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ON THE METRIC DIMENSION OF DIRECTED AND UNDIRECTED CIRCULANT GRAPHS

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Abstract

The undirected circulant graph $C_n(\pm 1, \pm 2, \dots, \pm t)$ consists of vertices $v_0, v_1, \dots, v_{n-1}$ and undirected edges $v_i v_{i+j}$, where $0 \le i \le n-1$, $1 \le j \le t$ $(2 \le t \le \frac{n}{2})$, and the directed circulant graph $C_n(1, t)$ consists of vertices $v_0, v_1, \dots, v_{n-1}$ and directed edges $v_i v_{i+1}, v_i v_{i+t}$, where $0 \le i \le n-1$ $(2 \le t \le n-1)$, the indices are taken modulo n. Results on the metric dimension of undirected circulant graphs $C_n(\pm 1, \pm t)$ are available only for special values of t. We give a complete solution of this problem for directed graphs $C_n(1, t)$ for every $t \ge 2$ if $n \ge 2t^2$. Grigorious *et al.* [*On the metric dimension of circulant and Harary graphs*, Appl. Math. Comput. 248 (2014) 47–54] presented a conjecture saying that $dim\big(C_n(\pm 1, \pm 2, \dots, \pm t)\big) = t + p - 1$ for $n = 2tk + t + p$, where $3 \le p \le t + 1$. We disprove it by showing that $dim\big(C_n(\pm 1, \pm 2, \dots, \pm t)\big) \le t + \frac{p+1}{2}$ for $n = 2tk + t + p$, where $t \ge 4$ is even, p is odd, $1 \le p \le t + 1$ and $k \ge 1$.

Keywords: metric dimension, resolving set, circulant graph, distance.

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1. Introduction

Let $V(G)$ be vertex set of a connected (undirected or directed) graph G. The distance $d(u, v)$ between two vertices u, v in an undirected graph is the number of edges in a shortest path between u and v. In a directed graph G the distance $d(u, v)$ from a vertex $u \in V(G)$ to a vertex $v \in V(G)$ is the length of a shortest directed path from u to v.

A vertex w resolves two vertices u and v if $d(u,w) \neq d(v,w)$. For an ordered set of vertices $W = \{w_1, w_2, \ldots, w_z\}$, the representation of distances of v with respect to W is the ordered z-tuple

$$r(v|W) = \big(d(v,w_1), d(v,w_2), \ldots, d(v,w_z)\big).$$

A set $W \subset V(G)$ is a resolving set of G if every two distinct vertices of G have different representations of distances with respect to W (if every two vertices of G are resolved by some vertex in W). The metric dimension of G is the number of vertices in a smallest resolving set and it is denoted by $dim(G)$. The i-th coordinate in $r\,(v|W)$ is 0 if and only if $v = w_i$. Thus in order to prove that W is a resolving set of G, it suffices to show that $r\,(u|W) \neq r\,(v|W)$ for every two different vertices $u, v \in V(G) \setminus W$.

The metric dimension is an invariant, which has applications in robot navigation [9], pharmaceutical chemistry [2], pattern recognition and image processing [10]. It has been extensively studied. For example, Imran [5] studied barycentric subdivisions of Cayley graphs and Saputro *et al.* [12] gave bounds on the metric dimension of the lexicographic product of graphs

Let n, m and $a_1, a_2, \ldots, a_m$ be positive integers such that $1 \leq a_1 < a_2 < \cdots < a_m \leq \frac{n}{2}$. The undirected circulant graph $C_n(\pm a_1, \pm a_2, \ldots, \pm a_m)$ consists of the vertices $v_0, v_1, \ldots, v_{n-1}$ and undirected edges $v_i v_{i+a_j}$, where $0 \leq i \leq n-1$, $1 \leq j \leq m$; the indices are taken modulo n.

For generators $a_1, a_2, \ldots, a_m$ such that $1 \leq a_1 < a_2 < \cdots < a_m \leq n-1$, the directed circulant graph $C_n(a_1, a_2, \ldots, a_m)$ consists of the vertices v_0, $v_1, \ldots, v_{n-1}$ and directed edges $v_i v_{i+a_j}$, where $0 \leq i \leq n-1$, $1 \leq j \leq m$; the indices are taken modulo n. The directed circulant graph $C_n(-a_1, -a_2, \ldots, -a_m)$ contains the directed edges $v_i v_{i-a_j}$.

