\rm pr2}(G) \le \frac{2}{3}n$, the upper bound of $\frac{2}{3}n$ on $\gamma_{\rm pr3}(G)$ is asymptotically best possible. One can in fact show using analogous proofs as in Theorems 4 and 5 that if G is a connected graph of order $n \ge 4$, then $\gamma_{\rm pr3}(G) \le \frac{2}{3}(n-1)$. Since this is only a very small improvement on the $\frac{2}{3}n$ upper bound, we omit the details of the proof.

If $k \ge \mathrm{diam}(G)$, then $\gamma_{\rm prk}(G) = \gamma(G)$ if $\gamma(G)$ is even and $\gamma_{\rm prk}(G) = \gamma(G)+1$ if $\gamma(G)$ is odd. Thus in this case, $\gamma_{\rm prk}(G) \le \gamma(G)+1 \le \frac{1}{2}n+1$, and the bound is tight as may be seen by taking G to be the corona of a connected graph of odd order. For $4 \le k \le \mathrm{diam}(G)-1$ and G a connected graph of order $n \ge k+1$, we have yet to determine a sharp upper bound on $\gamma_{\rm prk}(G)$. It is quite possible that $\gamma_{\rm prk}(G) \le \frac{1}{2}n+1$ also holds in this case.

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