

NEIGHBOR SUM DISTINGUISHING TOTAL CHOOSABILITY OF IC-PLANAR GRAPHS

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Abstract

Two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph G has a drawing in the plane such that every two crossings are independent, then we call G a plane graph with independent crossings or IC-planar graph for short. A proper total- k -coloring of a graph G is a mapping $c : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $\sum_c(v)$ denote the sum of the color of a vertex v and the colors of all incident edges of v . A total- k -neighbor sum distinguishing-coloring of G is a total- k -coloring of G such that for each edge $uv \in E(G)$, $\sum_c(u) \neq \sum_c(v)$. The least number k needed for such a coloring of G is the neighbor sum distinguishing total chromatic number, denoted by $\chi''_{\Sigma}(G)$. In this paper, it is proved that if G is an IC-planar graph with maximum degree $\Delta(G)$, then $ch''_{\Sigma}(G) \leq \max\{\Delta(G) + 3, 17\}$, where $ch''_{\Sigma}(G)$ is the neighbor sum distinguishing total choosability of G .

Keywords: neighbor sum distinguishing total choosability, maximum degree, IC-planar graph, Combinatorial Nullstellensatz.

2010 Mathematics Subject Classification: 05C15.

1. INTRODUCTION

All graphs considered are finite, simple and undirected. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For planar graph G , $F(G)$ denotes its face set, $d(v)$ denotes the *degree* of a vertex v in G . The *length* or *degree* of a face f , denoted by $d(f)$, is the length of the boundary walk of f in G . We call v a k -*vertex*, or a k^+ -*vertex*, or a k^- -*vertex* if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively and call f a k -*face*, or a k^+ -*face*, or a k^- -*face* if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$, respectively. Any undefined notation follows that of Bondy and Murty [3].

A proper total- k -coloring of a graph G is a mapping $c : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $\sum_c(v)$ be the sum of the color of a vertex v and the colors of all edges incident with v . If for each edge $uv \in E(G)$, $\sum_c(u) \neq \sum_c(v)$, then we say such total- k -coloring a *neighbor sum distinguishing total- k -coloring*, denoted by *tnsd- k -coloring* for short. The least number k needed for such a coloring of G is the *neighbor sum distinguishing total chromatic number*, denoted by $\chi''_{\Sigma}(G)$. For neighbor sum distinguishing total colorings, we have the following conjecture proposed by Piłśniak and Woźniak [11].

Conjecture 1. *For any graph G , $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$.*

Loeb and Tang [10] proved that this bound was asymptotically correct by showing that $\chi''_{\Sigma}(G) \leq \Delta(G)(1 + o(1))$. Piłśniak and Woźniak [11] proved that Conjecture 1 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. With the Combinatorial Nullstellensatz, neighbor sum distinguishing total coloring have been studied widely, see [4–6, 8, 9, 12, 19]

For a given graph G , let $L_x(x \in V \cup E)$ be a set of lists of real numbers and each of size k . The neighbor sum distinguishing total choosability of G is the least number k for which for any specified collection of such lists, there exists a neighbor sum distinguish total coloring with colors from L_x for each $x \in V \cup E$, and we denote it by $ch''_{\Sigma}(G)$. We call such a coloring of G *list neighbor sum distinguish total- k -coloring* and denote it by *ltnsd- k -coloring*. Ding *et al.* [4] proved that for any graph G , $ch''_{\Sigma}(G) \leq 2\Delta(G) + col(G) - 1$, where $col(G)$ is the coloring number of G . Later Ding *et al.* [5] improved the bound to $ch''_{\Sigma}(G) \leq 2\Delta(G) + col(G) - 2$. Recently, Lu *et al.* [20] improved the bound to $ch''_{\Sigma}(G) \leq \max\{\Delta(G) + \lfloor \frac{3col(G)}{2} \rfloor - 1, 3col(G) - 2\}$. The list neighbor sum distinguish total- k -coloring of some special classes of graphs were also investigated. Graphs with bounded maximum average degree (Yao and Kong [16]); d -degenerate graphs (Yao *et al.* [18]); planar graphs (Qu *et al.* [13], Wang *et al.* [15]).

In this paper, we consider IC-planar graphs and prove the following result.

Theorem 2. *Let G is an IC-planar graph with maximum degree $\Delta(G)$. Then $ch''_{\Sigma}(G) \leq \max\{\Delta(G) + 3, 17\}$.*

An *IC-plane graph* is a topological graph where every edge is crossed at most once and no two crossed edges share a vertex, i.e., two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph G has a drawing in the plane in which every two crossings are independent, then we call G a plane graph with independent crossings or IC-planar graph for short throughout this paper. This definition of IC-planar graph was introduced by Albertson [1] in 2008. Settling a conjecture of Albertson [1], Král and Stacho [7] showed that every IC-planar graph is 5-colorable. Obviously, every IC-planar graph also is a 1-planar graph. We call G a 1-planar graph if it can be drawn on a plane such that each edge is crossed by at most one other edge.

2. PRELIMINARIES

Every IC-planar graph G in this paper has been embedded on a plane such that all its crossings are independent and the number of crossings is as small as possible. In other words, we call G an IC-plane graph. *The associated plane graph G^\times* of G is obtained by turning all crossings of G into new 4-vertices on a plane. For convenience, a vertex in G^\times is called *false* if it is not a vertex of G and *real* otherwise. For a vertex $v \in V(G)$, we use $d_i(v)$ to denote the number of i -vertices which are adjacent to v . One can see that every real vertex in G^\times is adjacent to at most one false vertex and incident with at most two false faces in G^\times .

