Discussiones Mathematicae
Graph Theory xx (xxxx) 1–21

THE COMPARED COSTS OF DOMINATION, LOCATION-DOMINATION AND IDENTIFICATION

Olivier Hudry

LTCI, Télécom ParisTech
Université Paris-Saclay
46 rue Barrault, 75634 Paris Cedex 13 – France

e-mail: olivier.hudry@telecom-paristech.fr

and

Antoine Lobstein

Centre National de la Recherche Scientifique
Laboratoire de Recherche en Informatique, UMR 8623
Université Paris-Sud, Université Paris-Saclay
Bâtiment 650 Ada Lovelace, 91405 Orsay Cedex – France

e-mail: antoine.lobstein@lri.fr

Abstract

Let $G = (V, E)$ be a finite graph and $r \geq 1$ be an integer. For $v \in V$, let $B_r(v) = \{x \in V : d(v, x) \leq r\}$ be the ball of radius r centered at v. A set $C \subseteq V$ is an r-dominating code if for all $v \in V$, we have $B_r(v) \cap C \neq \emptyset$; it is an r-locating-dominating code if for all $v \in V$, we have $B_r(v) \cap C \neq \emptyset$, and for any two distinct non-codewords $x \in V \setminus C$, $y \in V \setminus C$, we have $B_r(x) \cap C \neq B_r(y) \cap C$; it is an r-identifying code if for all $v \in V$, we have $B_r(v) \cap C \neq \emptyset$, and for any two distinct vertices $x \in V$, $y \in V$, we have $B_r(x) \cap C \neq B_r(y) \cap C$. We denote by $\gamma_r(G)$ (respectively, $ld_r(G)$ and $id_r(G)$) the smallest possible cardinality of an r-dominating code (respectively, an r-locating-dominating code and an r-identifying code). We study how small and how large the three differences $id_r(G) - ld_r(G)$, $id_r(G) - \gamma_r(G)$ and $ld_r(G) - \gamma_r(G)$ can be.

Keywords: graph theory, dominating set, locating-dominating code, identifying code, twin-free graph.

2010 Mathematics Subject Classification: 68R10, 05C90, 94C12, 94B60, 94B65.

1. Introduction

1.1. Definitions and notation

For graph theory, we refer to, e.g., [1, 2] or [8]; for the vast topic of domination in graphs, see [13]. For locating-dominating codes, see the first papers [7] and [18], for identifying codes, see the seminal paper [14]; for both, see also the large bibliography at [15].

We shall denote by $G = (V, E)$ a finite, simple, undirected graph with vertex set V and edge set E, where an *edge* between $x \in V$ and $y \in V$ is indifferently denoted by xy or yx. The *order* of the graph is its number of vertices, $|V|$. Our graphs will generally be connected. The *distance* between two vertices $x \in V$, $y \in V$, will be denoted by $d_G(x, y)$, or $d(x, y)$ when there is no ambiguity.

For an integer $k \geq 2$, the *k-th transitive closure*, or *k-th power*, of $G = (V, E)$ is the graph $G^k = (V, E^k)$ defined by $E^k = \{uv : u \in V, v \in V, 0 < d_G(u, v) \leq k\}$. For a given graph G^*, any graph G such that $G^k = G^*$ is called a *k-th root* of G^*; such roots do not always exist.

For any integer $r \geq 1$, and for every vertex $x \in V$, we denote by $B_{G,r}(x)$ (and $B_r(x)$ when there is no ambiguity) the *ball of radius* r *centered at* x, i.e., the set of vertices at distance at most r from x:

$$B_r(x) = \{y \in V : d(x, y) \leq r\}.$$

Whenever $x \in B_r(y)$ (which is equivalent to $y \in B_r(x)$), we say that x and y *r-dominate* or *r-cover* each other. A vertex $x \in V$ is said to be *r-universal* if it r-dominates all the vertices, i.e., if $B_r(x) = V$. When three vertices x, y, z are such that $z \in B_r(x)$ and $z \notin B_r(y)$, we say that z *r-sepatares* x and y in G (note that $z = x$ is possible). A set of vertices is said to r-separate x and y if at least one of its element does.

Let $C \subseteq V$ be a set of vertices; the set C is called a *code*, and its elements *codewords*.

A code C is said to be an *r-dominating set* or an *r-dominating code* (r-D code for short) if for all $x \in V$, we have $B_r(x) \cap C \neq \emptyset$. One can also find the terminology *dominating set at distance* r, or *distance* r *dominating set*.

A code C is said to be *r-locating-dominating* (r-LD for short) if for all $x \in V$, we have $B_r(x) \cap C \neq \emptyset$, and for any two distinct non-codewords $x \in V \setminus C$, $y \in V \setminus C$, we have $B_r(x) \cap C \neq B_r(y) \cap C$.

A code C is said to be *r-identifying* (r-ID for short) if for all $x \in V$, we have $B_r(x) \cap C \neq \emptyset$, and for any two distinct vertices $x \in V$, $y \in V$, we have $B_r(x) \cap C \neq B_r(y) \cap C$.

In other words: every vertex must be r-dominated by at least one codeword for the three definitions; in addition, every pair of distinct non-codewords (respectively, vertices) must be r-separated by an r-LD (respectively, r-ID) code.

Two vertices $x \in V$, $y \in V$, $x \neq y$, are said to be *r-twins* if $B_r(x) = B_r(y)$. Dominating and locating-dominating codes exist for all graphs. On the other hand, it is easy to see that a graph G admits an r-identifying code if and only if

$$(1) \qquad \forall x \in V,\ \forall y \in V,\ x \neq y:\ B_r(x) \neq B_r(y).$$

A graph satisfying (1) is called *r-identifiable* or *r-twin-free*.

1.2. Aim of the paper

For all three concepts, we are often interested in finding the minimum sized codes. We denote by $\gamma_r(G)$ (respectively, $ld_r(G)$ and $id_r(G)$) the smallest possible cardinality of an r-dominating code (respectively, an r-locating-dominating code and an r-identifying code when G is r-twin-free). We call $\gamma_r(G)$ the *r-domination number* of G. Since obviously an r-ID code (when it exists) is an r-LD code which in turn is an r-D code, the following inequalities hold:

$$\gamma_r(G) \leq ld_r(G) \leq id_r(G).$$

In other words, location-domination is more "expensive" than domination, and identification is more expensive than location-domination. In this paper, we compare the respective "costs" for these three definitions.

More precisely, denoting

$$\mathcal{G}_{r,n} = \{G : G \text{ is } r\text{-twin-free, connected, with order } n \geq 2\},$$

$$\text{and } \mathcal{G}^{tw}_{r,n} = \{G : G \text{ has } r\text{-twins and is connected, with order } n \geq 2\},$$

we study the following maximum and minimum differences:

- $F_{id,ld}(r,n) = \max\{id_r(G) - ld_r(G) : G \in \mathcal{G}_{r,n}\}$,
- $f_{id,ld}(r,n) = \min\left\{id_r(G) - ld_r(G) : G \in \mathcal{G}_{r,n}\right\}$,
- $F_{id,\gamma}(r,n) = \max\{id_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\}$,
- $f_{id,\gamma}(r,n) = \min\{id_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\}$.

For D- and LD-codes, we have two cases, (a) and (b), which study graphs which are without or with twins, respectively:

(a)
- $F_{ld,\gamma}(r,n) = \max\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\}$,
- $f_{ld,\gamma}(r,n) = \min\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\}$;

these two functions are considered on the same set of graphs (the twin-free graphs) as the four functions involving identification, unlike the two functions below:

(b)
- $F^{tw}_{ld,\gamma}(r,n) = \max\left\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}^{tw}_{r,n}\right\}$,
- $f^{tw}_{ld,\gamma}(r,n) = \min\left\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}^{tw}_{r,n}\right\}$.

Finally, if we want to consider all the connected graphs of order n, twin-free or not, the result is obviously obtained by taking $\max\left\{F_{ld,\gamma}(r,n), F^{tw}_{ld,\gamma}(r,n)\right\}$ and $\min\left\{f_{ld,\gamma}(r,n), f^{tw}_{ld,\gamma}(r,n)\right\}$.

2. Some Earlier Results

The following easy four lemmas are as old as the definitions of dominating, locating-dominating or identifying codes. We give the proofs only for the first two.