Circulant graphs form an important family of Cayley graphs. The metric dimension of undirected circulant graphs $C_n(\pm 1, \pm t)$ was studied for special values of t. Javaid, Rahim and Ali [8] proved that if $n \equiv 0, 2, 3 \pmod 4$, then $dim\big(C_n(\pm 1, \pm 2)\big) = 3$. Borchert and Gosselin [1] showed that if $n \equiv 1 \pmod 4$, then $dim\big(C_n(\pm 1, \pm 2)\big) = 4$. The undirected circulant graphs $C_n(\pm 1, \pm 3)$ were considered in [7] and the graphs $C_n(\pm 1, \pm \frac{n}{2})$ for even n were investigated in [11]. We study the metric dimension for directed circulant graphs with 2 generators. We give a complete solution of this problem for directed graphs $C_n(1, t)$ for every $t \geq 2$ if $n \geq 2t^2$.

Exact values of the metric dimension of undirected graphs $C_n\big(\pm 1, \pm 2, \pm 3\big)$ were given in [1] and [6]. Grigorious *et al.* [4] showed that $t+1$ vertices $v_0, v_1, \ldots, v_t$ resolve the graph $C_n(\pm 1, \pm 2, \ldots, \pm t)$ if $n \equiv r \pmod{2t}$, where $2 \leq r \leq t+2$ and they gave the bound $dim\big(C_n(\pm 1, \pm 2, \ldots, \pm t)\big) \leq r-1$ if $n \equiv r \pmod{2t}$, where $t+3 \leq r \leq 2t+1$. They presented a conjecture saying that $dim\big(C_n(\pm 1, \pm 2, \ldots, \pm t)\big) = t + p - 1$ for $n = 2tk + t + p$, where

$3 \leq p \leq t+1$. We disprove it for even $t \geq 4$ and odd $p \geq 5$ by showing that $dim\big(C_n(\pm 1, \pm 2, \ldots, \pm t)\big) \leq t + \frac{p+1}{2}$ for $n = 2tk+t+p$ where $t \geq 4$ is even, p is odd, $1 \leq p \leq t+1$ and $k \geq 1$. Note that Chau and Gosselin [3] recently proved that $dim\big(C_n(\pm 1, \pm 2, \ldots, \pm t)\big) = t+1$ if $n \equiv 2 \pmod{2t}$ and $n \equiv t+1 \pmod{2t}$. They also showed that $dim\big(C_n(\pm 1, \pm 2, \ldots, \pm t)\big) = dim\big(C_{n+2t}(\pm 1, \pm 2, \ldots, \pm t)\big)$ for large n, which implies that the metric dimension of the graphs $C_n(\pm 1, \pm 2, \ldots, \pm t)$ is completely determined by the congruence class of n modulo $2t$.

2. Directed Circulant Graphs

We study the metric dimension of directed circulant garphs $C_n(1,t)$. It is easy to see that the graph $C_n(1,t)$ is isomorphic to the graph $C_n(-1,-t)$ for $2 \leq t \leq n-1$. We present Theorems 1 and 2 for the graph $C_n(-1,-t)$, because it is easier to express distances from vertices in a graph to vertices in chosen resolving sets if we consider $C_n(-1,-t)$ (especially in the proof of Theorem 2).

The distance from the vertex v_j to the vertex v_i in $C_n(-1,-t)$, where $i, j \in \{0, 1, \ldots, n-1\}$, is

$$d(v_j, v_i) = \begin{cases} \left\lfloor \dfrac{j-i}{t} \right\rfloor + p, & p \equiv (j-i) \pmod{t}, \quad \text{if } j \geq i, \quad (1) \\ \left\lfloor \dfrac{n+j-i}{t} \right\rfloor + p, & p \equiv (n+j-i) \pmod{t}, \quad \text{if } j < i, \quad (2) \end{cases}$$

where $0 \leq p \leq t-1$.