Lemma 3 [17]. *Let G be a 1-plane graph and G^\times be its associated plane graph. If $d_G(u) = 3$ and v is a crossing vertex in G^\times , then either $uv \notin E(G^\times)$ or uv is not incident with two 3-faces.*

We define that a graph G' is *smaller* than a graph G if $|E(G')| < |E(G)|$. We call a graph *minimal* for a property when no smaller graph satisfies it. Let from now on $G = (V, E)$ be a minimal counterexample to Theorem 2. We set $k = \max\{\Delta(G) + 3, 17\}$. For each 5⁻-vertex $v \in V(G)$, it is obvious that v has at most five neighbors and five incident edges, so v has at most 15 forbidden colors. Since $k \geq 17$, we can first erase the color of vertex v and finally recolor it after arguing. In other words, we may omit the coloring for all 5⁻-vertices of G in the following discussion.

Theorem 4 (Combinatorial Nullstellensatz [2]). *Let \mathbb{F} be an arbitrary field, and let $P = P(x_1, x_2, \dots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \dots, x_n]$. Suppose the degree $\deg(P)$ of P equals $\sum_{i=1}^n k_i$, where each k_i is a nonnegative integer, and suppose the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ in P is non-zero. Then if S_1, S_2, \dots, S_n*

are subsets of \mathbb{F} with $|S_i| > k_i$, there are $s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n$ so that $P(s_1, s_2, \dots, s_n) \neq 0$.

Lemma 5 [14]. *If $P(x_1, x_2, \dots, x_n) \in \mathbb{R}[x_1, x_2, \dots, x_n]$ is of degree $\leq s_1 + s_2 + \dots + s_n$, where s_1, s_2, \dots, s_n are nonnegative integers, then*

$$\begin{aligned} & \left(\frac{\partial}{\partial x_1} \right)^{s_1} \left(\frac{\partial}{\partial x_2} \right)^{s_2} \cdots \left(\frac{\partial}{\partial x_n} \right)^{s_n} P(x_1, x_2, \dots, x_n) \\ &= \sum_{x_1=0}^{s_1} \cdots \sum_{x_n=0}^{s_n} (-1)^{s_1+x_1} \binom{s_1}{x_1} \cdots (-1)^{s_n+x_n} \binom{s_n}{x_n} P(x_1, x_2, \dots, x_n). \end{aligned}$$

Lemma 6 [13]. *Let L_i be the sets of real numbers, with $|L_i| = l_i$, where $i = 1, 2, \dots, p$. Let $L = \left\{ \sum_{i=1}^p x_i \mid x_i \in L_i \text{ and } \prod_{1 \leq i < j \leq p} (x_i - x_j) \neq 0 \right\}$. Then $|L| \geq \sum_{i=1}^p (l_i - p + 1) - (p - 1) = \sum_{i=1}^p l_i - p^2 + 1$.*

3. PROOF OF THEOREM 2

3.1. Unavoidable configurations

In the following, we will often delete some edges to get a proper subgraph G' of G , then by the minimality of G , there exists an ltnsd- k -coloring c of G' . Let $W_G(v) = \sum_{e \ni v, e \in E(G)} c(e) + c(v)$. We may extend this coloring c to the whole graph G . For any $x \in V(G) \cup E(G)$, the *available* colors are the remaining colors after excluding the colors of its adjacent edges and vertices in G' from L_x .

Claim 7. *For any vertex $v \in V(G)$, it holds that*

$$\sum_{j=1}^t [d_j(v)(\Delta(G) + 4 - d(v) - j)] \leq d(v) - 1, \quad (1 \leq t \leq 5).$$

Claim 8. *For any vertex $v \in V(G)$, $d_{2-}(v) \leq \frac{d_{6+}(v)-1}{\Delta(G)-d(v)+1}$. Moreover, if $d(v) = \Delta(G)$, then $d_{2-}(v) \leq d_{6+}(v) - 1$.*

The proof of Claim 7 and 8 are similar to that of Claim 3.1 and Claim 3.2 in [13], we omit it here. By Claim 7, we can easily get the following Corollaries.

Corollary 9. *If $d(v) = 8$, then $d_{5-}(v) \leq 1$.*

Corollary 10. *If $d(v) = 9$, then $d_{5-}(v) \leq 2$.*

Corollary 11. *If $d(v) = 10$, then $d_{5-}(v) \leq 3$.*

Claim 12. *If $d(v) = 11$ and $d_{6+}(v) \leq 6$, then $d_{3-}(v) \leq 1$.*

Proof. Suppose to the contrary that v is adjacent to two 3^- -vertices. Without loss of generality, we assume that $N(v) = \{v_1, v_2, \dots, v_{11}\}$, $d(v_1) = d(v_2) = 3$ and $d(v_j) \geq 6$, ($6 \leq j \leq 11$). Consider $G' = G - vv_1 - vv_2$, then G' admits an ltnsd- k -coloring c . Now we will color the edges vv_1, vv_2 and recolor vertices v_1, v_2 . Let S_1, S_2 be the sets of available colors for vv_1, vv_2 , respectively. It is easy to obtain that $|S_i| = 17 - 12 = 5$, ($i = 1, 2$). By Lemma 6, $|L| \geq |S_1| + |S_2| - 4 + 1 = 7 > 6$. We can choose a pair, say $(x, y) \in S_1 \times S_2$ with $x \neq y$, such that $x + y \notin \{W_G(v_j) - W_G(v) \mid 6 \leq j \leq 11\}$. Finally, we can recolor v_1, v_2 to get an ltnsd- k -coloring of G , a contradiction. ■