Lemma 1. (a) *For any graph $G=(V,E)$ of order n and any integer $r \geq 1$, we have*

$$ld_r(G) \geq \lceil \log_2(n - ld_r(G) + 1) \rceil . \tag{2}$$

(b) *For any integer $r \geq 1$ and any r-twin-free graph $G=(V,E)$ of order n, we have*

$$id_r(G) \geq \lceil \log_2(n+1) \rceil . \tag{3}$$

Proof. (a) Let C be any r-LD code in G. All the $n-|C|$ non-codewords $v \in V \setminus C$ must be given nonempty and distinct sets $B_r(v) \cap C$ constructed with the $|C|$ codewords, so $2^{|C|} - 1 \geq n - |C|$, from which (2) follows when C is optimal;

(b) the argument is the same, but we have to consider all the n vertices $v \in V$, so $2^{|C|} - 1 \geq n$. ■

Lemma 2. *Let $r \geq 2$ be any integer and $G=(V,E)$ be a graph.*

(a) *A code C is 1-locating-dominating in G^r, the r-th power of G, if and only if it is r-locating-dominating in G.*

(b) *A code C is 1-identifying in G^r if and only if it is r-identifying in G.*

(c) *A code C is 1-dominating in G^r if and only if it is r-dominating in G.*

Proof. (a) For every vertex $v \in V$, we have

$$\{c \in C : d_G(v,c) \leq r\} = \{c \in C : d_{G^r}(v,c) \leq 1\},$$

so if for all $v \in V \setminus C$, the sets on the left-hand side of the equality are nonempty and distinct, then the sets on the right side also are, and *vice-versa*; (b) the same proof, for all $v \in V$; (c) the same proof, for all $v \in V$, with only nonemptiness to be checked. ■

Lemma 3. (a) *For any integer $r \geq 1$, if G is a connected graph of order n, then*

$$ld_r(G) \leq n-1. \tag{4}$$

(b) *If G is an r-twin-free graph of order n, then $n \geq 2r+1$, and the only r-twin-free graph of order $2r+1$ is the path.*
(c) *If G is an r-twin-free cycle of order n, then $n \geq 2r+2$.*

The following obvious lemma is often used implicitly.

Lemma 4. *Let $r \geq 1$ be any integer and $G = (V, E)$ be a graph.*
(a) *If C is r-dominating in G, so is any set $S \supset C$.*
(b) *If C is r-locating-dominating in G, so is any set $S \supset C$.*
(c) *If C is r-identifying in G, so is any set $S \supset C$.*

Proposition 5. (a) [16], [13, p. 41] *If G has no isolated vertices (in particular, if G is connected) and has order n, then $\gamma_1(G) \leq \frac{n}{2}$.*
(b) [11] *If G is a 1-twin-free graph, then $id_1(G) \leq 2ld_1(G)$.*

The following result is from [3], but a shorter proof can be found in [12].

Proposition 6. *If G is a connected 1-twin-free graph of order n, then $id_1(G) \leq n-1$.*

Corollary 7. *Let $r \geq 1$ be any integer.*
(i) *If G is a connected graph of order n, then $\gamma_r(G) \leq \frac{n}{2}$.*
(ii) *If G is an r-twin-free graph, then $id_r(G) \leq 2ld_r(G)$.*
(iii) *If G is a connected r-twin-graph of order n, then*

$$id_r(G) \leq n-1. \tag{5}$$

Proof. Use the r-th power of G, together with the previous two propositions. ∎

Both lower bounds (2), (3) and upper bounds (4), (5) for r-LD and r-ID codes can be reached [6], as well as all intermediate values [4], [5].

The graphs G of order n such that $id_1(G) = n-1$ have been characterized in [10], but the case $r \geq 2$ remains open.

3. SOME IMPORTANT GRAPHS

The following three lemmas describe three useful graphs, which have been used in previous papers. The first graph is the "star".

Lemma 8. *For $n \geq 3$, let G_n be the tree consisting of n vertices $v_0, v_1, \ldots, v_{n-1}$, and $n-1$ edges v_0v_i, $1 \leq i \leq n-1$. Then*

$$\gamma_1(G_n) = 1, \ ld_1(G_n) = n-1 \text{ and } id_1(G_n) = n-1.$$

Proof. (a) Since v_0 is a 1-universal vertex, we have $\gamma_1(G_n) = 1$.

(b) It is quite straightforward to check that taking for codewords any set of $n-1$ vertices is necessary and sufficient to obtain a 1-LD or 1-ID code, except for $n = 3$, when only $\{v_1, v_2\}$ is a 1-ID code. ■

The second graph, denoted G^*_{2p}, has even order and is the complete graph (or clique) minus a perfect matching; see Figure 1.

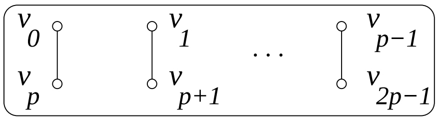

Figure 1. The complement of G^*_{2p}: only the missing edges $v_0v_p, \ldots, v_{p-1}v_{2p-1}$ are represented.

Lemma 9. *Let $p \geq 2$ and $G^*_{2p} = \left(V^*_{2p}, E^*_{2p}\right)$, with $V^*_{2p} = \{v_0, v_1, \ldots, v_{2p-1}\}$, $E^*_{2p} = \left\{v_iv_j : v_i \in V^*_{2p}, v_j \in V^*_{2p},\ i \neq j,\ i \neq j + p \bmod 2p\right\}$. Then*

$$\gamma_1\left(G^*_{2p}\right) = 2,\ ld_1\left(G^*_{2p}\right) = p \text{ and } id_1\left(G^*_{2p}\right) = 2p-1.$$

Proof. For every $v_i \in V^*_{2p}$, we have $B_1(v_i) = V^*_{2p} \setminus \left\{v_{i+p \bmod 2p}\right\}$, and for every pair of distinct vertices $v_i \in V^*_{2p}$, $v_j \in V^*_{2p}$, we have $B_1(v_i)\Delta B_1(v_j) = \left\{v_{i+p \bmod 2p}, v_{j+p \bmod 2p}\right\}$, where Δ stands for the symmetric difference.

(a) The fact that $\gamma_1(G^*_{2p}) = 2$ is easy to check.

(b) Obviously, $C = \{v_0, \ldots, v_{p-1}\}$ is a 1-LD code, of size p. Assume that there is a minimum 1-LD code C with fewer than p elements. Then there is at least one j such that $v_j \notin C$ and $v_{j+p \bmod 2p} \notin C$. Without loss of generality, we may assume that $v_0 \notin C$, $v_p \notin C$. Then $B_1(v_0)\Delta B_1(v_p) = \{v_0, v_p\}$ leads to $C \cap (B_1(v_0)\Delta B_1(v_p)) = \emptyset$, contradicting the definition of a 1-LD code: v_0 and v_p are non-codewords not 1-separated by any codeword.

(c) We know that at most $2p-1$ codewords are necessary in any minimum 1-ID code C; therefore, assume, without loss of generality, that $v_0 \notin C$. Then for all $j \neq p$, $B_1(v_p)\Delta B_1(v_j) = \left\{v_0, v_{j+p \bmod 2p}\right\}$, and, since v_p and v_j are 1-separated by at least one codeword, we have $\emptyset \neq (B_1(v_p)\Delta B_1(v_j)) \cap C \subseteq \left\{v_{j+p \bmod 2p}\right\}$. So for all values of j but one, the $2p-1$ distinct vertices $v_{j+p \bmod 2p}$ are codewords, and $|C| \geq 2p-1$, i.e., $|C| = 2p-1$. ■

The third graph is obtained from the previous one by adding one 1-universal vertex, and its order is odd.

Lemma 10. *Let $p \geq 2$ and $G^*_{2p+1} = \left(V^*_{2p+1}, E^*_{2p+1}\right)$, with $V^*_{2p+1} = \{v_0, v_1, \ldots, v_{2p}\}$, $E^*_{2p+1} = \left\{v_iv_j : v_i \in V^*_{2p+1} \setminus \{v_{2p}\}, v_j \in V^*_{2p+1} \setminus \{v_{2p}\},\ i \neq j,\ i \neq j + p \bmod 2p\right\} \cup \left\{v_{2p}v_j : v_j \in V^*_{2p+1} \setminus \{v_{2p}\}\right\}$. Then*

$$\gamma_1\left(G^*_{2p+1}\right) = 1,\ ld_1\left(G^*_{2p+1}\right) = p \text{ and } id_1\left(G^*_{2p+1}\right) = 2p.$$

Proof. (a) The fact that v_{2p} is 1-universal shows that $\gamma_1(G^*_{2p+1}) = 1$.

