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BOUNDS ON THE LOCATING-TOTAL DOMINATION NUMBER IN TREES

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Abstract

Given a graph $G = (V, E)$ with no isolated vertex, a subset S of V is called a total dominating set of G if every vertex in V has a neighbor in S. A total dominating set S is called a locating-total dominating set if for each pair of distinct vertices u and v in $V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The minimum cardinality of a locating-total dominating set of G is the locating-total domination number, denoted by $\gamma_t^L(G)$. We show that, for a tree T of order $n \geq 3$ and diameter d, $\frac{d+1}{2} \leq \gamma_t^L(T) \leq n - \frac{d-1}{2}$, and if T has l leaves, s support vertices and s_1 strong support vertices, then $\gamma_t^L(T) \geq \max\left\{\frac{n+l-s+1}{2} - \frac{s+s_1}{4}, \frac{2(n+1)+3(l-s)-s_1}{5}\right\}$. We also characterize the extremal trees achieving these bounds.

Keywords: tree, total dominating set, locating-total dominating set, locating-total domination number.

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1. Introduction

In [5, 8], the authors introduced the concept of a locating-total dominating set in a graph. Locating-total dominating set has been studied, for example, in [1, 2, 3, 4, 9] and elsewhere. The problem of placing monitoring devices in a system such that every site (including the monitors themselves) in the system is adjacent to a monitor can be modelled by total domination in graphs. Applications where it is also important that if there is a problem in a device, its location can be uniquely identified by the set of monitors, can be modelled by a combination of total dominating sets and locating sets in graphs. In this paper, we consider locating-total domination in trees.

For notation and graph theory terminology in general we follow [6, 7]. Let $G = (V, E)$ be a graph with n vertices. For a vertex v in G, the set $N(v) = \{u \in V : uv \in E\}$ is called the *open neighborhood* of v and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. The *degree* of v in G, denoted by $d(v)$, is equal to $|N(v)|$. A vertex of degree one is a *leaf* and the edge incident with a leaf is a *pendent edge*. A vertex adjacent to a leaf is a *support vertex* and a support vertex adjacent to at least two leaves is a *strong support vertex*. We will use $L(G)$, $S(G)$ and $S_1(G)$ to denote the set of leaves, support vertices and strong support vertices of G, respectively. The *distance* between two vertices u and v, denoted by $d(u, v)$, is the number of edges in a shortest path joining u and v. The *diameter* of G, denoted by diam(G), is the maximum distance over all pairs of vertices of G. For two disjoint subsets A and B of V, let $[A, B] = \{uv \in E(G) : u \in A, v \in B\}$. Suppose G and H are two disjoint graphs, then the *disjoint union* of G and H, denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G_1 \cong \cdots \cong G_k$, we simply write kG_1 for $G_1 + \cdots + G_k$.

For a subset $S \subseteq V$, let $G[S]$ be the subgraph induced by S. The *open neighborhood* of S is $N(S) = \bigcup_{v \in S} N(v)$ and the *closed neighborhood* of S is $N[S] = N(S) \cup S$. S is called a *total dominating set* (TDS) of G if $N(S) = V$. A TDS S is a *locating-total dominating set* (LTDS) if for each pair of distinct vertices u and v in $V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The minimum cardinality of an LTDS of G is the *locating-total domination number* of G, denoted by $\gamma_t^L(G)$. An LTDS of cardinality $\gamma_t^L(G)$ is called a *$\gamma_t^L(G)$-set*.

Let P_n and S_n be a path of order n and a star of order n, respectively. A *double star* $S_{p,q}$ is a tree obtained from S_{p+2} and S_{q+1} by identifying a leaf of S_{p+2} with the center of S_{q+1}, where $p, q \geq 1$.

Locating-total domination in trees has been studied in [2, 4, 8]. In this paper, we continue the study of it. We show that, for a tree T of order $n \geq 3$ and diameter d, $\frac{d+1}{2} \leq \gamma_t^L(T) \leq n - \frac{d-1}{2}$, and if T has l leaves, s support vertices and s_1 strong support vertices, then $\gamma_t^L(T) \geq \max\left\{\frac{n+l-s+1}{2} - \frac{s+s_1}{4}, \frac{2(n+1)+3(l-s)-s_1}{5}\right\}$. We also characterize the extremal trees achieving these bounds.