Theorem 1. *Let $t \geq 2$ and $n \geq 2t^2$. Then $dim\big(C_n(-1,-t)\big) \geq t$.*

Proof. We prove the result by contradiction. Assume that $dim\big(C_n(-1,-t)\big) \leq t-1$. Let $W = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{t-1}}\}$ be a resolving set of $C_n(-1,-t)$, where $0 \leq i_1 \leq i_2 \leq \cdots \leq i_{t-1}$. Since we have at most $t-1$ different vertices in W and the graph $C_n(-1,-t)$ has at least $2t^2$ vertices, $C_n(-1,-t)$ contains a set of $2t$ consecutive vertices $V' = \{v_j, v_{j+1}, \ldots, v_{j+2t-1}\}$, where $0 \leq j \leq n-1$, such that no vertex of W is in V'. Without loss of generality we can assume that $j = n-2t$, which means that $V' = \{v_{n-2t}, v_{n-2t+1}, \ldots, v_{n-1}\}$ and $i_{t-1} < n-2t$.

Since $|W| \leq t-1$, there is a $k \in \{0, 1, \ldots, t-1\}$, such that no vertex $v_i \in W$ satisfies $i \equiv k \pmod{t}$. So we can write any vertex of W in the form v_{tr+s}, where $0 \leq s \leq t-1$, $s \neq k$ and $r \geq 0$.

Let v_l be any vertex in the set of t vertices $\{v_{n-2t}, v_{n-2t+1}, \ldots, v_{n-t-1}\}$, such that $l \equiv k \pmod{t}$. Then we can write $l = tx + k$, where $0 \leq k \leq t-1$. We show that the vertices $v_{tx+k}, v_{tx+k+t-1} \in V'$ are not resolved by W. Note that

$tx + k > tr + s$. By (1) we have

$$d(v_{tx+k}, v_{tr+s}) = \begin{cases} \left\lfloor \frac{tx+k-(tr+s)}{t} \right\rfloor + k - s = x - r + \lfloor \frac{k-s}{t} \rfloor + k - s \\ = x - r + k - s & \text{if } k > s, \\ x - r + \lfloor \frac{k-s}{t} \rfloor + k - s + t \\ = x - r + k - s + t - 1 & \text{if } k < s, \end{cases}$$

$$d(v_{tx+k+t-1}, v_{tr+s}) = \begin{cases} \left\lfloor \frac{tx+k+t-1-(tr+s)}{t} \right\rfloor + k - 1 - s \\ = x - r + k - s & \text{if } k > s, \\ x + 1 - r + \lfloor \frac{k-1-s}{t} \rfloor + k - 1 - s + t \\ = x - r + k - s + t - 1 & \text{if } k < s. \end{cases}$$

Since $d(v_{tx+k}, v_{tr+s}) = d(v_{tx+k+t-1}, v_{tr+s})$ for any vertex $v_{tr+s} \in W$, the graph $C_n(-1, -t)$ is not resolved by W. A contradiction. ■

Let us present an upper bound on the metric dimension of directed circulant graphs with 2 generators.

Theorem 2. *Let $2 \leq t < n$. Then $dim\big(C_n(-1, -t)\big) \leq t$.*

Proof. We prove that $W = \{v_0, v_1, \dots, v_{t-1}\}$ is a resolving set of $C_n(-1, -t)$. First we find all vertices v_j $(1 \leq j \leq n-1)$ of $C_n(-1, -t)$ such that $d(v_j, v_0) = x$ for any $x \geq 1$. We can write $j = tr + p$ where $r \geq 0$ and $0 \leq p \leq t - 1$. Since by (1), $d(v_{tr+p}, v_0) = r + p$, we have $r + p = x$. Thus $r = x - p$ (≥ 0) and then $v_{t(x-p)+p}$ for $0 \leq p \leq t-1$ and $1 \leq t(x-p)+p \leq n-1$ are the vertices of $C_n(1, t)$ such that $d\big(v_{t(x-p)+p}, v_0\big) = x$.