(b) For 1-LD codes, the argument of the Case (b) of the previous proof can be applied *mutatis mutandis*, because the 1-universal vertex does not change anything when considering symmetric differences of balls of radius one.

(c) For $i \in \{0, \dots, 2p-1\}$, we have $B_1(v_{2p})\Delta B_1(v_i) = \left\{v_{i+p \bmod 2p}\right\}$, therefore all vertices but v_{2p} must be codewords. ■

Now, what is more difficult and interesting is that the two graphs G^*_{2p} and G^*_{2p+1} just described in Lemmas 9 and 10 admit r-th roots for any r, if p is sufficiently large [6]. More precisely:

Proposition 11. *Let $r \geq 2$ and $p \geq 2$ be integers.*

(a) [6, Theorem 5] *If $2p \geq 3r^2$, then there exists a graph G_{2p} of order $2p$ such that $(G_{2p})^r = G^*_{2p}$.*

(b) [6, Theorem 6] *If $2p \geq 3r^2$, then there exists a graph G_{2p+1} of order $2p+1$ such that $(G_{2p+1})^r = G^*_{2p+1}$.*

(c) *For $n \geq 3r^2$, there exists a graph G_n of even order n such that $\gamma_r(G_n) = 2$, $ld_r(G_n) = \frac{n}{2}$ and $id_r(G_n) = n-1$.*

(d) *For $n \geq 3r^2+1$, there exists a graph G_n of odd order n such that $\gamma_r(G_n) = 1$, $ld_r(G_n) = \frac{n-1}{2}$ and $id_r(G_n) = n-1$.*

Proof. (c)–(d). Use the properties of r-th powers of graphs (Lemma 2). ■

See also the constructions presented and discussed immediately after Proposition 23 in Section 7.1.

4. THE VERY SMALL CASES: $n \leq 4$

Here, we denote by $T_r(G)$ the triple $(\gamma_r(G), ld_r(G), id_r(G))$, with the convention that $id_r(G) = ?$ if G is not r-twin-free. Figure 2 gives all the nonisomorphic unlabeled connected graphs with two, three or four vertices, together with their triples for $r = 1$.

For $r = 2$, the triples are, for the nine graphs of Figure 2, respectively: $(1,1,?)$; $(1,2,?)$ and $(1,2,?)$; $(1,2,?)$, $(1,3,?)$, $(1,3,?)$, $(1,3,?)$, $(1,3,?)$, and $(1,3,?)$. For $r \geq 3$, the triples are $(1, n-1, ?)$ for all nine graphs. From this, we have the following result.

Proposition 12. *We have*

(a) $r = 1$

$n = 2 : F^{tw}_{ld,\gamma}(1,2) = f^{tw}_{ld,\gamma}(1,2) = 0;$

$n=3: F_{id,ld}(1,3)=f_{id,ld}(1,3)=0;\ F_{id,\gamma}(1,3)=f_{id,\gamma}(1,3)=1;$
$F_{ld,\gamma}(1,3)=f_{ld,\gamma}(1,3)=1;\ F^{tw}_{ld,\gamma}(1,3)=f^{tw}_{ld,\gamma}(1,3)=1;$

$n=4: F_{id,ld}(1,4)=1,\ f_{id,ld}(1,4)=0;\ F_{id,\gamma}(1,4)=2,\ f_{id,\gamma}(1,4)=1;$
$F_{ld,\gamma}(1,4)=2,\ f_{ld,\gamma}(1,4)=0;\ F^{tw}_{ld,\gamma}(1,4)=2,\ f^{tw}_{ld,\gamma}(1,4)=1;$

(b) $r=2$

$n=2: F^{tw}_{ld,\gamma}(2,2)=f^{tw}_{ld,\gamma}(2,2)=0;$

$n=3: F^{tw}_{ld,\gamma}(2,3)=f^{tw}_{ld,\gamma}(2,3)=1;$

$n=4: F^{tw}_{ld,\gamma}(2,4)=2,\ f^{tw}_{ld,\gamma}(2,4)=1;$

(c) $r\geq 3$

$n\in\{2,3,4\}: F^{tw}_{ld,\gamma}(r,n)=f^{tw}_{ld,\gamma}(r,n)=n-2.$

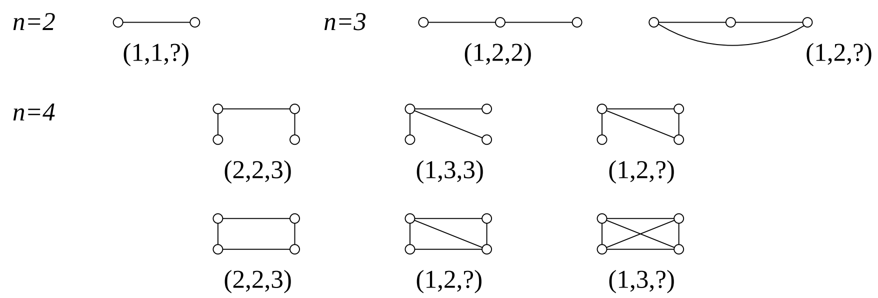

Figure 2. Small graphs, $r=1$.

5. Identification vs Domination

First, we construct an infinite family of graphs G^*_n such that G^*_n has order n and satisfies $id_r(G^*_n)=\gamma_r(G^*_n)$.

These graphs will have order $n=k(r+1)$ and consist of one cycle of order k and k strings with r vertices each: $G^*_n=(V^*_n,E^*_n)$, with $V^*_n=V_0\cup\left(\bigcup_{1\leq i\leq k}V_i\right)$ and $E^*_n=E_0\cup\left(\bigcup_{1\leq i\leq k}E_i\right)$, where $V_0=\{v_{1,0},v_{2,0},\dots,v_{k,0}\}$, $V_i=\{v_{i,j}:1\leq j\leq r\}$ for $i\in\{1,2,\dots,k\}$, $E_0=\{v_{i,0}v_{i+1,0}:1\leq i\leq k-1\}\cup\{v_{k,0}v_{1,0}\}$ and $E_i=\left\{v_{i,j}v_{i,j+1}:0\leq j\leq r-1\right\}$ for $i\in\{1,2,\dots,k\}$ (see Figure 3(a)).

Proposition 13. *For all $r\geq 1$ and $k\geq 2r+2$, the graph G^*_n is such that*

$$\gamma_r(G^*_n)=id_r(G^*_n).$$

Proof. The k leaves $v_{i,r}$ must be r-dominated by at least one codeword, and no vertex can r-dominate two leaves, so $\gamma_r(G^*_n)\geq k$. On the other hand, the code $C=V_0$ represented by the black vertices in Figure 3(a) has cardinality k, and it is

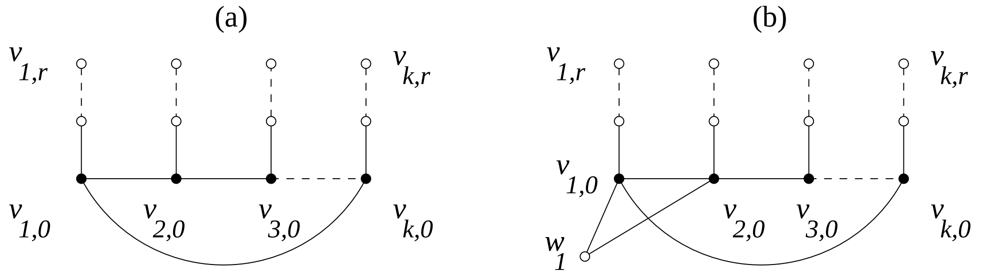

Figure 3. (a) The graph G_n^*. (b) The graph G_{n+1}. The k black vertices represent codewords.

straightforward to check that it is r-identifying. Note in particular that vertices in V_0 are r-dominated by exactly $2r+1$ codewords (this is where the assumption $k \geq 2r+2$ is crucial, cf. Lemma 3(c)), and vertices $v_{i,j} \in V_i$ are r-dominated by exactly $2r-2j+1$ codewords. See also the proof of Proposition 28 for r-LD codes, which is analogous but more intricate.