2. LOWER BOUNDS ON THE LOCATING-TOTAL DOMINATION NUMBER IN TREES

The locating-total domination number of P_n was given in [8].

Theorem 1 [8]**.** *For* $n \geq 2$, $\gamma_t^L(P_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.

In [9], a lower bound of $\gamma_t^L(G)$ involving diameter was given.

Theorem 2 [9]**.** *If G is a connected graph of order at least 2, then* $\gamma_t^L(G) \geq \frac{\text{diam(G)}+1}{2}$.

If G is a tree, we characterize all trees which achieve the lower bound.

Corollary 3. *Suppose T is a tree of order at least 2, then* $\gamma_t^L(T) \geq \frac{\text{diam(T)}+1}{2}$ *and the equality holds if and only if* $T = P_n$, *where* $n \equiv 0 \pmod 4$.

Proof. Let $d = \text{diam(T)}$. From Theorem 2, $\gamma_t^L(T) \geq \frac{d+1}{2}$. If $T = P_n$, where $n \equiv 0 \pmod 4$, then by Theorem 1, we have $\gamma_t^L(P_n) = \frac{n}{2} = \frac{d+1}{2}$.

Now assume T is a tree of order $n \geq 2$ and $\gamma_t^L(T) = \frac{d+1}{2}$. From the proof of Theorem 2, we have $d + 1 \equiv 0 \pmod 4$.

If $d = 3$, then $T = S_{a,b}$ for some $a, b \geq 1$. Since $\gamma_t^L(S_{a,b}) = n - 2$ and $\gamma_t^L(T) = \frac{d+1}{2} = 2$, we have $n = 4$ and $T = P_4$. Thus, we may assume $d \geq 7$.

Let D be a $\gamma_t^L(T)$-set of T that contains a minimum number of leaves. Then for every support vertex v, exactly one leaf adjacent to v is not in D. Suppose $x, y \in V(T)$ with $d(x, y) = d$ and $P = v_0v_1 \cdots v_d$ is the unique path joining x and y, where $v_0 = x$ and $v_d = y$. Then $d(x) = d(y) = 1$. For $i = 1, 2, \ldots, \frac{d+1}{4}$, let T_i be the component of $T \setminus \bigcup_{i=1}^{(d-3)/4} \{v_{4i-1}, v_{4i}\}$ containing the vertex v_{4i-1} and let $V(T_i) = D_i$. Then $|D \cap D_i| \geq 2$ because $\{v_{4i-3}, v_{4i-2}\} \subseteq N(D)$. Thus, $|D| \geq \frac{2(d+1)}{4} = \frac{d+1}{2}$. Since $|D| = \gamma_t^L(T) = \frac{d+1}{2}$, we obtain $|D_i \cap D| = 2$ for $i = 1, 2, \ldots, \frac{d+1}{4}$. Obviously, we have $v_1, v_{d-1} \in D$.

Fact 1. $d(v_1) = 2$.

Proof of Fact 1. Suppose $d(v_1) \geq 3$, then v_1 is a strong support vertex which is adjacent to exactly two leaves because $|D_1 \cap D| = 2$. Let z be the other leaf adjacent to v_1. Thus we may assume $D \cap D_1 = \{z, v_1\}$. Now, for $v_0, v_2 \notin D$, we have $N(v_0) \cap D = N(v_2) \cap D = \{v_1\}$, a contradiction. □

Fact 2. $D = \bigcup_{i=1}^{(d+1)/4} \{v_{4i-3}, v_{4i-2}\}$.

Proof of Fact 2. By Fact 1, we have $D \cap D_1 = \{v_1, v_2\}$ in order to totally dominate v_1.

Suppose $v_4 \in D$. Then $D \cap D_2 = \{v_4, v_5\}$ in order to totally dominate v_4 and v_6. Consequently, we have $D = \{v_1, v_2\} \cup (\bigcup_{i=2}^{(d+1)/4} \{v_{4i-4}, v_{4i-3}\})$, which induces $v_d \notin N(D)$, a contradiction. Thus, $v_4 \notin D$.

Suppose $v_5 \notin D$. In order to totally dominate v_4, there must be two vertices $z_1, z_2 \in (V(T) \setminus V(P)) \cap D$ with $z_1 \in N(v_4)$ and $z_2 \in N(z_1)$. Since $|D_2 \cap D| = 2$, we have $v_6 \notin N(D)$, a contradiction. Thus, we have $v_5 \in D$.