Claim 13. *If $d(v) = 12$ and $d_{6^+}(v) \leq 6$, then $d_{3^-}(v) \leq 2$.*

Proof. Suppose to the contrary that v is adjacent to three 3^- -vertices. Without loss of generality, we assume that $N(v) = \{v_1, v_2, \dots, v_{12}\}$, $d(v_1) = d(v_2) = d(v_3) = 3$ and $d(v_j) \geq 6$, ($7 \leq j \leq 12$). Consider $G' = G - \{vv_i \mid i = 1, 2, 3\}$, then G' admits an ltnsd- k -coloring c . Now we will color the edges vv_1, vv_2, vv_3 and recolor vertices v_1, v_2, v_3 . Let S_1, S_2, S_3 be the sets of available colors for vv_1, vv_2, vv_3 , respectively. It is easy to obtain that $|S_i| = 17 - 12 = 5$, ($1 \leq i \leq 3$). By Lemma 6, $|L| \geq |S_1| + |S_2| + |S_3| - 9 + 1 = 7 > 6$. We can choose a triple, say $(x, y, z) \in S_1 \times S_2 \times S_3$ with x, y, z distinct colors, such that $x + y + z \notin \{W_G(v_j) - W_G(v) \mid 7 \leq j \leq 12\}$. Finally, we can recolor v_1, v_2, v_3 to get an ltnsd- k -coloring of G , a contradiction. ■

By Lemma 5, if $P(x_1, x_2, \dots, x_n)$ is a polynomial with $\deg(P) = n$, k_1, k_2, \dots, k_m are non-negative integers with $\sum_{i=1}^m k_i = n$ and $cp \left(x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} \right)$ is the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$ in P , then $\frac{\partial^n P}{\partial x_1^{k_1} \dots \partial x_m^{k_m}} = cp \left(x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} \right) \prod_{i=1}^m k_i!$. In the following, we use MATLAB to calculate the coefficients of specific monomials. Moreover, we will list the codes in Appendix.

Claim 14. *Every 5^- -vertex is not adjacent to 7^- -vertex in G .*

Proof. Suppose to the contrary that there exists a 5^- -vertex u adjacent to a 7^- -vertex v . Without loss of generality, we assume that $d(u) = 5$, $d(v) = 7$, $N(u) = \{v, u_1, \dots, u_4\}$, $N(v) = \{u, v_1, \dots, v_6\}$. Consider $G' = G - uv$, then G' admits an ltnsd- k -coloring c . Now we will recolor the vertices u, v and color the edge uv . Let S_1, S_2, S_3 be the sets of available colors for u, uv, v , respectively. Notice that the colors in $\{c(uu_i) \mid 1 \leq i \leq 4\} \cup \{c(u_i) \mid 1 \leq i \leq 4\}$ are forbidden for u , the colors in $\{c(uu_i) \mid 1 \leq i \leq 4\} \cup \{c(vv_i) \mid 1 \leq i \leq 6\}$ are forbidden for uv , and the colors in $\{c(vv_i) \mid 1 \leq i \leq 6\} \cup \{c(v_i) \mid 1 \leq i \leq 6\}$ are forbidden for v . Thus, $|S_1| = 17 - 8 = 9 > 8$, $|S_2| = 17 - 10 = 7 > 6$, $|S_3| = 17 - 12 = 5 > 4$. We associate that u, uv, v with the variables x_1, x_2, x_3 , respectively. Then we

consider the following polynomial.

$$P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3) \left(x_2 - x_3 \left(x_1 + \sum_{l=1}^4 c(uu_l) - x_3 - \sum_{k=1}^6 c(vv_k) \right) \right. \\ \left. \prod_{i=1}^4 \left(x_1 + x_2 + \sum_{l=1}^4 c(uu_l) - W(u_i) \right) \right. \\ \left. \prod_{j=1}^6 \left(x_2 + x_3 + \sum_{k=1}^6 c(vv_k) - W(v_j) \right) \right).$$

We have $cp(x_1^6 x_2^4 x_3^4) = -25$. According to Theorem 4, there exists $x_i \in S_i$, ($1 \leq i \leq 3$) such that $P(x_1, x_2, x_3) \neq 0$. We color u, uv, v correspondingly. Finally, we can get an ltnsd- k -coloring of the graph G , a contradiction. ■