So $k \leq \gamma_r(G_n^*) \leq id_r(G_n^*) \leq k$. ■

Second, if we want n to reach all intermediate values between $k(r+1)$ and $(k+1)(r+1)-1$, we can do so by adding $p \in \{0, \ldots, r\}$ vertices to G_n^* in the following way: since $p < \frac{k}{2}$, we can add the set of p vertices $W_p = \{w_1, \ldots, w_p\}$ together with the set of edges $X_p = \{w_1v_{1,0}, w_1v_{2,0}, w_2v_{3,0}, w_2v_{4,0}, \ldots, w_pv_{2p-1,0}, w_pv_{2p,0}\}$, see Figure 3(b) for $p=1$. Setting $G_{n+p} = (V_n^* \cup W_p, E_n^* \cup X_p)$, we obtain a graph of order $n+p$, for which, due to the assumption $k \geq 2r+2$ and the remark in the proof of Proposition 13 stating that all vertices in G_n^* are r-dominated by an odd number of codewords, it is again straightforward to check that $C = V_0$ is still a (minimum) r-ID code. Therefore we have the following.

Proposition 14. *For all $r \geq 1$, $k \geq 2r+2$ and $p \in \{0, \ldots, r\}$, the graph $G_{k(r+1)+p}$ is such that $\gamma_r(G_{k(r+1)+p}) = id_r(G_{k(r+1)+p})$. As a consequence, for all $r \geq 1$ and $n \geq (2r+2)(r+1)$, we have*

$$f_{id,\gamma}(r,n) = 0.$$

In advance of the next sections, we have the following obvious consequence.

Corollary 15. *For all $r \geq 1$ and $n \geq (2r+2)(r+1)$, we have*

$$f_{id,ld}(r,n) = f_{ld,\gamma}(r,n) = 0.$$

For $r=1$, the construction for Propositions 13 and 14 works for $n \geq 8$; however, we have the exact value of $f_{id,\gamma}(1,n)$ for all n, due to an alternative construction.

Proposition 12(a) has already settled the cases $n=3$, $n=4$. Lemma 1(b) and Proposition 5(a) establish that any (1-twin-free) graph G with five vertices is such that $id_1(G) \geq 3$ and $\gamma_1(G) \leq 2$; on the other hand it is easy to find graphs G of order five with $id_1(G) = 3$ and $\gamma_1(G) = 2$, e.g., the path, so that $f_{id,\gamma}(1,5) = 1$. For even n, $n \geq 6$, and odd n, $n \geq 7$, it is easy to see that Figure 4 gives graphs G such that $id_1(G) = \gamma_1(G) = k$.

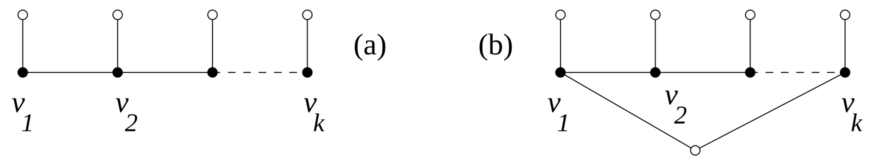

Figure 4. (a) n even. (b) n odd. The k black vertices represent codewords constituting both a minimum 1-identifying and 1-dominating code.

Proposition 16. (a) *For all* $n \geq 6$, *we have* $f_{id,\gamma}(1,n) = 0$; *consequently,* $f_{id,ld}(1,n) = f_{ld,\gamma}(1,n) = 0$.
(b) *For* $n \in \{3,4,5\}$, *we have* $f_{id,\gamma}(1,n) = 1$.

Now how large can the difference $id_r(G) - \gamma_r(G)$ be? By Corollary 7(iii), it is at most $n-2$, obtained by graphs G with $id_r(G) = n-1$ and $\gamma_r(G) = 1$.

We first treat the case $r = 1$, which is easy: the star on n vertices (Lemma 8) is an example of a graph G with $id_1(G) = n-1$ and $\gamma_1(G) = 1$.

Proposition 17. *For all* $n \geq 3$, *we have* $F_{id,\gamma}(1,n) = n-2$.

We now turn to the case $r \geq 2$. When n is odd, the answer is given by Proposition 11(d). Again, we can reach $n-2$ for the difference $id_r(G) - \gamma_r(G)$. When n is even, the study of all the graphs G of even order n such that $id_1(G) = n-1$ [10] shows that none of them contains a 1-universal vertex, i.e., none of them is such that $\gamma_1(G) = 1$, except the star; but the star cannot be the power of any graph. Therefore, for $r \geq 2$, there can exist no graph G with even order n such that $id_r(G) = n-1$ and $\gamma_r(G) = 1$, since the r-th power of this graph would contradict the characterization from [10]; consequently the difference $id_r(G) - \gamma_r(G)$ is at most $n-3$. On the other hand, Proposition 11(c) gives an example achieving $n-3$, and we have proved the following.

Proposition 18. (a) *For all* $r \geq 2$ *and even* $n \geq 3r^2$, *we have* $F_{id,\gamma}(r,n) = n-3$.
(b) *For all* $r \geq 2$ *and odd* $n \geq 3r^2+1$, *we have* $F_{id,\gamma}(r,n) = n-2$.

6. IDENTIFICATION VS LOCATION-DOMINATION

We have already seen in Corollary 15 that, for $r \geq 1$ and $n \geq (2r+2)(r+1)$, we have $f_{id,ld}(r,n) = 0$.

For $r = 1$, and for all values of n, Propositions 12(a) and 16(a) completely settle this case except when $n = 5$, where $f_{id,ld}(1, 5) = 0$ thanks to the graph G_5 with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_5\}$, $id_1(G_5) = ld_1(G_5) = 3$. Therefore we have the following.

Proposition 19. *For all $n \geq 3$, we have $f_{id,ld}(1, n) = 0$.*

What about $F_{id,ld}(r, n)$? We can use Corollary 7(ii) and obtain that any (connected) r-twin-free graph G is such that $id_r(G) \leq 2ld_r(G)$. Therefore, $-ld_r(G) \leq -\frac{id_r(G)}{2}$, and $id_r(G) - ld_r(G) \leq id_r(G) - \frac{id_r(G)}{2} \leq \frac{n-1}{2}$, leading to $id_r(G) - ld_r(G) \leq \lceil \frac{n}{2} \rceil - 1$. On the other hand, Proposition 11(c)–(d) gives examples of graphs reaching $\lceil \frac{n}{2} \rceil - 1$.

Proposition 20. *For all $r \geq 1$ and $n \geq 3r^2 + 1$, we have $F_{id,ld}(r, n) = \lceil \frac{n}{2} \rceil - 1$.*

Proposition 21. (a) *For all $n \geq 4$, we have $F_{id,ld}(1, n) = \lceil \frac{n}{2} \rceil - 1$.*
(b) $F_{id,ld}(1, 3) = 0$.

Proof. Proposition 12(a) settles the case $n = 3$. ■

7. LOCATION-DOMINATION VS DOMINATION

7.1. Twin-free graphs

We have already seen in Corollary 15 that, for $r \geq 1$ and $n \geq (2r+2)(r+1)$, we have $f_{ld,\gamma}(r, n) = 0$. Moreover, for $r = 1$, Propositions 12(a) and 16(a) treat all values of n but $n = 5$, for which the path shows that $f_{ld,\gamma}(1, 5) = 0$.

Proposition 22. (a) *For all $n \geq 4$, we have $f_{ld,\gamma}(1, n) = 0$.*
(b) $f_{ld,\gamma}(1, 3) = 1$.
(c) *For all $r \geq 1$ and $n \geq (2r + 2)(r + 1)$, we have $f_{ld,\gamma}(r, n) = 0$.*

We know, using the example of the star (Lemma 8), that $F_{ld,\gamma}(1, n) = n - 2$. What about $F_{ld,\gamma}(r, n)$ for general r?

On the one hand, Proposition 11(c)–(d) immediately gives examples proving that $F_{ld,\gamma}(r, n) \geq \lceil \frac{n}{2} \rceil - 2$, for all $r \geq 2$ and $n \geq 3r^2 + 1$. On the other hand, the characterization [10] of the graphs G of order n such that $id_1(G) = n - 1$ gives graphs which, apart from the star which is not the power of any graph, are such that $ld_1(G) \leq n - 2$. This allows to conclude that $F_{ld,\gamma}(r, n) \leq n - 3$. Indeed, $F_{ld,\gamma}(r, n) = n - 2$ is possible only if a graph G of order n satisfies $\gamma_r(G) = 1$ and $ld_r(G) = n - 1$, which implies $\gamma_1(G^r) = 1$ and $ld_1(G^r) = n - 1 = id_1(G^r)$, contradicting the previous sentence.