Suppose $v_6 \notin D$. In order to totally dominate v_5, there must be a vertex $z \in N(v_5) \cap D \setminus V(P)$. Then $D \cap D_2 = \{v_5, z\}$ and $N(v_4) \cap D = N(v_6) \cap D = \{v_5\}$, a contradiction. Thus, $v_6 \in D$ and $D_2 \cap D = \{v_5, v_6\}$.

By induction on i, we have $D \cap D_i = \{v_{4i-3}, v_{4i-2}\}$ for $i = 2, 3, \ldots, \frac{d+1}{4}$. Thus, $D = \bigcup_{i=1}^{(d+1)/4} \{v_{4i-3}, v_{4i-2}\}$. □

Fact 3. $V(T) = V(P)$.

Proof of Fact 3. Suppose $V(T) \setminus V(P) \neq \emptyset$. Since $D_i \cap D = \{v_{4i-3}, v_{4i-2}\}$ for $i = 1, 2, \ldots, \frac{d+1}{4}$, there are no vertices in $V(T) \setminus V(P)$ adjacent to v_{4i-1} or v_{4i} for $i = 1, 2, \ldots, \frac{d+1}{4}$.

Suppose there is $z \in V(T) \setminus V(P)$ with $zv \in E(T)$, where $v \in D_i \cap D$. Without loss of generality, we may assume that $z \in N(v_{4i-3})$ for some $i \in \{1, 2, \ldots, \frac{d+1}{4}\}$. Then $N(v_{4i-4}) \cap D = N(z) \cap D = \{v_{4i-3}\}$, a contradiction. □

Thus, $T = P = P_n$, where $n = d + 1 \equiv 0 \pmod 4$. ■

Let $\mathcal{F}$ be the family of trees obtained from t disjoint copies of P_4 and P_3 by first adding $t-1$ edges in such a way that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once. Let ξ be the family of trees T that can be obtained from any tree T' by first attaching at least two leaves to each vertex of T', and then subdividing each edge of T' exactly once if T' is nontrivial.

Theorem 4 [2]**.** *If T is a tree of order $n \geq 3$, $|L(T)| = l$ and $|S(T)| = s$, then*

$$\gamma_t^L(T) \geq \frac{2(n+l-s+1)}{5},$$

with equality if and only if $T \in \mathcal{F}$.

Theorem 5 [4]**.** *If T is a tree of order $n \geq 3$ with l leaves and s support vertices, then $\gamma_t^L(T) \geq \frac{n+l+1}{2} - s$ and the equality holds if and only if $T \in \xi$.*

In the following, we give two new lower bounds on the locating-total domination number in trees. We also characterize the trees achieving those lower bounds. First, we need the following lemma. Let $T = (V, E)$ be a tree of order $n \geq 3$. Let $L(T) = L$, $S(T) = S$, $S_1(T) = S_1$, $S \setminus S_1 = S_2$ and A be a $\gamma_t^L(T)$-set of

T that contains a minimum number of leaves. Then $S \subseteq A$ and for every $v \in S$, exactly one leaf adjacent to v is not in A. Let $B = \{v \notin A : |N(v) \cap A| = 1\}$ and $C = \{v \notin A : |N(v) \cap A| \geq 2\}$. Let $L_1 = L \cap A$, $Q_1 = A \setminus (L_1 \cup S)$, $L_2 = L \setminus L_1$ and $Q_2 = B \setminus L_2$. Then $A = L_1 \cup S \cup Q_1$, $B = L_2 \cup Q_2$, $V = A \cup B \cup C$. We have the following lemma.

Lemma 6. *Let $|L| = l$, $|S| = s$ and $|S_1| = s_1$. Then*
(1) $|[A, B \cup C]| \geq |B| + 2|C| = 2n - 2|A| - |B|$;
(2) $|[A, B \cup C]| = n - 1 - |E(T[A])| - |E(T[Q_2 \cup C])|$;
(3) $|L_1| = l - s$, $|L_2| = s$, $|Q_1| = |A| - l$, $|Q_2| = |B| - s$;
(4) $|Q_2| \leq |Q_1|$, $|B| \leq |A| - l + s$;
(5) $|E(T[Q_2 \cup C])| \geq \frac{|Q_2|}{2}$ *and the equality holds if and only if* $T[Q_2 \cup C] \cong \frac{|Q_2|}{2}K_2 + |C|K_1$ *and* C *is an independent set in* $T[Q_2 \cup C]$;
(6) $|E(T[S \cup Q_1])| \geq \frac{1}{2}(s - s_1 + |A| - l)$ *and the equality holds if and only if* $T[S \cup Q_1] \cong s_1K_1 + \frac{|S_2 \cup Q_1|}{2}K_2$ *and* S_1 *is an independent set in* $T[S \cup Q_1]$;
(7) $|E(T[A])| \geq \frac{|A|}{2}$ *and the equality holds if and only if* $T[A] \cong \frac{|A|}{2}K_2$.