Claim 15. *Every 6^- -vertex is not adjacent to 6^- -vertex in G .*

Proof. Suppose to the contrary that there exists a 6^- -vertex u adjacent to a 6^- -vertex v . Without loss of generality, we assume that $d(u) = 6$, $d(v) = 6$, $N(u) = \{v, u_1, \dots, u_5\}$, $N(v) = \{u, v_1, \dots, v_5\}$. Consider $G' = G - uv$, then G' admits an ltnsd- k -coloring c . Now we will recolor the vertices u, v and color the edge uv . Let S_1, S_2, S_3 be the sets of available colors for u, uv, v , respectively. Notice that the colors in $\{c(uu_i) \mid 1 \leq i \leq 5\} \cup \{c(u_i) \mid 1 \leq i \leq 5\}$ are forbidden for u , the colors in $\{c(uu_i) \mid 1 \leq i \leq 5\} \cup \{c(vv_i) \mid 1 \leq i \leq 5\}$ are forbidden for uv , and the colors in $\{c(vv_i) \mid 1 \leq i \leq 5\} \cup \{c(v_i) \mid 1 \leq i \leq 5\}$ are forbidden for v . Thus, $|S_1| = 17 - 10 = 7 > 6$, $|S_2| = 17 - 10 = 7 > 6$, $|S_3| = 17 - 10 = 7 > 6$. We associate that u, uv, v with the variables x_1, x_2, x_3 , respectively. Then we consider the following polynomial.

$$P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \left(x_1 + \sum_{l=1}^5 c(uu_l) - x_3 - \sum_{k=1}^5 c(vv_k) \right) \\ \prod_{i=1}^5 \left(x_1 + x_2 + \sum_{l=1}^5 c(uu_l) - W(u_i) \right) \\ \prod_{j=1}^5 \left(x_2 + x_3 + \sum_{k=1}^5 c(vv_k) - W(v_j) \right).$$

We have $cp(x_1^6 x_2^4 x_3^4) = -20$. According to Theorem 4, there exists $x_i \in S_i$, ($1 \leq i \leq 3$) such that $P(x_1, x_2, x_3) \neq 0$. We color u, uv, v correspondingly. Finally, we can get an ltnsd- k -coloring of the graph G , a contradiction. ■

Claim 16. *Let $d(v) = 13$ and $d_{6^+}(v) \leq 6$, then $d_{3^-}(v) \leq 5$. Moreover, if $d_{2^-}(v) \geq 1$, then $d_{3^-}(v) \leq 4$.*

Proof. Suppose to the contrary that there exists a 13-vertex v adjacent to six 3^- -vertices. Without loss of generality, assume that $N(v) = \{v_1, v_2, \dots, v_{13}\}$, $d(v_i) = 3$, ($1 \leq i \leq 6$) and $d(v_j) \geq 6$, ($8 \leq j \leq 13$). Consider $G' = G - \{vv_i \mid i = 1, 2, \dots, 6\}$, then G' admits an ltnsd- k -coloring c . Now we will color the edges vv_i and recolor vertices v_i , ($1 \leq i \leq 6$). Let S_i , ($1 \leq i \leq 6$) be the sets of available colors for vv_i , ($1 \leq i \leq 6$), respectively. It is easy to obtain that $|S_i| = 17 - 7 - 1 - 2 = 7 > 6$, ($1 \leq i \leq 6$). We associate that vv_i , ($1 \leq i \leq 6$) with the variables x_i , ($1 \leq i \leq 6$), respectively. Then we consider the following polynomial.

$$P(x_1, x_2, x_3, x_4, x_5, x_6) = \prod_{1 \leq i < j \leq 6} (x_i - x_j) \prod_{k=8}^{13} \left(\sum_{t=1}^6 x_t + \sum_{l=7}^{13} c(vv_l) + c(v) - W(v_k) \right).$$

We have $cp(x_1^6 x_2^5 x_3^4 x_4^3 x_5^2 x_6^1) = 1$. According to Theorem 4, there exists $x_i \in S_i$, ($1 \leq i \leq 6$) such that $P(x_1, x_2, x_3, x_4, x_5, x_6) \neq 0$. We color vv_i , ($1 \leq i \leq 6$) correspondingly. Finally, we can recolor vertices v_i , ($1 \leq i \leq 6$) to get an ltnsd- k -coloring of the graph G , a contradiction.

Moreover, if $d(v_1) = 2$, $d(v_i) = 3$, ($2 \leq i \leq 5$) and $d(v_j) \geq 6$, ($8 \leq j \leq 13$). Consider $G' = G - \{vv_i \mid i = 1, 2, \dots, 5\}$, then G' admits an ltnsd- k -coloring c . Now we will color the edges vv_i and recolor vertices v_i , ($1 \leq i \leq 5$). Let S_i , ($1 \leq i \leq 5$) be the sets of available colors for vv_i , ($1 \leq i \leq 5$), respectively. It is easy to obtain that $|S_1| = 17 - 8 - 1 - 1 = 7 > 6$, $|S_i| = 17 - 8 - 1 - 2 = 6 > 5$, ($2 \leq i \leq 5$). We associate that vv_i , ($1 \leq i \leq 5$) with the variables x_i , ($1 \leq i \leq 5$), respectively. Then we consider the following polynomial.

$$P(x_1, x_2, x_3, x_4, x_5) = \prod_{1 \leq i < j \leq 5} (x_i - x_j) \prod_{k=8}^{13} \left(\sum_{t=1}^5 x_t + \sum_{l=6}^{13} c(vv_l) + c(v) - W(v_k) \right).$$

We have $cp(x_1^6 x_2^4 x_3^3 x_4^2 x_5^1) = -5$. According to Theorem 4, there exists $x_i \in S_i$, ($1 \leq i \leq 5$) such that $P(x_1, x_2, x_3, x_4, x_5) \neq 0$. We color vv_i , ($1 \leq i \leq 5$) correspondingly. Finally, we can recolor vertices v_i , ($1 \leq i \leq 5$) to get an ltnsd- k -coloring of the graph G , a contradiction. ■

Claim 17. Let $d(v) = \Delta(G) \geq 14$ and $d_{6^+}(v) \leq 6$. If $d_{2^-}(v) \geq 1$, then $d_{3^-}(v) \leq 5$.