Proposition 23. (a) *For all $n \geq 3$, we have $F_{ld,\gamma}(1, n) = n - 2$.*

(b) *For all* $r \geq 2$ *and* $n \geq 3r^2+1$, *we have* $F_{ld,\gamma}(r,n) \geq \lceil \frac{n}{2} \rceil - 2$.
(c) *For all* $r \geq 2$ *and* $n \geq 2r+1$, *we have* $F_{ld,\gamma}(r,n) \leq n-3$.

We now present a general framework using Theorem 5 in [6], and, to a lesser extent, Theorem 6 in [6], cf. Section 3, Proposition 11(a)–(b). We shall use it in the case $r=2$, when this gives a lower bound for $F_{ld,\gamma}(2,n)$ which is better than $\lceil \frac{n}{2} \rceil - 2$, for all $n \geq 24$; for $r=3$, $n=30$, this gives no improvement, and we shall informally explain why for $r=3$ and larger n, or for larger r, this method is doomed to fail.

Let $m = 2p \geq 3r^2+1$. We consider the Euclidean division of p by r: $p = rQ+R$, $0 \leq R \leq r-1$, and set $k = Q+1$, $A = r-R$, so that $p = rk-A$ with $A \in \{1,2,\ldots,r\}$. We build $G_m = (V_m, E_m)$ in the following way:

$$V_m = \{v_i : 0 \leq i \leq m-1\}, \tag{6}$$

$$E_m = \left\{v_i v_{i+j \bmod m} : 0 \leq i \leq m-1, j \in J = \{1,2,\ldots,k-A-1,k\}\right\}. \tag{7}$$

The graph G_m can be seen as a cycle with chords added according to the set J, where every vertex plays the same role, see Figure 5(a). Theorem 5 from [6] states that the r-th power of G_m is the graph G_m^* of Lemma 9, with $m=2p$. We now need to be more specific with respect to r.

In the case $r=2$, we can improve on Proposition 23(b) and build, for n large enough, graphs of order n proving that $\frac{F_{ld,\gamma}(2,n)}{n}$ tends to $\frac{5}{8}$ when n goes to infinity. The first step is the study of graphs with order a multiple of eight.

Proposition 24. *For* $n = 8t \geq 24$, *there exists a* 2*-twin-free graph* G_n *of order* n, *with* $\gamma_2(G_n) = 2$ *and* $ld_2(G_n) = 5t-1$.

Proof. Let $m = 6t = 2p \geq 18$ and $p = 2k-A$ with $A \in \{1,2\}$. Because $A \in \{1,2\}$ and $p \geq 9$, we have $p \geq 3A+3 \Rightarrow p-A-1 \geq \frac{2p}{3} = 2t \Rightarrow 2k-2A-1 \geq 2t$, and finally

$$k-A \geq 2t-(k-A-1). \tag{8}$$

When $p=9$, then $A=1$ and $\frac{p}{3} \geq A+2$, which also holds whenever $p \geq 12$. Therefore, $\frac{2k-A}{3} \geq A+2 \Rightarrow k \geq 2A+3 \Rightarrow 3k-3A-3 \geq 2k-A \Rightarrow k-A-1 \geq \frac{p}{3}$, and finally

$$2t+k-A-1 = \frac{2p}{3}+k-A-1 \geq p. \tag{9}$$

These two inequalities, (8) and (9), will be used later on. We have already seen that the graph $G_m = (V_m, E_m)$ defined by (6) and (7) is such that the square power of G_m is the graph G_m^* of Lemma 9, with $m=2p$. This means that

$B_{G_m,2}(v_i) = V_m \setminus \{v_{i+p \bmod m}\}$, for all $i \in \{0,\dots,m-1\}$; in other words, from any v_i we can go in two steps to any vertex in V_m but $v_{i+p \bmod m}$. Next, we diverge from Theorem 6 in [6]: we add $\frac{m}{3} = 2t$ vertices Z_j, $0 \le j \le 2t-1$; every Z_j is linked to three vertices in V_m, namely v_j, v_{j+2t} and v_{j+4t}. By setting $G_n = (V_n, E_n)$ with $V_n = V_m \cup \{Z_j : 0 \le j \le 2t-1\}$ and $E_n = E_m \cup \{Z_j v_j, Z_j v_{j+2t}, Z_j v_{j+4t} : 0 \le j \le 2t-1\}$, we obtain a graph of order $8t$, see Figure 5(b).

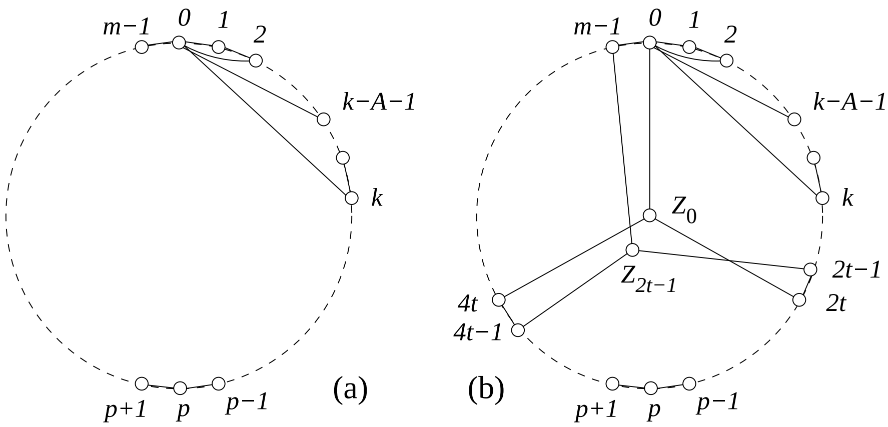

Figure 5. (a) The graph G_m. (b) The graph G_n ($r = 2$). Not all vertices nor edges are represented. Only the indices of the vertices v_i are given.

We claim that

(a) $B_{G_n,2}(Z_j) = V_m \setminus \{Z_\ell : 0 \le \ell \le 2t-1, \ell \ne j\}$, for all $j \in \{0,\dots,2t-1\}$;

(b) $B_{G_n,2}(v_i) = V_m \setminus \{v_{i+p \bmod m}\}$, for all $i \in \{0,\dots,m-1\}$.

(a) That we cannot go in two steps from Z_j to Z_ℓ is obvious since $B_{G_n,1}(Z_j) \cap B_{G_n,1}(Z_\ell) = \emptyset$. Note already that this could not be directly transposed to the case $r \ge 3$, since then the existence of paths such as $Z_j v_j, v_j v_{j+1}, v_{j+1} Z_{j+1}$ would lead to a contradiction; see the discussion below for $r = 3$.

Next, we show that we can go in two moves from any Z_j to any v_i; because of the symmetries of the graph, we need to do it only for, say, Z_0, and the vertices going from v_0 to v_p. Thanks to the edge $Z_0 v_0$, Z_0 can reach $v_1, v_2, \dots, v_{k-1-A}$ (and v_k, but we do not need it) in two moves; thanks to the edge $Z_0 v_{2t}$, Z_0 can also reach $v_{2t+1}, \dots, v_{2t+k-1-A}$ (and v_{2t+k}), as well as $v_{2t-1}, \dots, v_{2t-(k-1-A)}$ (and v_{2t-k}). Using (8), we can see that in the worst case, $v_{2t-(k-1-A)} = v_{k-A}$ and all the vertices between v_0 and v_{2t} can be reached, including in particular v_{k-1-A} and v_{k-A}. In other words, the areas reached in one move by going clockwise from v_0 or anticlockwise from v_{2t} do meet. Similarly, by (9), we have in the worst case $v_{2t+k-A-1} = v_p$ and all the vertices between v_{2t+1} and v_p can be reached in two moves from Z_0. Claim (a) is proved.

(b) The proof is the same as for the graph G_m; we just have to check that the additional vertices Z_j and their edges do not make it possible to go in two moves from v_0 to v_p (this is sufficient for reasons of symmetry).

Claims (a) and (b) show that G_n is 2-twin-free; they also show that the square power of G_n is the following graph. $(G_n)^2$ has vertex set V_n and edge set all the possible edges except the edges $v_i v_{i+p \bmod m}$, $0 \le i \le p$, and $Z_j Z_\ell$, $\{j, \ell\} \subset \{0, \dots, 2t-1\}$, $j \neq \ell$, see Figure 6.