It remains to show that these vertices are resolved by v_i, $i = 1, 2, \dots, t-1$. It suffices to consider only those vertices $v_{t(x-p)+p}$ which are not in W, so we can assume that $t(x-p) + p > i$. For $i = 1, 2, \dots, t-1$, by (1),

$$(3) \qquad d(v_{t(x-p)+p}, v_i) = \begin{cases} x - p + \left\lfloor \frac{p-i}{t} \right\rfloor + p - i = x - i & \text{if } p \geq i, \\ x - p + \left\lfloor \frac{p-i}{t} \right\rfloor + p - i + t = x + t - 1 - i & \text{if } p < i. \end{cases}$$

We know that the first entry of $r\big(v_{t(x-p)+p}|W\big)$ is x. From (3) it follows that the next p entries (where $0 \leq p \leq t-1$) are $x - i$ and the last $t - 1 - p$ entries of $r\big(v_{t(x-p)+p}|W\big)$ are $x + t - 1 - i$.

So if $p = 0$ (and if v_{tx} exists), the first entry of $r\big(v_{tx}|W\big)$ is x and the other entries are $x + t - 1 - i$ which means that $r\big(v_{tx}|W\big) = \big(x, x + t - 2, x + t - 3, \dots, x + t - 1 - (t-1)\big)$. If $p = 1$, the first entry of $r\big(v_{t(x-1)+1}|W\big)$ is x, the second entry is $x-1$ and the other entries are $x+t-1-i$, so $r\big(v_{t(x-1)+1}|W\big) =$

$\big(x, x-1, x+t-3, x+t-4, \ldots, x+t-1-(t-1)\big)$. Similarly $r\big(v_{t(x-2)+2}|W\big) = \big(x, x-1, x-2, x+t-4, \ldots, x+t-1-(t-1)\big), \ldots, r\big(v_{t(x-(t-1))+(t-1)}|W\big) = \big(x, x-1, x-2, \ldots, x-(t-1)\big)$.

Since all vertices v_j, $1 \le j \le n-1$, such that $d(v_j, v_0) = x$ are resolved by W, we have $dim\big(C_n(-1, -t)\big) \le |W| = t$. ■

From Theorems 1 and 2 we obtain Corollary 3.

Corollary 3. *Let* $t \ge 2$ *and* $n \ge 2t^2$. *Then* $dim\big(C_n(-1, -t)\big) = t$.

Since the graphs $C_n(-1, -t)$ and $C_n(1, t)$ are isomorphic, we get the following corollary.

Corollary 4. *Let* $t \ge 2$ *and* $n \ge 2t^2$. *Then* $dim\big(C_n(1, t)\big) = t$.

3. Undirected Circulant Graphs

We give an upper bound on the metric dimension of undirected circulant graphs $C_n(\pm1, \pm2, \ldots, \pm t)$ for $n \equiv r \pmod{2t}$, where $r = 1$ and $r = t+1, t+3, \ldots, 2t-1$.

The distance between two vertices v_i and v_j in $C_n(\pm1, \pm2, \ldots, \pm t)$, where $0 \le i < j < n$, is

$$d(v_i, v_j) = \min\left\{ \left\lceil \frac{j-i}{t} \right\rceil, \left\lceil \frac{n-(j-i)}{t} \right\rceil \right\}. \tag{4}$$

This equation can be simplified as

$$d(v_i, v_j) = \begin{cases} \left\lceil \dfrac{j-i}{t} \right\rceil & \text{if } 0 \le j-i \le \frac{n}{2}, \quad (5) \\ \left\lceil \dfrac{n-(j-i)}{t} \right\rceil & \text{if } \frac{n}{2} < j-i < n. \quad (6) \end{cases}$$

Theorem 5. *Let* $n = 2tk + t + p$ *where* $t \ge 4$ *is even,* p *is odd,* $1 \le p \le t+1$ *and* $k \ge 1$. *Then*

$$dim\big(C_n(\pm1, \pm2, \ldots, \pm t)\big) \le t + \frac{p+1}{2}.$$

Proof. Let $n = 2tk + t + p$ where $k \ge 1$, $t \ge 4$ is even and $p = 1, 3, \ldots, t+1$. Let

$$W_1 = \{v_0, v_2, \ldots, v_{t-2}\}, \quad W_2 = \{v_{t-1}, v_{t+1}, \ldots, v_{2t-3}\},$$
$$W_3 = \{v_{tk+t-1}, v_{tk+t+1}, \ldots, v_{tk+t+p-2}\}.$$

We have $|W_1| = |W_2| = \frac{t}{2}$ and $|W_3| = \frac{p+1}{2}$. Let us prove that $W = W_1 \cup W_2 \cup W_3$ is a resolving set of the graph $C_n(1, 2, \ldots, t)$.