Proof. (1)–(5) and (7) can be obtained by applying an argument similar to that of Lemma 3 we gave in [11] and can also be seen in [10].

(6) For every $v \in S_2 \cup Q_1$, $N(v) \cap (S \cup Q_1) \neq \emptyset$ by the definition of an LTDS. Thus,

$$|E(T[S \cup Q_1])| \geq \frac{1}{2} \sum_{v \in S_2 \cup Q_1} d_{T[S \cup Q_1]}(v) \geq \frac{1}{2}|S_2 \cup Q_1| = \frac{1}{2}(s - s_1 + |A| - l),$$

and the equality holds if and only if $T[S \cup Q_1] \cong s_1K_1 + \frac{|S_2 \cup Q_1|}{2}K_2$ and S_1 is an independent set in $T[S \cup Q_1]$. ∎

Let $\mathcal{T}_1$ denote the set $\{P_4\} \cup \{S_a : a \geq 3\}$. Let $\mathcal{F}_1$ be the family of trees obtained from r disjoint copies of trees in $\mathcal{T}_1$ by first adding $r - 1$ edges so that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 7. *Suppose T is a tree of order $n \geq 3$, $|L(T)| = l$, $|S(T)| = s$ and $|S_1(T)| = s_1$. Then*

$$\gamma_t^L(T) \geq \frac{2(n+1) + 3(l - s) - s_1}{5},$$

with equality if and only if $T \in \mathcal{F}_1$.

Proof. From Lemma 6(1) and (4), we obtain $|[A, B\cup C]| \geq 2n-3|A|+l-s$. By Lemma 6(2), (3) and (6), $|[A, B\cup C]| \leq n-1-|E(T[A])| = n-1-|L_1|-|E(T[S\cup Q_1])| \leq n-1-(l-s)-\frac{1}{2}(s-s_1+|A|-l)$. Thus $\gamma_t^L(T) = |A| \geq \frac{2(n+1)+3(l-s)-s_1}{5}$.

The equality $\gamma_t^L(T) = \frac{2(n+1)+3(l-s)-s_1}{5}$ holds if and only if $|E(T[Q_2\cup C])| = 0$, $|N(v)\cap A| = 2$ for every vertex $v \in C$, $|Q_1| = |Q_2|$, $T[S\cup Q_1] \cong s_1K_1 + \frac{|S_2\cup Q_1|}{2}K_2$ and S_1 is an independent set in $T[S\cup Q_1]$. The equality $|E(T[Q_2\cup C])| = 0$ implies $|Q_1| = |Q_2| = 0$ by Lemma 6(5). Thus, $A = L_1 \cup S$ and $T[S] \cong s_1K_1 + \frac{s-s_1}{2}K_2$. Consequently, every connected component of $T[A \cup B]$ is either a P_4, or a S_a, where $a \geq 3$. Thus, we have $T \in \mathcal{F}_1$. ■

Remark 8. The lower bound in Theorem 7 is no less than the lower bound in Theorem 4 because $\frac{2(n+1)+3(l-s)-s_1}{5} - \frac{2(n+l-s+1)}{5} = \frac{l-s-s_1}{5} \geq 0$. Note that we have the fact $\mathcal{F} \subset \mathcal{F}_1$.