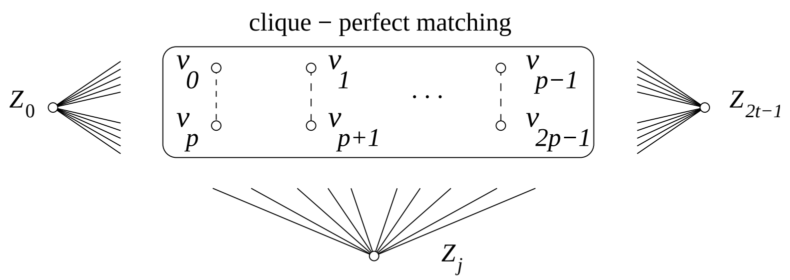

Figure 6. A partial representation of $(G_n)^2$. Dotted lines are non-edges.

Now what are $\gamma_1\big((G_n)^2\big)$ and $ld_1((G_n)^2)$ (or equivalently, $\gamma_2(G_n)$ and $ld_2(G_n)$)? Obviously, $\gamma_1\big((G_n)^2\big) = 2$. Next, the argument of Case (b) of the proofs of Lemmas 9 and 10 can be used to show that it is necessary to take half of the vertices v_i in V_m for a 2-LD code. Then, for $j \neq \ell$, we have $B_{(G_n)^2,1}(Z_j) \Delta B_{(G_n)^2,1}(Z_\ell) = \{Z_j, Z_\ell\}$, which implies that we have to take all vertices Z_j but one as codewords, together with p vertices in V_m, and this is sufficient, $ld_1\big((G_n)^2\big) = p + (2t-1) = 5t - 1$. ■

In order to reach the values of n other than multiples of eight, we might consider $m = 6t+2$ or $6t+4$ instead of $m = 6t$, but it is more efficient to stick to $m = 6t$ and simply add a number of vertices Z_j smaller (by a number between 1 and 7) than $2t$. From $m = 6t \ge 18$ we constructed a graph with $8t$ vertices; now, we start from $6(t+1)$, and, instead of building a graph with order $6(t+1) + 2(t+1) = 8(t+1)$, we build a graph with $6(t+1) + [2(t+1) - q] = 8t + (8-q)$ vertices, by adding only $2(t+1) - q$ vertices Z_j, with $1 \le q \le 7$. The resulting graph has its 2-domination number equal to 1 (in the unique case when $t = 3$, $q = 7$ and we add only one vertex, Z_0) or 2; any minimum 2-LD code has size $3(t+1) + [2(t+1) - q - 1] = 5(t+1) - q - 1$, including when $2(t+1) - q = 1$.

So, letting $i = 8-q$, $1 \le i \le 7$, we obtain graphs G_n with order $n = 8t+i$ and $ld_2(G_n) = 5t + i - 4$ (the borderline case $i = 0$, i.e., dropping eight vertices Z_j, logically leads to a worse result, namely $5t - 4$, than if we start from $6t$ to reach $8t$, in which case we have just seen that we obtain $5t-1$). Since the 2-domination number of these graphs is at most 2, we have the following result.

Proposition 25. (a) *Let* $n = 8t \geq 24$. *Then* $F_{ld,\gamma}(2,n) \geq 5t - 3$.
(b) *Let* $n = 8t + i \geq 24$, *with* $1 \leq i \leq 7$. *Then* $F_{ld,\gamma}(2,n) \geq 5t + i - 6$.

The least favorable case is when $i = 1$, which leads to

$$F_{ld,\gamma}(2,n) \geq \frac{5n-45}{8}. \tag{10}$$

The case $m = 6t$ works best because we have a miraculously large number of Z_j's, namely $2t$, which is advantageous when we look for a "large" LD-code, since we have to take all of them but one in a 1-LD code in $(G_n)^2$. If we consider $m = 6t+2$ or $m = 6t + 4$, we cannot take as many vertices Z_j; yet, if we can take a number of Z_j's which is only a fraction $\frac{m}{\beta}$ with $\beta > 3$, then we obtain a graph G_n with order $n = m + \frac{m}{\beta}$ and $ld_2(G_n) = ld_1\big((G_n)^2\big) = \frac{m}{2} + \frac{m}{\beta} - 1$, leading to the ratio $\frac{F_{ld,\gamma}(2,n)}{n}$ greater than $\frac{ld_2(G_n)-2}{n} \approx \frac{\beta+2}{2\beta+2}$, which is not as good as $\frac{5}{8}$.

For $r = 3$, we consider $m = 30 = 2p = 2(3k - A)$, leading to $p = 15$, $k = 6$, $A = 3$, $J = \{1, 2, 6\}$. To the graph G_{30} defined by (6) and (7), whose third power, by [6, Theorem 5], is G^*_{30}, we add first the vertex Z_0 together with the edges Z_0v_0, Z_0v_{10} and Z_0v_{20}. Then we add the vertex Z_1. Because we want no path of the type $Z_0v_0v_iZ_1$ for some i, among the vertices $\{v_1, \dots, v_{15}\}$, Z_1 cannot be linked to v_1, v_2 nor v_6; because of Z_0v_{10}, this also excludes v_4, v_8, v_9, v_{11} and v_{12} as neighbours of Z_1. Finally we can take, e.g., Z_1 with the edges Z_1v_3, Z_1v_{13}, Z_1v_{23}, Z_2 with Z_2v_7, Z_2v_{17}, Z_2v_{27}, and no more. Exactly as before, this leads to a graph G_{33} whose third power is a graph of the type given by Figure 6, with 33 vertices, $ld_3(G_{33}) = 15 + (3 - 1) = 17$, 3-domination number equal to 2, and $ld_3(G_{33}) - \gamma_3(G_{33}) = 15 = \lceil \frac{33}{2} \rceil - 2$, i.e., not better than Proposition 23(b).

It is impossible to take fewer than three neighbours for each vertex Z_ℓ. On the other hand, as discussed above when studying the possible neighbours of Z_1, if v_0 is the neighbour of Z_0, the "first" neighbour of Z_1 will be v_i with $i \geq k - A$, for Z_2 it will be v_j with $j \geq i + (k - A) \geq 2(k - A)$, ... So, roughly speaking, the total number of possible neighbours for the vertices Z_ℓ is at most

$$\frac{m}{k-A} = \frac{m}{\frac{p}{3} - \frac{2A}{3}} = \frac{6p}{p-2A}, \tag{11}$$

and therefore, the number of vertices Z_ℓ is at most $\frac{2p}{p-2A}$. When $p = 15$, this leads to at most three vertices Z_ℓ, and things only worsen when m, hence p increases. Anyway, with only three additional vertices Z_ℓ, all we can reach is a graph G_n with $n = m + 3$ vertices and

$$ld_3(G_n) - \gamma_3(G_n) = (p + (3 - 1)) - 2 = p = \frac{n-3}{2}.$$

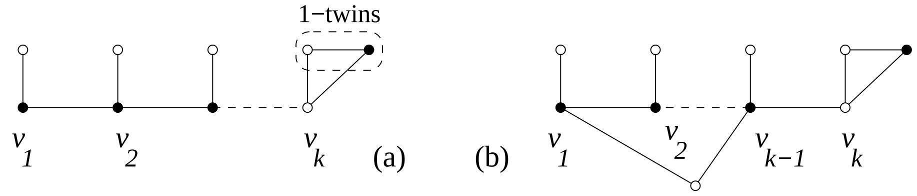

Figure 7. (a) n odd: $n = 2k + 1 \geq 5$. (b) n even: $n = 2k + 2 \geq 8$. The k black vertices represent codewords constituting both a minimum 1-dominating and 1-locating-dominating code.

When we place ourselves again in the general case for r, we must have the "first" neighbour of Z_1, say v_i, such that $i \geq (r-2)(k-A)$, in order to avoid a path of length r between Z_0 and Z_1, and equalities (11) now read

$$\frac{m}{(r-2)(k-A)} = \frac{m}{(r-2)\left(\frac{p}{r} - \frac{(r-1)A}{r}\right)} = \frac{2rp}{(r-2)(p-(r-1)A)}.$$

Even with $p = \frac{3r^2}{2}$ and $A = r$, this can lead only to approximately $\frac{3r^3}{(r-2)\frac{r^2}{2}} \approx 6$, hence at most two vertices Z_ℓ, and again, things only worsen when p increases. Therefore, other constructions should be invented—that is, if improvements do exist in Proposition 23(b).