We divide the vertex set of $C_n(\pm 1, \pm 2, \dots, \pm t)$ into four disjoint sets:

$$V_1 = \{v_0, v_1, \dots, v_t\}, \qquad V_2 = \{v_{t+1}, v_{t+2}, \dots, v_{tk+t}\},$$
$$V_3 = \{v_{tk+t+1}, v_{tk+t+2}, \dots, v_{tk+t+p-1}\}, \quad V_4 = \{v_{tk+t+p}, v_{tk+t+p+1}, \dots, v_{n-1}\}.$$

First we prove that any two vertices of V_2 have different representations of distances with respect to W. For $x = 1, 2, \dots, k-1$; $j = 1, 2, \dots, t$; $i = 0, 2, \dots, t-2$, we have $v_i \in W_1$ and by (5),

$$d(v_{tx+j}, v_i) = x + \left\lceil \frac{j-i}{t} \right\rceil = \begin{cases} x+1 & \text{if } i < j, \\ x & \text{if } i \geq j, \end{cases}$$

and if $x = k$; $j = 1, 2, \dots, t$, by (4), we get

$$\begin{aligned} d(v_{tk+j}, v_i) &= \min\left\{ \left\lceil \frac{(tk+j)-i}{t} \right\rceil, \left\lceil \frac{n-[(tk+j)-i]}{t} \right\rceil \right\}, \\ &= \min\left\{ k + \left\lceil \frac{j-i}{t} \right\rceil, k+1+\left\lceil \frac{p+i-j}{t} \right\rceil \right\} = \begin{cases} k+1 & \text{if } i < j, \\ k & \text{if } i \geq j. \end{cases} \end{aligned}$$

Since j (where $1 \leq j \leq t$) is greater than $\lceil \frac{j}{2} \rceil$ elements from the set $\{0, 2, \dots, t-2\}$, the first $\lceil \frac{j}{2} \rceil$ entries of $r\big(v_{tx+j}|W_1\big)$ for $x = 1, 2, \dots, k$ are equal to $x+1$ and the other $\frac{t}{2} - \lceil \frac{j}{2} \rceil$ entries are x; $r(v_{tx+j}|W_1) = (x+1, \dots, x+1, x, \dots, x)$. Therefore the only vertices in V_2 with the same representations of distances with respect to W_1 are the pairs $(v_{t+1}, v_{t+2}), (v_{t+3}, v_{t+4}), \dots, (v_{tk+t-1}, v_{tk+t})$. But since for $x = 1, 2, \dots, k$ and $j = 1, 3, \dots, t-3$, we obtain $v_{t+j} \in W_2$ and by (5),

$$d(v_{tx+j}, v_{t+j}) = x - 1, \quad d(v_{tx+j+1}, v_{t+j}) = x - 1 + \left\lceil \frac{1}{t} \right\rceil = x,$$

and for $v_{t-1} \in W_2$, we have

$$d(v_{tx+t-1}, v_{t-1}) = x, \quad d(v_{tx+t}, v_{t-1}) = x + \left\lceil \frac{1}{t} \right\rceil = x+1,$$

vertices in W_2 resolve the pairs $(v_{t+1}, v_{t+2}), (v_{t+3}, v_{t+4}), \dots, (v_{tk+t-1}, v_{tk+t})$. Thus no two vertices in V_2 have the same representations of distances with respect to W.