Proof. Let $d(v) = d$. Suppose to the contrary that there exists a d -vertex v adjacent to six 3^- -vertices. Without loss of generality, assume that $N(v) = \{v_1, v_2, \dots, v_d\}$, $d(v_1) = 2$, $d(v_i) = 3$, ($2 \leq i \leq 6$) and $d(v_j) \geq 6$, ($d - 5 \leq j \leq d$). Consider $G' = G - \{vv_i \mid i = 1, 2, \dots, 6\}$, then G' admits an ltnsd- k -coloring c . Now we will color the edges vv_i and recolor vertices v_i , ($1 \leq i \leq 6$). Let S_i , ($1 \leq i \leq 6$) be the sets of available colors for vv_i ($1 \leq i \leq 6$), respectively.

It is easy to obtain that $|S_1| = (\Delta(G) + 3) - (\Delta(G) - 6) - 1 - 1 = 7 > 6$, $|S_i| = (\Delta(G) + 3) - (\Delta(G) - 6) - 1 - 2 = 6 > 5$, ($2 \leq i \leq 6$). We associate that vv_i , ($1 \leq i \leq 6$) with the variables x_i , ($1 \leq i \leq 6$), respectively. Then we consider the following polynomial.

$$P(x_1, x_2, x_3, x_4, x_5, x_6) = \prod_{1 \leq i < j \leq 6} (x_i - x_j) \prod_{k=d-5}^d \left(\sum_{t=1}^6 x_t + \sum_{l=7}^d c(vv_l) + c(v) - W(v_k) \right).$$

We have $cp(x_1^6 x_2^5 x_3^4 x_4^3 x_5^2 x_6^1) = 1$. According to Theorem 4, there exists $x_i \in S_i$, ($1 \leq i \leq 6$) such that $P(x_1, x_2, x_3, x_4, x_5, x_6) \neq 0$. We color vv_i , ($1 \leq i \leq 6$) correspondingly. Finally, we can recolor vertices v_i , ($1 \leq i \leq 6$) to get an ltnsd- k -coloring of the graph G , a contradiction. \blacksquare

3.2. Discharging process

Let T be the graph obtained by removing all 2^- -vertices from the graph G and T^\times be the associated plane graph of T . We have $d_T(v) = d(v) - d_{2^-}(v)$.

Corollary 18. *For any vertex v with $d(v) \geq 7$, it holds that $d_T(v) \geq 7$.*

Proof. If $7 \leq d(v) \leq 10$, we can easily get $d_T(v) \geq 7$ by Claim 14 and Corollaries 9–11. When $d(v) > 10$, we just consider the situation $d_{6^+}(v) \leq 6$. By Claim 8, $d_T(v) = d(v) - d_{2^-}(v) \geq d(v) - \frac{d_{6^+}(v)-1}{\Delta(G)-d(v)+1} \geq 11 - \frac{5}{14-11+1} \geq 9$. \blacksquare

We apply the discharging method on associated plane graph T^\times of T and complete the proof by contradiction. Since T^\times is a plane graph, we have

$$\begin{aligned} & \sum_{v \in V(T^\times)} (d_{T^\times}(v) - 6) + \sum_{f \in F(T^\times)} (2d_{T^\times}(f) - 6) \\ &= \sum_{v \in V(T)} (d_T(v) - 6) + \sum_{v \in V(T^\times) \setminus V(T)} (d_{T^\times}(v) - 6) + \sum_{f \in F(T^\times)} (2d_{T^\times}(f) - 6) \\ &= \sum_{v \in V(T)} (d(v) - d_{2^-}(v) - 6) + \sum_{v \in V(T^\times) \setminus V(T)} (d_{T^\times}(v) - 6) + \sum_{f \in F(T^\times)} (2d_{T^\times}(f) - 6) \\ &= -12. \end{aligned}$$

Now we define the initial charge function $ch(x)$ of $x \in V(T^\times) \cup F(T^\times)$. Let $ch(v) = d_T(v) - 6 = d(v) - d_{2^-}(v) - 6$ if $v \in V(T)$, $ch(v) = d_{T^\times}(v) - 6$ if $v \in V(T^\times) \setminus V(T)$ and $ch(f) = 2d_{T^\times}(f) - 6$ if $f \in F(T^\times)$. Then we define suitable discharging rules to change the initial charge function $ch(x)$ to the final charge function $ch'(x)$ on $V(T^\times) \cup F(T^\times)$ such that $ch'(x) \geq 0$ for all $x \in V(T^\times) \cup F(T^\times)$. Notice that our discharging rules only move charge around and do not affect the sum. Thus we have $0 \leq \sum_{x \in V(T^\times) \cup F(T^\times)} ch'(x) =$

$\sum_{x \in V(T^\times) \cup F(T^\times)} ch(x) = -12$, a contradiction. Since for every vertex $v \in V(T)$, $ch(v) = d_G(v) - d_{2^-}(v) - 6$, in the discharging process, we use $d_G(v)$ instead of $d_T(v)$. Similarly, for every vertex $v \in V(T)$, when check $ch'(v) \geq 0$, we split the proof into cases depending on the size of $d_G(v)$.