Open Problem. Reduce the gap between lower and upper bounds for $F_{ld,\gamma}(r,n)$, when $r > 1$.

7.2. Graphs with twins

The study of $F^{tw}_{ld,\gamma}(r,n)$ is trivial, because of the clique, or complete graph on n vertices, K_n, which obviously contains r-twins, and is such that $\gamma_r(K_n) = 1$ and $ld_r(K_n) = n - 1$.

We are going to prove that (i) for $r = 1$ and $n \in \{2, 5\}$ or $n \geq 7$ (Proposition 26) and (ii) for any $r \geq 2$ and n large enough (Proposition 28), we have $f^{tw}_{ld,\gamma}(r,n) = 0$.

Proposition 26. (a) *For $n = 2$, $n = 5$ and all $n \geq 7$, we have* $f^{tw}_{ld,\gamma}(1,n) = 0$;
(b) *For $n \in \{3,4,6\}$, we have* $f^{tw}_{ld,\gamma}(1,n) = 1$.

Proof. We already know by Proposition 12(a) that

$$f^{tw}_{ld,\gamma}(1,2) = 0;\ f^{tw}_{ld,\gamma}(1,3) = 1;\ f^{tw}_{ld,\gamma}(1,4) = 1.$$

For $n = 6$, Lemma 1(a) and Proposition 5(a) state that for any connected graph G with six vertices, $ld_1(G) \geq 3$ and $\gamma_1(G) \leq 3$; but a study of the graphs with 1-twins shows that for them, $\gamma_1(G) \leq 2$ (alternatively, one can use the characterization of the graphs with even order and 1-domination number half their order [13, p. 42], [9, 17]), and eventually $f^{tw}_{ld,\gamma}(1, 6) = 1$. For $n = 5$ and $n \geq 7$, we consider the graphs in Figure 7, obtained from the graphs in Figure 4 by a slight modification, intended to create one pair of 1-twins. The study of these graphs is straightforward and gives the desired result. ■

We now turn to the case $r \geq 2$ (even if the results below are also valid for $r = 1$); first, we give an analogue of Proposition 13 for r-LD codes. We take the graphs $G^*_n = (V^*_n, E^*_n)$ represented in Figure 3(a) and described just before Proposition 13, and transform them into graphs G^y_{n+1} by applying the same type of modification just performed for $r = 1$. We simply add one vertex y which is the r-twin of $v_{k,r}$, see Figure 8(a). The order of G^y_{n+1} is $n + 1 = k(r + 1) + 1$.

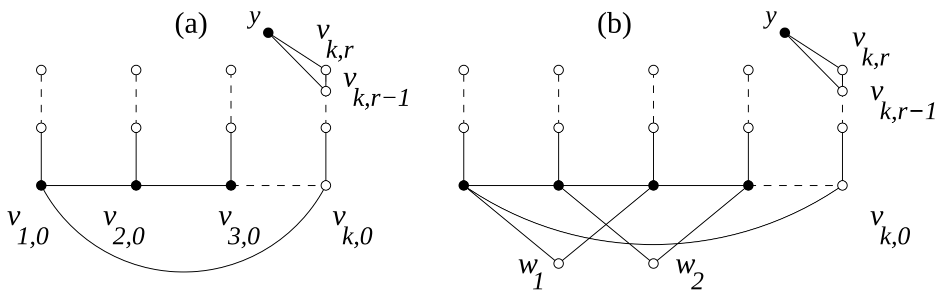

Figure 8. The k black vertices represent codewords.

Observation 27. *Because here we deal with r-LD codes, not r-ID codes like in Proposition 13, the bound for k could be lowered, down to $k \geq 2r$. For simplicity and because this does not represent a significant improvement, we keep the bound $k \geq 2r + 2$.*

Proposition 28. *For all $r \geq 2$ and $k \geq 2r + 2$, the graphs G^y_{n+1} are such that $\gamma_r(G^y_{n+1}) = ld_r(G^y_{n+1})$.*

Proof. Obviously, $k \leq \gamma_r(G^y_{n+1})$ (and $k \leq ld_r(G^y_{n+1})$), and the code $C = \{v_{1,0}, v_{2,0}, \ldots, v_{k-1,0}, y\}$, with k codewords, is an r-D code. We are going to prove that C is also r-LD. In spite of the fact that all we have to check is that any two distinct non-codewords are r-separated by C, the proof is a little more intricate than the proof of Proposition 13 for r-ID codes, because of the "missing" codeword $v_{k,0}$, so we present it in detail.

(a) The non-codewords $v_{k,j}$, $0 \leq j \leq r$, are the only non-codewords r-dominated by y, so they all are r-separated by $y \in C$ from other non-codewords; each of them is r-dominated by a different number of codewords, because k is large enough, and therefore they are pairwise r-separated by C.

(b) Consider any two non-codewords $v_{i,j}$, $v_{i,t}$ on the same string i, $1 \leq i \leq k-1$, $\{j,t\} \subseteq \{1,\ldots,r\}$, $j < t$; then $v_{i,j}$ is r-dominated by at least one codeword more than $v_{i,t}$ (it would be at least two if we had all k elements of the cycle in the code), and so these two vertices are r-separated by C.

(c) Let us consider two non-codewords $v_{i,j}$ and $v_{s,t}$ belonging to two different strings, other than the k-th string: $\{i,s\} \subset \{1,\ldots,k-1\}$, $i \neq s$, $\{j,t\} \subseteq \{1,\ldots,r\}$; without loss of generality, we may assume that $j \leq t$.

If $j < t$, then again, $v_{i,j}$ is r-dominated by at least one codeword more than $v_{s,t}$; so from now on, we assume that $j = t$. The set of codewords r-dominating $v_{i,j}$ has cardinality $2r - 2j + 1$ or $2r - 2j$, and consists, with computations performed modulo k, of $v_{i,0}$, $v_{i-1,0}, \ldots, v_{i-r+j,0}$, $v_{i+1,0}, \ldots, v_{i+r-j,0}$, or of the same set without $v_{k,0}$, which is not a codeword. In both cases, it cannot be the same as the set of codewords r-dominating $v_{s,j}$.

We have just proved that C r-separates the non-codewords $v_{k,j}$ between themselves and from the other non-codewords; the non-codewords belonging to the same string; the non-codewords belonging to different strings. Therefore C is an r-LD code. ■

Now, like in Proposition 14, we want to reach all intermediate values between $k(r+1)+1$ and $(k+1)(r+1)$. We do so by adding a set $W_p = \{w_1,\ldots,w_p\}$ of p vertices, $p \in \{0,\ldots,r\}$. However, if we proceed exactly as for Proposition 14 by creating the edge set $X_p = \{w_1v_{1,0}, w_1v_{2,0}, w_2v_{3,0}, w_2v_{4,0}, \ldots, w_pv_{2p-1,0}, w_pv_{2p,0}\}$ but now considering the code $C = \{v_{1,0},\ldots,v_{k-1,0}\} \cup \{y\}$, we might have one or two pairs of vertices not r-separated by C. We show one such pair ($v_{4,1}$ and w_2) in Figure 9 when $r = 4$, $k = 11$, $p = 4$; more generally, this may occur whenever r is even. Moreover, a symmetrical situation appears when $k - r$ is odd, see the same Figure with w_4 and $v_{7,1}$. The existence of both pairs is due to the fact that $v_{k,0} \notin C$.

So we choose another way of linking the vertices w_i to the vertices $v_{s,0}$: $X_p = \{w_1v_{1,0}, w_1v_{3,0}, w_2v_{2,0}, w_2v_{4,0}, \ldots\}$, see Figure 8(b) for $p = 2$. Setting $G^y_{n+1+p} = \left(V^*_n \cup \{y\} \cup W_p, E^*_n \cup \{yv_{k,r}, yv_{k,r-1}\} \cup X_p\right)$, we obtain a graph of order $n+1+p = k(r+1)+1+p$.

Proposition 29. *For all $r \geq 2$, $k \geq 2r+2$ and $p \in \{0,\ldots,r\}$, the graph $G^y_{k(r+1)+1+p}$ is such that $\gamma_r\left(G^y_{k(r+1)+1+p}\right) = ld_r\left(G^y_{k(r+1)+1+p}\right)$.*

Proof. Again, we take $C = \{v_{i,0} : 1 \leq i \leq k-1\} \cup \{y\}$. Using anew the proof of the previous proposition, we can see that we have only to prove in addition the following two assertions about the w_i's.