Now let $\mathcal{T}_2$ denote the set $\{S_a : a \geq 3\} \cup \{P_b : b \geq 4 \text{ and } b \equiv 0 \pmod 4\}$. For every $T \in \mathcal{T}_2$, if $T = P_b = v_1v_2\cdots v_b$ for some $b \geq 4$ and $b \equiv 0 \pmod 4$, then we define $D_T = \bigcup_{i=1}^{b/4}\{v_{4i-2}, v_{4i-1}\}$; if $T = S_a$ for some $a \geq 3$, then we define $D_T = S(S_a)$. Let $\mathcal{F}_2$ be the family of trees obtained from r disjoint copies of trees in $\mathcal{T}_2$ by first adding $r-1$ edges so that they are only incident with vertices in $\bigcup_{T\in\mathcal{T}_2} D_T$ and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 9. *Suppose T is a tree of order $n \geq 3$, $|L(T)| = l$, $|S(T)| = s$ and $|S_1(T)| = s_1$. Then*

$$\gamma_t^L(T) \geq \frac{n+l-s+1}{2} - \frac{s+s_1}{4},$$

with equality if and only if $T \in \mathcal{F}_2$.

Proof. From Lemma 6(1), we obtain $|[A, B\cup C]| \geq 2n-2|A|-|B|$. On the other hand,

$$\begin{aligned}
|[A, B\cup C]| &= n-1-|E\left(T[A]\right)|-|E\left(T[Q_2\cup C]\right)| \quad \text{by Lemma 6(2)}\\
&\leq n-1-|E\left(T[A]\right)|-\frac{|Q_2|}{2} \quad \text{by Lemma 6(5)}\\
&= n-1-\frac{|B|-s}{2}-\left(|L_1|+|E\left(T[S\cup Q_1]\right)|\right) \quad \text{by Lemma 6(3)}\\
&\leq n-1-\frac{|B|-s}{2}-\left((l-s)+\frac{1}{2}(s-s_1+|A|-l)\right) \quad \text{by Lemma 6(6).}
\end{aligned}$$

Combining this with $|[A, B\cup C]| \geq 2n-2|A|-|B|$, we have

$$\frac{3}{2}|A| \geq n+1-\frac{|B|}{2}-s+\frac{l}{2}-\frac{s_1}{2}.$$

By Lemma 6(4), we have $2|A| \geq n + 1 + l - s - \frac{s+s_1}{2}$, which implies $\gamma_t^L(T) = |A| \geq \frac{n+l-s+1}{2} - \frac{s+s_1}{4}$.

The equality holds if and only if $|Q_1| = |Q_2|$, $T[Q_2 \cup C] \cong \frac{|Q_2|}{2}K_2 + |C|K_1$ and C is an independent set in $T[Q_2 \cup C]$, $T[S \cup Q_1] \cong s_1K_1 + \frac{|S_2 \cup Q_1|}{2}K_2$ and S_1 is an independent set in $T[S \cup Q_1]$, and $|N(u_2) \cap A| = 2$ for every $u_2 \in C$. For every $u_1 \in Q_2 \subseteq B$, $N(u_1) \cap A \subseteq Q_1$ by the definition of an LTDS and $|N(u_1) \cap Q_1| = 1$.

If $|Q_2| = 0$, then $T \in \mathcal{F}_1$ (by the same argument as that in the proof of Theorem 7) and therefore $T \in \mathcal{F}_2$ as $\mathcal{F}_1 \subset \mathcal{F}_2$.

Now we consider the case $|Q_1| = |Q_2| \neq 0$. Let $T_1, T_2, \ldots, T_{\omega_1}$ be the components of $T[Q_1 \cup Q_2 \cup S_2]$. Note that T is a tree. Then for $i = 1, 2, \ldots, \omega_1$, T_i is a path of order a_i with two leaves in S_2 and the other vertices in $Q_1 \cup Q_2$, where $a_i \equiv 2 \pmod 4$. Thus, every component of $T[A \cup B]$ is in $\mathcal{T}_2$. Suppose $X_1, X_2, \ldots, X_{\omega_2}$ are the components of $T[A \cup B]$. For every X_j, if $X_j = P_{b_j} = v_1v_2 \cdots v_{b_j}$ for some $b_j \geq 4$ and $b_j \equiv 0 \pmod 4$, then we define $D_{X_j} = \bigcup_{i=1}^{b_j/4} \{v_{4i-2}, v_{4i-1}\}$, but if $X_j = S_{a_j}$ for some $a_j \geq 3$, then we define $D_{X_j} = S(X_j)$. Thus we have $S \cup Q_1 = \bigcup_{j=1}^{\omega_2} D_{X_j}$. Note that for every vertex $u \in C$, $|N(u) \cap A| = |N(u) \cap (S \cup Q_1)| = 2$ and T is a tree. Thus, $T \in \mathcal{F}_2$. ■

Remark 10. The lower bound in Theorem 9 is not less than the lower bound in Theorem 5 because $\frac{n+l-s+1}{2} - \frac{s+s_1}{4} - (\frac{n+l+1}{2} - s) = \frac{s-s_1}{4} \geq 0$. We also have $\xi \subset \mathcal{F}_2$, where ξ is defined in Theorem 5. On the other hand, if $n > \frac{3s+2l-s_1-2}{2}$, the lower bound in Theorem 9 is better than the lower bound in Theorem 7.