Let us study representations of distances of the vertices in V_4. For $x = 1, 2, \dots, k-1$; $j = 0, 1, \dots, t-1$; $i = 0, 2, \dots, t-2$; we have $v_i \in W_1$ and by (6),

$$d(v_{n-tx+j}, v_i) = \left\lceil \frac{n - [(n-tx+j)-i]}{t} \right\rceil = x + \left\lceil \frac{i-j}{t} \right\rceil = \begin{cases} x & \text{if } i \leq j, \\ x+1 & \text{if } i > j, \end{cases}$$

and if $x = k$, we get

$$d(v_{n-tk+j}, v_i) = \min\left\{ \left\lceil \frac{(n-tk+j)-i}{t} \right\rceil, \left\lceil \frac{n - [(n-tk+j)-i]}{t} \right\rceil \right\}$$
$$= \min\left\{ k+1+\left\lceil \frac{p+j-i}{t} \right\rceil, k + \left\lceil \frac{i-j}{t} \right\rceil \right\} = \begin{cases} k & \text{if } i \le j, \\ k+1 & \text{if } i > j. \end{cases}$$

Since j (where $0 \le j \le t-1$) is greater than or equal to $\lfloor \frac{j}{2} \rfloor + 1$ elements from the set $\{0, 2, \ldots, t-2\}$, the first $\lfloor \frac{j}{2} \rfloor + 1$ entries of $r(v_{n-tx+j}|W_1)$ (for $x = 1, 2, \ldots, k$) are equal to x and the other entries are $x+1$. The only vertices in V_4 with the same representations of distances with respect to W_1 are the pairs (v_{n-tk}, v_{n-tk+1}), $(v_{n-tk+2}, v_{n-tk+3}), \ldots, (v_{n-2}, v_{n-1})$. We show that most of these pairs are resolved by vertices in W_2. For $x = 1, 2, \ldots, k-1$ and $j = 1, 3, \ldots, t-3$, we have $v_{t+j} \in W_2$ and by (6),

$$d(v_{n-tx+j}, v_{t+j}) = x+1, d(v_{n-tx+j-1}, v_{t+j}) = x+1+\lceil \tfrac{1}{t} \rceil = x+2,$$

and for $v_{t-1} \in W_2$, $x = 1, 2, \ldots, k$, by (6),

$$d(v_{n-tx+t-1}, v_{t-1}) = x, \; d(v_{n-tx+t-2}, v_{t-1}) = x + \lceil \tfrac{1}{t} \rceil = x+1,$$

so vertices of W_2 resolve all pairs of vertices $(v_{n-tk+t-2}, v_{n-tk+t-1})$, $(v_{n-tk+t}, v_{n-tk+t+1}), \ldots, (v_{n-2}, v_{n-1})$, which are the pairs $(v_{tk+2t+p-2}, v_{tk+2t+p-1})$, $(v_{tk+2t+p}, v_{tk+2t+p+1}), \ldots, (v_{n-2}, v_{n-1})$. It remains to resolve the pairs $(v_{tk+t+p}, v_{tk+t+p+1})$, $(v_{tk+t+p+2}, v_{tk+t+p+3}), \ldots, (v_{tk+2t+p-4}, v_{tk+2t+p-3})$.

For $j = 0, 2, \ldots, t-p-3$, we have $v_{t+p+j} \in W_2$ and by (5),

$$d(v_{tk+t+p+j}, v_{t+p+j}) = k, \quad d(v_{tk+t+p+j+1}, v_{t+p+j}) = k + \lceil \tfrac{1}{t} \rceil = k+1,$$

so the pairs $(v_{tk+t+p}, v_{tk+t+p+1}), \ldots, (v_{tk+2t-3}, v_{tk+2t-2})$ are resolved by W_2.

For $j = t-p-1, t-p+1, \ldots, t-4$, we have $v_{tk+p+j} \in W_3$ and by (5),

$$d(v_{tk+t+p+j}, v_{tk+p+j}) = 1, \; d(v_{tk+t+p+j+1}, v_{tk+p+j}) = 1 + \lceil \tfrac{1}{t} \rceil = 2,$$

so the pairs $(v_{tk+2t-1}, v_{tk+2t}), \ldots, (v_{tk+2t+p-4}, v_{tk+2t+p-3})$ are resolved by W_3. Thus all pairs of vertices in V_4 are resolved by W.