(a) If $p \geq 2$, any two non-codewords w_i and w_s, $\{i,s\} \subseteq \{1,2,\ldots,p\}$, $i < s$, are r-separated by C. If w_i is linked to $v_{\ell,0}$ and $v_{\ell+2,0}$, then the set of codewords r-dominating w_i has size $3+2(r-1)$ or $2+2(r-1)$, and consists (with computations modulo k) of $v_{\ell,0}$, $v_{\ell+1,0}$, $v_{\ell+2,0}$, $v_{\ell-1,0}, \ldots, v_{\ell-r+1,0}, v_{\ell+3,0}, \ldots, v_{\ell+2+r-1,0}$, or of the same set without $v_{k,0}$. In both cases, it cannot be the same as the set of codewords r-dominating w_s.

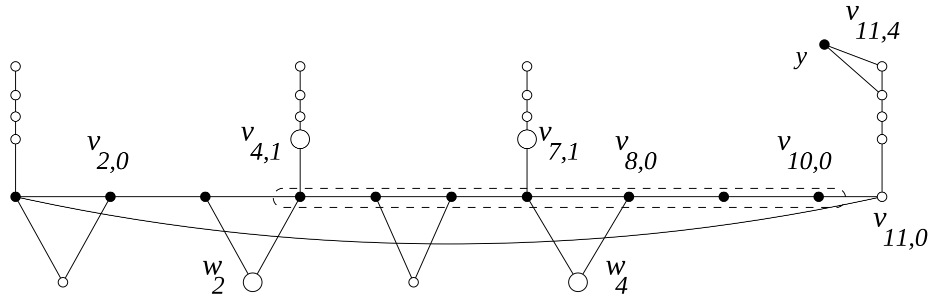

Figure 9. $r = 4$, $n = 60$. The eleven black vertices represent codewords. Not all strings are shown. The vertices w_2 and $v_{4,1}$ are not 4-separated by C; neither are w_4 and $v_{7,1}$, which are both 4-dominated by $v_{i,0}$, $4 \le i \le 10$, as indicated by the dotted-line box.

(b) Two non-codewords w_i, $i \in \{1, \ldots, p\}$, and $v_{s,t}$, $1 \le s \le k-1$, $1 \le t \le r$, are r-separated by C. If w_i is linked to $v_{\ell,0}$ and $v_{\ell+2,0}$, the most crucial cases are when $s \in \{\ell, \ell+1, \ell+2\}$ and $t = 1$, but even here, w_i is r-dominated by more codewords than $v_{s,1}$ (note that this "W-construction" would not have worked for r-ID codes, because then w_i and $v_{\ell+1,0}$ would not be r-separated by the code). ■

Corollary 30. *For all $r \ge 2$ and $n \ge (2r+2)(r+1)+1$, we have $f^{tw}_{ld,\gamma}(r,n) = 0$.*

8. Conclusion

In the following tables, we recapitulate our results on the different minimum and maximum differences between cardinalities of minimum dominating, locating-dominating or identifying codes in connected graphs, first for $r = 1$, then for $r \ge 2$. For $r = 1$, we have exact values for all n and all functions.

n	2	3	4	5	6	≥ 7	Proposition
$f_{id,\gamma}(1,n)$	×	1	1	1	0	0	16
$F_{id,\gamma}(1,n)$	×	1	2	3	4	$n-2$	17
$f_{id,ld}(1,n)$	×	0	0	0	0	0	19
$F_{id,ld}(1,n)$	×	0	1	2	2	$\lceil \frac{n}{2} \rceil - 1$	21
$f_{ld,\gamma}(1,n)$	×	1	0	0	0	0	22(a)–(b)
$F_{ld,\gamma}(1,n)$	×	1	2	3	4	$n-2$	23(a)
$f^{tw}_{ld,\gamma}(1,n)$	0	1	1	0	1	0	26
$F^{tw}_{ld,\gamma}(1,n)$	0	1	2	3	4	$n-2$	(clique)

For $r \geq 2$, most results are valid for n large (typically, n is in r^2).

id vs γ	$\forall r \geq 2,\ f_{id,\gamma}(r,n) = 0$ [Proposition 14] n even, $\forall r \geq 2,\ F_{id,\gamma}(r,n) = n-3$ [Proposition 18(a)] n odd, $\forall r \geq 2,\ F_{id,\gamma}(r,n) = n-2$ [Proposition 18(b)]
id vs *ld*	$\forall r \geq 2,\ f_{id,ld}(r,n) = 0$ [Corollary 15] $\forall r \geq 2,\ F_{id,ld}(r,n) = \lceil \frac{n}{2} \rceil - 1$ [Proposition 20]
ld vs γ (twin-free graphs)	$\forall r \geq 2,\ f_{ld,\gamma}(r,n) = 0$ [Corollary 15, Proposition 22(c)] $F_{ld,\gamma}(2,n) \geq \frac{5n-45}{8} \approx \frac{5n}{8}$ [Prop. 25(b), case $i=1$, ineq. (10)] $\forall r \geq 3,\ F_{ld,\gamma}(r,n) \geq \lceil \frac{n}{2} \rceil - 2$ [Proposition 23(b)] $\forall r \geq 2,\ F_{ld,\gamma}(r,n) \leq n-3$ [Proposition 23(c)]
ld vs γ (with twins)	$\forall r \geq 2,\ f^{tw}_{ld,\gamma}(r,n) = 0$ [Corollary 30] $\forall r \geq 2,\ F^{tw}_{ld,\gamma}(r,n) = n-2$ (clique)

References

[1] C. Berge, Graphes (Gauthier-Villars, Paris, 1983).

[2] C. Berge, Graphs (North-Holland Publishing Co., Amsterdam, 1985).

[3] N. Bertrand, *Codes identifiants et codes localisateurs-dominateurs sur certains graphes*, Mémoire de stage de maîtrise, ENST (Paris, France, 2001).

[4] I. Charon, O. Hudry and A. Lobstein, *Possible cardinalities for identifying codes in graphs*, Australas. J. Combin. **32** (2005) 177–195.

[5] I. Charon, O. Hudry and A. Lobstein, *Possible cardinalities for locating-dominating codes in graphs*, Australas. J. Combin. **34** (2006) 23–32.

[6] I. Charon, O. Hudry and A. Lobstein, *Extremal cardinalities for identifying and locating-dominating codes in graphs*, Discrete Math. **307** (2007) 356–366. doi:10.1016/j.disc.2005.09.027

[7] C.J. Colbourn, P.J. Slater and L.K. Stewart, *Locating dominating sets in series parallel networks*, Congr. Numer. **56** (1987) 135–162.

[8] R. Diestel, Graph Theory (Springer-Verlag, Berlin, 2005).

[9] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, *On graphs having domination number half their order*, Period. Math. Hungar. **16** (1985) 287–293. doi:10.1007/BF01848079

[10] F. Foucaud, E. Guerrini, M. Kovše, R. Naserasr, A. Parreau and P. Valicov, *Extremal graphs for the identifying code problem*, European J. Combin. **32** (2011) 628–638. doi:10.1016/j.ejc.2011.01.002

[11] S. Gravier, R. Klasing and J. Moncel, *Hardness results and approximation algorithms for identifying codes and locating-dominating codes in graphs*, Algorithmic Oper. Res. **3** (2008) 43–50.

[12] S. Gravier and J. Moncel, *On graphs having a* $V \setminus \{x\}$ *set as an identifying code*, Discrete Math. **307** (2007) 432–434.
doi:10.1016/j.disc.2005.09.035

[13] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[14] M.G. Karpovsky, K. Chakrabarty and L.B. Levitin, *On a new class of codes for identifying vertices in graphs*, IEEE Trans. Inform. Theory **44** (1998) 599–611.
doi:10.1109/18.661507

[15] A. Lobstein, *Watching systems, identifying, locating-dominating and discriminating codes in graphs, a bibliography.*
https://www.lri.fr/∼lobstein/ debutBIBidetlocdom.pdf

[16] O. Ore, Theory of Graphs (American Mathematical Society, Providence, 1962).
doi:10.1090/coll/038

[17] C. Payan and N.H. Xuong, *Domination-balanced graphs*, J. Graph Theory **6** (1982) 23–32.
doi:10.1002/jgt.3190060104

[18] P.J. Slater, Domination and location in graphs, Research Report 93 (National University of Singapore, 1983).

Received 21 July 2017
Revised 15 February 2018
Accepted 15 February 2018