3. Upper Bounds on the Locating-Total Domination Number in Trees

The next theorem gives an upper bound on $\gamma_t^L(T)$ of a tree of fixed order and diameter.

Theorem 11. *Suppose T is a tree of order $n \geq 3$ and diameter $d \geq 2$. Then $\gamma_t^L(T) \leq n - \frac{d-1}{2}$ and the equality holds if and only if $T = P_n$, where $n \equiv 2 \pmod 4$.*

Proof. We first use an induction on the order n of T to show that $\gamma_t^L(T) \leq n - \frac{d-1}{2}$. If $n = 3$, then $\gamma_t^L(T) = 2 < n - \frac{d-1}{2}$. Next we assume that every tree T' of order $3 \leq n' < n$ and diameter $d' \geq 2$ satisfies $\gamma_t^L(T') \leq n' - \frac{d'-1}{2}$. Let T be a tree of order $n > 3$ and diameter $d \geq 2$.

Let $P = v_0v_1v_2 \cdots v_d$ be a path of length d in T. If $T = P$, then $d = n-1$ and $\gamma_t^L(T) \leq n - \frac{d-1}{2}$ by Theorem 1. Now suppose $T \neq P$. Then there is a vertex v of P with $d(v) \geq 3$. Let u be a vertex of $T \setminus V(P)$ such that $d(u, v)$ is maximum. Then $u \in L(T)$. Let $N(u) = \{w\}$, $T' = T - u$ and D be a $\gamma_t^L(T')$-set of T'. Then

$n' = n-1$ and $d' = d$. By the inductive hypothesis, $\gamma_t^L(T') \le n' - \frac{d'-1}{2}$. If $w \ne v$, then $w \in L(T')$ and $D \cup \{w\}$ is an LTDS of T; if $w = v$ and $v \in D$, then $D \cup \{u\}$ is an LTDS of T; if $w = v$ and $v \notin D$, then $D \cup \{v\}$ is an LTDS of T. In each case, we can find an LTDS of T with no more than $\gamma_t^L(T') + 1$ elements. Thus, $\gamma_t^L(T) \le \gamma_t^L(T') + 1 \le n' - \frac{d'-1}{2} + 1 = n - \frac{d-1}{2}$. This completes the proof of $\gamma_t^L(T) \le n - \frac{d-1}{2}$.

By Theorem 1, if $T = P_n$, where $n \ge 4$ and $n \equiv 2 \pmod 4$, then $\gamma_t^L(T) = \frac{n+2}{2} = n - \frac{d-1}{2}$. Conversely, suppose T is a tree with $\gamma_t^L(T) = n - \frac{d-1}{2}$, then $d \ge 4$ and d is odd. In order to prove $T = P_n$, where $n \ge 4$ and $n \equiv 2 \pmod 4$, we proceed by induction on n. If $n \le 6$, then $T = P_6$. Assume every tree T' of order $6 \le n' < n$ and diameter $d' \ge 2$ with $\gamma_t^L(T') = n' - \frac{d'-1}{2}$ satisfies $T' = P_{n'}$, where $n' \ge 4$ and $n' \equiv 2 \pmod 4$.

If T has a strong support vertex v, let $T' = T - y$, where y is a leaf adjacent to v. Then $n' = n-1, d' = d$, $\gamma_t^L(T) \le \gamma_t^L(T')+1 \le n' - \frac{d'-1}{2} + 1 = n - \frac{d-1}{2}$. Since $\gamma_t^L(T) = n - \frac{d-1}{2}$, we have $\gamma_t^L(T') = n' - \frac{d'-1}{2}$. By induction, $T' = P_{n'}$, where $n' \ge 4$ and $n' \equiv 2 \pmod 4$. Suppose $T' = P_{n'} = v_1 v_2 \cdots v_{n'}$, where $v_2 = v$. Then $\{v_1, v_2\} \cup (\bigcup_{i=1}^{\lfloor n'/4 \rfloor} \{v_{4i}, v_{4i+1}\})$ is an $LTDS$ of T. Thus, $\gamma_t^L(T) \le 2 + 2 \cdot \lfloor \frac{n}{4} \rfloor = n' - \frac{d'-1}{2} < n - \frac{d-1}{2}$, a contradiction. Therefore, every support vertex in T is not strong.