A vertex $v \in V_2$ and a vertex in V_4 can have the same representations of distances with respect to W_1 only if all entries of $r(v|W_1)$ are the same numbers. For $x = 1, 2, \ldots, k$, we have $v_{tx+t-1}, v_{tx+t} \in V_2$ and $r(v_{tx+t-1}|W_1) = r(v_{tx+t}|W_1) = (x+1, \ldots, x+1)$. For $v_{n-tx+t-2}, v_{n-tx+t-1} \in V_4$ we have $r(v_{n-tx+t-2}|W_1) = r(v_{n-tx+t-1}|W_1) = (x, \ldots, x)$, which implies that for $x = 1, 2, \ldots, k-1$, we obtain $r(v_{tx+t-1}|W_1) = r(v_{tx+t}|W_1) = r(v_{n-tx-2}|W_1) = r(v_{n-tx-1}|W_1)$. Since for $v_{2t-3} \in W_2$, by (5),

$$d(v_{tx+t-1}, v_{2t-3}) = x-1+\lceil \tfrac{2}{t} \rceil = x, \quad d(v_{tx+t}, v_{2t-3}) = x-1+\lceil \tfrac{3}{t} \rceil = x,$$

and by (6),

$$d(v_{n-tx-2}, v_{2t-3}) = x+2+\left\lceil \tfrac{-1}{t} \right\rceil = x+2,\ d(v_{n-tx-1}, v_{2t-3}) = x+2+\left\lceil \tfrac{-2}{t} \right\rceil = x+2,$$

any vertex in V_2 and any vertex in V_4 have different representations of distances with respect to W.

We study representations of the vertices in V_3. For $j = 1, 2, \dots, p-1$ and $i = 0, 2, \dots, t-2$, we have $v_i \in W_1$ and by (4),

$$d(v_{tk+t+j}, v_i) = \min\left\{k + 1 + \left\lceil \tfrac{j-i}{t} \right\rceil, k + \left\lceil \tfrac{p+i-j}{t} \right\rceil\right\} = k + 1,$$

thus $r(v_{tk+t+j}|W_1) = (k+1, \dots, k+1)$. The only vertices in $V_2 \cup V_4$ with the same representations of distances with respect to W_1 are v_{tk+t-1} and v_{tk+t}.

Let us prove that any two vertices in $V_3 \cup \{v_{tk+t-1}, v_{tk+t}\}$ have different representations with respect to W. It suffices to consider the vertices in $V' = (V_3 \cup \{v_{tk+t-1}, v_{tk+t}\}) \setminus W_3 = \{v_{tk+t}, v_{tk+t+2}, \dots, v_{tk+t+p-1}\}$. For $j = 0, 2, \dots, p-1$ and $i = 1, 3, \dots, t-3$, we have $v_{t+i} \in W_2$ and by (5)

$$d(v_{tk+t+j}, v_{t+i}) = k + \left\lceil \tfrac{j-i}{t} \right\rceil = \begin{cases} k & \text{if } i \geq j, \\ k+1 & \text{if } i < j. \end{cases}$$

Since j (for $j \leq t-2$) is greater than $\frac{j}{2}$ elements from the set $\{1, 3, \dots, t-3\}$, the first $\frac{j}{2}$ entries of $r\big(v_{tk+t+j}|W_2'\big)$ where $W_2' = W_2 \setminus \{v_{t-1}\}$ are equal to $k+1$ and the other $\frac{t}{2} - \frac{j}{2} - 1$ entries are k. If $p = t+1$ and $j = t$, we obtain $r\big(v_{tk+t+j}|W_2'\big) = r\big(v_{tk+2t}|W_2'\big) = (k+1, \dots, k+1)$. It follows that the only vertices of V' having the same representations of distances with respect to W_2' are v_{tk+2t} and $v_{tk+2t-2}$ if $p = t+1$. These vertices are resolved by $v_{tk+t-1} \in W_3$, since by (5), $d(v_{tk+2t},\ v_{tk+t-1}) = 1 + \left\lceil \frac{1}{t} \right\rceil = 2$ and $d(v_{tk+2t-2},\ v_{tk+t-1}) = 1 + \left\lceil \frac{-1}{t} \right\rceil = 1$. Thus all vertices of V_3 are resolved by W.