For $v \in V(T^\times)$ and $f \in F(T^\times)$, we define the discharging rules as follows. Note that within all the degree of a real vertex shall refer to its degree in G and the faces and their degrees correspond to the graph T^\times .

- (R1): If the edge uv belongs to two 3-faces and $d(v) = 3$, then u sends 1 to v .
- (R2): If the edge uv belongs to exactly one 3-face and $d(v) = 3$, then u sends $\frac{1}{2}$ to v .
- (R3): If the edge uv belongs to two 3-faces and $d(v) = 4$, then u sends $\frac{1}{2}$ to v .
- (R4): If the edge uv belongs to two 3-faces and $d(v) = 5$, then u sends $\frac{1}{5}$ to v .
- (R5): Every 4-face sends 1 to each incident real 5^- -vertex in T^\times .
- (R6): Every 5^+ -face sends 2 to each incident real 5^- -vertex in T^\times .
- (R7): Let v be a false vertex crossed by edge uw and xy in T^\times . If $d(u) \geq 7$, then u sends 1 to v . Moreover, if $d(w) = 3$, then u sends $\frac{3}{2}$ to v .

By Corollary 18 and the discharging rules, we obtain the following facts easily.

Fact 1. For any $f \in F(T^\times)$, f is incident with at most $\lfloor \frac{d(f)}{2} \rfloor$ real 5^- -vertices in T^\times .

Fact 2. Each vertex v gives at most $\frac{d_{3^+}(v)}{2} + 1$ away.

Let f be a face of T^\times . Clearly, if $d(f) = 3$, then $ch'(f) = ch(f) = 2d(f) - 6 = 0$. If $d(f) = 4$, then $ch'(f) \geq ch(f) - 2 = 0$ by Fact 1 and (R5). If $d(f) \geq 5$, then $ch'(f) \geq ch(f) - \lfloor \frac{d(f)}{2} \rfloor \times 2 = 0$ by Fact 1 and (R6).

We next check the final charge of the vertex $v \in V(T^\times)$. Obviously, $d(v) \geq 3$. Recall that v has an initial weight of $d(v) - d_{2^-}(v) - 6$.

Suppose $d(v) = 3$. If v is incident with three 3-faces, then $ch'(v) \geq ch(v) + 3 = 0$ by (R1). If v is incident with two 3-faces, then $ch'(v) \geq ch(v) + 1 + \frac{1}{2} \times 2 + 1 = 0$ by (R1), (R2), (R5) and (R6). If v is incident with one 3-face, then $ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 1 \times 2 = 0$ by (R2), (R5) and (R6). Otherwise, v is incident with three 4^+ -faces, then $ch'(v) \geq ch(v) + 1 \times 3 = 0$ by (R5) and (R6).

Suppose $d(v) = 4$ and v is a real vertex. We have $d_{2^-}(v) = 0$. If v is incident with four 3-faces, then $ch'(v) \geq ch(v) + \frac{1}{2} \times 4 = 0$ by (R3). If v is incident with three 3-faces, then $ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 1 = 0$ by (R3), (R5) and (R6). If v is incident with at most two 3-faces, then $ch'(v) \geq ch(v) + 1 \times 2 = 0$ by (R5) and (R6).

Suppose $d(v) = 4$ and v is a false vertex crossed by edge uw and xy . By Claim 14 and 15, v is adjacent to at most two 6^- -vertices.

If $d_{6^-}(v) = 2$, without loss of generality, we assume that $d(u) \leq 6$ and $d(x) \leq 6$. By Claim 15, if $4 \leq d(u) \leq 6$ and $4 \leq d(x) \leq 6$, then $ux \notin E(T)$, v gives no weight away by (R3) and (R4). By the same claim, v is also adjacent to two 7^+ -vertices. So v receives at least $1 \times 2 = 2$ from its 7^+ -neighbors by (R7). Thus, we have $ch'(v) \geq ch(v) + 2 = 0$. If one of the vertices x, u is a 3-vertex, without loss of generality, we assume that $d(u) = 3$. Then, by Claim 14, w is a 8^+ -vertex. v may receives $\frac{3}{2}$ from vertex w and 1 from vertex y by (R7) and gives at most $\frac{1}{2}$ away by (R3) and (R4). Thus, we have $ch'(v) \geq ch(v) + \frac{3}{2} + 1 - \frac{1}{2} = 0$. Otherwise, $d(u) = d(x) = 3$. By Claim 14, v receives at least $\frac{3}{2} \times 2 = 3$ from its 8^+ -neighbors by (R7). And v gives at most $\frac{1}{2} \times 2 = 1$ away by Lemma 3, Claim 15 and (R2). Thus, we have $ch'(v) \geq ch(v) + 3 - 1 = 0$.

If $d_{6^-}(v) = 1$, without loss of generality, we assume that $d(u) \leq 6$. Then v is adjacent to three 7^+ -vertices. So v receives at least $1 \times 3 = 3$, and v gives at most $\frac{1}{2}$ away by Lemma 3 and (R2). Thus, we have $ch'(v) \geq ch(v) + 3 - \frac{1}{2} > 0$.

If v is adjacent to four 7^+ -vertices, v receives at least $1 \times 4 = 4$ from its 7^+ -neighbors by (R7) and gives no weight away. So we have $ch'(v) \geq ch(v) + 1 \times 4 > 0$.