Let $P = v_0 v_1 v_2 \cdots v_d$ be a path of length d in T. We root T at the vertex v_0. Then we have the following two facts.

Fact 1. $d(v_2) = 2$.

Proof of Fact 1. Suppose $d(v_2) \ge 3$. If v_2 has a child $b \ne v_1$ which is a support vertex, let $T' = T \setminus \{v_0, v_1\}$. Then $n' = n-2$ and $d' = d$. Let D' be a $\gamma_t^L(T')$-set of T' that contains a minimum number of leaves. Then $v_2, b \in D'$ and $D' \cup \{v_1\}$ is an LTDS of T. Thus, $\gamma_t^L(T) \le \gamma_t^L(T') + 1 \le n' - \frac{d'-1}{2} + 1 = n - 1 - \frac{d-1}{2} < n - \frac{d-1}{2}$, a contradiction. Therefore, every child of v_2 except v_1 is a leaf. Since T has no strong support vertices, $d(v_2) = 3$. Let c be a leaf adjacent to v_2 and $T' = T \setminus \{v_0, v_1, v_2, c\}$, then $n' = n - 4 \ge 3$ and $d - 3 \le d' \le d$. Let D' be a $\gamma_t^L(T')$-set of T', then $D' \cup \{v_1, v_2\}$ is an LTDS of T. Thus, $\gamma_t^L(T) \le \gamma_t^L(T')+2 \le n' - \frac{d'-1}{2} + 2 \le n' - \frac{d-4}{2} + 2 = n - \frac{d}{2} < n - \frac{d-1}{2}$, a contradiction. □

Fact 2. $d(v_3) = 2$.

Proof of Fact 2. Suppose $d(v_3) \ge 3$. Let $T' = T \setminus \{v_0, v_1, v_2\}$, then $n' = n-3$ and $d-2 \le d' \le d$. Let D' be a $\gamma_t^L(T')$-set of T', then $D' \cup \{v_1, v_2\}$ is an LTDS of T. Therefore, $\gamma_t^L(T) \le \gamma_t^L(T') + 2 \le n' - \frac{d'-1}{2} + 2 \le n - 1 - \frac{d-3}{2} = n - \frac{d-1}{2}$. Since $\gamma_t^L(T) = n - \frac{d-1}{2}$, we have $n' = n-3$, $d' = d-2$, $\gamma_t^L(T') = n - \frac{d'-1}{2}$ and v_3 is a support vertex in T. By induction, $T' = P_{n'}$, where $n' \ge 4$ and $n' \equiv 2$

(mod 4). Now the set $\{v_1, v_2, v_3\} \bigcup_{i=1}^{(d-3)/4} \{v_{4i+1}, v_{4i+2}\}$ is a $\gamma_t^L(T)$-set of T. Thus, $\gamma_t^L(T) = \gamma_t^L(T') + 1 = \frac{n+1}{2} < \frac{n+3}{2} = n - \frac{d-1}{2}$, a contradiction. □

Now let $T' = T \setminus \{v_0, v_1, v_2, v_3\}$. Then $n' = n - 4 \geq 3$ and $d - 4 \leq d' \leq d$. Let D' be a $\gamma_t^L(T')$-set of T', then $D' \cup \{v_1, v_2\}$ is an LTDS of T. Thus, $\gamma_t^L(T) \leq \gamma_t^L(T') + 2 \leq n' - \frac{d'-1}{2} + 2 \leq n - 2 - \frac{d-5}{2} = n - \frac{d-1}{2}$. Since $\gamma_t^L(T) = n - \frac{d-1}{2}$, we have $n' = n - 4$, $d' = d - 4$, $\gamma_t^L(T') = n' - \frac{d'-1}{2}$ and $d_T(v_4) = 2$. By induction, $T' = P_{n'} = P_{n-4}$, where $n' \geq 4$ and $n' \equiv 2 \pmod 4$. Therefore, $T = P_n$, where $n \equiv 2 \pmod 4$. ■

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