We consider the vertices in V_1. For $j = 1, 3, \dots, t-1$ and t; $i = 0, 2, \dots, t-2$, we have $v_i \in W_1$ and $d(v_j, v_i) = \left\lceil \frac{|j-i|}{t} \right\rceil = 1$, thus $r(v_j|W_1) = (1, \dots, 1)$ for $v_j \in V_1 \setminus W_1$. From the previous part of this proof we know that the only vertices in $V_2 \cup V_3 \cup V_4$ having the representation with respect to W_1 equal to $(1, \dots, 1)$ are v_{n-2} and v_{n-1}. Since $v_{t-1} \in W_2$, it remains to resolve all pairs of vertices in the set $V'' = \{v_1, v_3, \dots, v_{t-3}; v_t, v_{n-2}, v_{n-1}\}$.

We study their representations with respect to W_2. For $j = 1,\ 3, \dots, t-3$ and $i = -1, 1, \dots, t-3$, we have $v_{t+i} \in W_2$ and by (5),

$$d(v_j, v_{t+i}) = 1 + \left\lceil \tfrac{i-j}{t} \right\rceil = \begin{cases} 1 & \text{if } i \leq j, \\ 2 & \text{if } i > j. \end{cases}$$

Since j is greater than or equal to $\frac{j+3}{2}$ elements from the set $\{-1, 1, \dots, t-3\}$, the first $\frac{j+3}{2}$ entries of $r(v_j|W_2)$ are equal to 1 and the other $\frac{t}{2} - \frac{j+3}{2}$ entries are 2. The first two entries of $r(v_j|W_3)$ are always 1. For v_t and any $v_{t+i} \in W_2$, $d(v_t, v_{t+i}) = \left\lceil \frac{|i|}{t} \right\rceil = 1$, therefore $r(v_t|W_2) = (1, \ldots, 1)$.

For $i = -1, 1, \ldots, t-3$, by (6),

$$d(v_{n-1}, v_{t+i}) = 1 + \left\lceil \frac{i+1}{t} \right\rceil = \begin{cases} 1 & \text{if } i = -1, \\ 2 & \text{if } i \geq 1, \end{cases}$$

so $r(v_{n-1}|W_2) = (1, 2, \ldots, 2)$. We have $d(v_{n-2}, v_{t+i}) = 1 + \left\lceil \frac{i+2}{t} \right\rceil = 2$, thus $r(v_{n-2}|W_2) = (2, \ldots, 2)$.

The only pair of vertices in V'' having the same representations with respect to W_2 is (v_{t-3}, v_t), which is resolved by $v_{tk+t-1} \in W_3$, since by (5) we have $d(v_{t-3}, v_{tk+t-1}) = k + \left\lceil \frac{2}{t} \right\rceil = k+1$ and $d(v_t, v_{tk+t-1}) = k + \left\lceil \frac{-1}{t} \right\rceil = k$.

Every two distinct vertices of the graph $C_n(\pm 1, \pm 2, \ldots, \pm t)$ have different representations of distances with respect to W, thus W is a resolving set of $C_n(\pm 1, \pm 2, \ldots, \pm t)$. Hence $dim\big(C_n(\pm 1, \pm 2, \ldots, \pm t)\big) \leq |W| = t + \frac{p+1}{2}$. ■

4. Conclusion

We studied the metric dimension of undirected and directed circulant graphs. Results on the metric dimension of undirected circulant graphs $C_n(\pm 1, \pm t)$ are available only for special values of t. In Section 2 we found exact values of the metric dimension for directed circulant graphs $C_n(1, t)$ by showing that if $t \geq 2$ and $n \geq 2t^2$, then $dim\big(C_n(1, t)\big) = t$.

In Section 3 we presented a bound on the metric dimension of undirected circulant graphs. We proved that for $n = 2tk + t + p$, where $t \geq 4$ is even, p is odd, $1 \leq p \leq t+1$ and $k \geq 1$, $dim\big(C_n(\pm 1, \pm 2, \ldots, \pm t)\big) \leq t + \frac{p+1}{2}$. Note that by [13], $dim\big(C_n(\pm 1, \pm 2, \ldots, \pm t)\big) \leq t + \frac{p}{2}$ if t and p are even, $2 \leq p \leq t$, thus we have $dim\big(C_n(\pm 1, \pm 2, \ldots, \pm t)\big) \leq t + \left\lceil \frac{p}{2} \right\rceil$ for $n = 2tk + t + p$, where $t \geq 4$ is even, $1 \leq p \leq t+1$ and $k \geq 1$,

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