Suppose $d(v) = 5$. If v is not incident with any 4^+ -faces, then by (R4), $ch'(v) \geq ch(v) + \frac{1}{5} \times 5 = 0$. Otherwise, if v is incident with at least one 4^+ -faces, then by (R5) and (R6), $ch'(v) \geq ch(v) + 1 = 0$.

Suppose $d(v) = 6$. v gives no weight away to any other vertex by the discharging rules. So $ch'(v) = ch(v) = 0$.

Suppose $d(v) = 7$. v gives at most 1 to the false neighbor in T^\times by (R7), then $ch'(v) = ch(v) - 1 = 0$.

Suppose $d(v) = 8$. By Corollary 9, $ch'(v) = ch(v) - \max\{2, \frac{3}{2}\} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 9$. By Corollary 10, $ch'(v) = ch(v) - \max\{3, 1 + \frac{3}{2}\} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 10$. By Corollary 11, $ch'(v) = ch(v) - \max\{4, 2 + \frac{3}{2}\} \geq 0$ by (R1)–(R4) and (R7).

Next we check the final charge of the vertices with $d(v) \geq 11$. Let w be a false vertex crossed by edge uv and edge xy . According to the discharging rules, if $d(u) \leq 5$, then v gives at most $d_{5^-}(v) + \frac{1}{2}$ away. Otherwise, v gives at most $d_{5^-}(v) + 1$ away. Therefore, for every vertex v with $d_{6^+}(v) \geq 7$, $ch'(v) \geq 0$. In the following discussion, we only consider the vertex with $d(v) \geq 11$ and $d_{6^+}(v) \leq 6$.

Suppose $d(v) = 11$. By Claim 12, we have that $d_{3^-}(v) \leq 1$. If $d_{2^-}(v) = 0$, then $d_3(v) \leq 1$. We have $ch'(v) = ch(v) - \max\{1 + d_3(v) + (\lfloor \frac{11-1}{2} \rfloor - d_3(v)) \times \frac{1}{2}, \frac{3}{2} + \lfloor \frac{11-1}{2} \rfloor \times \frac{1}{2}\} = 5 - \max\{\frac{7+d_3(v)}{2}, 4\} > 0$ by (R1)–(R4) and (R7). If $d_{2^-}(v) = 1$, then $d_3(v) = 0$. We have $ch'(v) = ch(v) - 1 - \lfloor \frac{11-1}{2} \rfloor \times \frac{1}{2} = 3 - \frac{5}{2} > 0$ by (R3) and (R7).

Suppose $d(v) = 12$. By Claim 13, we have that $d_{3^-}(v) \leq 2$.

If $d_{2^-}(v) = 0$, then $d_3(v) \leq 2$. If $d_3(v) \geq 1$, we have $ch'(v) = ch(v) -$

$\max \left\{ 1 + d_3(v) + \left(\left\lfloor \frac{12-1}{2} \right\rfloor - d_3(v) \right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\left\lfloor \frac{12-1}{2} \right\rfloor - (d_3(v) - 1) \right) \times \frac{1}{2} \right\} = 6 - \frac{7+d_3(v)}{2} > 0$ by (R1)–(R4) and (R7). If $d_3(v) = 0$, then we have $ch'(v) = ch(v) - 1 - \left\lfloor \frac{12-1}{2} \right\rfloor \times \frac{1}{2} > 0$ by (R3) and (R7).

If $d_{2-}(v) \geq 1$ and $d_3(v) \geq 1$, we have $ch'(v) = ch(v) - \max \left\{ 1 + d_3(v) + \left(\left\lfloor \frac{12-d_{2-}(v)-1}{2} \right\rfloor - d_3(v) \right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\left\lfloor \frac{12-d_{2-}(v)-1}{2} \right\rfloor - (d_3(v) - 1) \right) \times \frac{1}{2} \right\} \geq \frac{9-3d_{3-}(v)+d_3(v)}{4} > 0$ by (R1)–(R4) and (R7). Otherwise $d_3(v) = 0$, then $d_{2-}(v) \leq 2$. So $ch'(v) = ch(v) - 1 - \left\lfloor \frac{12-d_{2-}(v)-1}{2} \right\rfloor \times \frac{1}{2} \geq \frac{9-3d_{2-}(v)}{4} > 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 13$. By Claim 16, we have that $d_{3-}(v) \leq 5$. Moreover, if $d_{2-}(v) \geq 1$, then $d_{3-}(v) \leq 4$.

If $d_{2-}(v) = 0$, then $d_3(v) \leq 5$. If $d_3(v) \geq 1$, then we have $ch'(v) = ch(v) - \max \left\{ 1 + d_3(v) + \left(\left\lfloor \frac{13-1}{2} \right\rfloor - d_3(v) \right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\left\lfloor \frac{13-1}{2} \right\rfloor - (d_3(v) - 1) \right) \times \frac{1}{2} \right\} = 3 - \frac{d_3(v)}{2} > 0$ by (R1)–(R4) and (R7). If $d_3(v) = 0$, then we have $ch'(v) = ch(v) - 1 - \left\lfloor \frac{13-1}{2} \right\rfloor \times \frac{1}{2} > 0$ by (R3) and (R7).

If $d_{2-}(v) \geq 1$, then $d_{3-}(v) \leq 4$. If $d_3(v) \geq 1$, we have $ch'(v) = ch(v) - \max \left\{ 1 + d_3(v) + \left(\left\lfloor \frac{13-d_{2-}(v)-1}{2} \right\rfloor - d_3(v) \right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\left\lfloor \frac{13-d_{2-}(v)-1}{2} \right\rfloor - (d_3(v) - 1) \right) \times \frac{1}{2} \right\} \geq 3 - \frac{3d_{3-}(v)}{4} + \frac{d_3(v)}{4} > 0$ by (R1)–(R4) and (R7). Otherwise $d_3(v) = 0$, then $d_{2-}(v) \leq 4$. So $ch'(v) = ch(v) - 1 - \left\lfloor \frac{13-1-d_{2-}(v)}{2} \right\rfloor \times \frac{1}{2} \geq 3 - \frac{3d_{2-}(v)}{4} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = \Delta(G) \geq 14$. If $d_{2-}(v) = 0$, then by Fact 2, we have $ch'(v) = ch(v) - \frac{d_3+(v)}{2} - 1 \geq 0$ by (R1)–(R4) and (R7).

If $d_{2-}(v) \geq 1$, then by Claim 17, $d_{3-}(v) \leq 5$. If $d_3(v) \geq 1$, then we have $ch'(v) = ch(v) - \max \left\{ 1 + d_3(v) + \left(\left\lfloor \frac{14-d_{2-}(v)-1}{2} \right\rfloor - d_3(v) \right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\left\lfloor \frac{14-d_{2-}(v)-1}{2} \right\rfloor - (d_3(v) - 1) \right) \times \frac{1}{2} \right\} \geq \frac{15-3d_{3-}(v)+d_3(v)}{4} > 0$ by (R1)–(R4) and (R7). Otherwise $d_3(v) = 0$, then $d_{2-}(v) \leq 5$. So $ch'(v) = ch(v) - d_{2-}(v) - 1 - \left\lfloor \frac{14-1-d_{2-}(v)}{2} \right\rfloor \times \frac{1}{2} \geq \frac{15-3d_{2-}(v)}{4} \geq 0$ by (R1)–(R4) and (R7).

This completes the proof.

4. REMARK

By the definition of IC-planar graphs, we know that every planar graphs are special IC-planar graphs. In [13], the authors proved that $ch''_{\Sigma}(G) \leq \max\{\Delta(G) + 3, 16\}$. So we can easily obtain the following question.

Question 1. Is it true that $ch''_{\Sigma}(G) \leq \Delta(G) + 3$ for IC-planar graphs with $\Delta = 13$?

Acknowledgements

The authors would like to express their thanks to the referee for his valuable corrections and suggestions of the manuscript that greatly improve the format and correctness of it. This work was supported by National Natural Science Foundation of China (11771443).

APPENDIX A

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%% The m. file of Matlab to compute the coefficients.
% INPUT
function coefficients ()
syms x1 x2 x3 x4 x5 x6 x7 % Variables used in the following.

% Claim 3.7 %To calculate the coefficient of  $x_1^6 x_2^4 x_3^4$ 
P=(x1-x2)*(x2-x3)*(x1-x3)^2*(x1+x2)^4*(x2+x3)^6; % The polynomial
cp1=diff(diff(diff(P,x1,6),x2,4),x3,4)/factorial(6)/factorial(4)
    /factorial(4)

% Claim 3.8
P=(x1-x2)*(x2-x3)*(x1-x3)^2*(x1+x2)^5*(x2+x3)^5;
cp2=diff(diff(diff(P,x1,6),x2,4),x3,4)/factorial(6)/factorial(4)
    /factorial(4)

% Claim 3.9
P=(x1-x2)*(x1-x3)*(x1-x4)*(x1-x5)*(x1-x6)*(x2-x3)*(x2-x4)*(x2-x5)
    *(x2-x6)*(x3-x4)*(x3-x5)*(x3-x6)...*(x4-x5)*(x4-x6)*(x5-x6)
    *(x1+x2+x3+x4+x5+x6)^6;
cp3=diff(diff(diff(diff(diff(P,x1,6),x2,5),x3,4),x4,3),x5,2),
    x6,1)/factorial(6)/factorial(5)/factorial(4)/factorial(3)
    /factorial(2)/factorial(1)

P=(x1-x2)*(x1-x3)*(x1-x4)*(x1-x5)*(x2-x3)*(x2-x4)*(x2-x5)*(x3-x4)
    *(x3-x5)*(x4-x5)*(x1+x2+x3+x4+x5+x6)^6;
cp4=diff(diff(diff(diff(P,x1,6),x2,4),x3,3),x4,2),x5,1)
    /factorial(6)/factorial(4)/factorial(3)/factorial(2)
    /factorial(1)

% Claim 3.10
P=(x1-x2)*(x1-x3)*(x1-x4)*(x1-x5)*(x1-x6)*(x2-x3)*(x2-x4)*(x2-x5)
    *(x2-x6)*(x3-x4)*(x3-x5)*(x3-x6)...*(x4-x5)*(x4-x6)*(x5-x6)
    *(x1+x2+x3+x4+x5+x6)^6;
cp5=diff(diff(diff(diff(diff(P,x1,6),x2,5),x3,4),x4,3),x5,2),
    x6,1)/factorial(6)/factorial(5)/factorial(4)/factorial(3)
    /factorial(2)/factorial(1)

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Received 30 June 2017
Revised 21 March 2018
Accepted 21 March 2018