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ON TOTAL H-IRREGULARITY STRENGTH OF THE DISJOINT UNION OF GRAPHS

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Abstract

A simple graph G admits an H-covering if every edge in $E(G)$ belongs to at least to one subgraph of G isomorphic to a given graph H. For the subgraph $H \subseteq G$ under a total k-labeling we define the associated H-weight as the sum of labels of all vertices and edges belonging to H. The total k-labeling is called the *H-irregular total k-labeling* of a graph G admitting

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an H-covering if all subgraphs of G isomorphic to H have distinct weights. The *total H-irregularity strength* of a graph G is the smallest integer k such that G has an H-irregular total k-labeling.

In this paper, we estimate lower and upper bounds on the total H-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

Keywords: H-covering, H-irregular labeling, total H-irregularity strength, copies of graphs, union of graphs.

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1. Introduction

Consider a simple and finite graph G with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is $V(G) \cup E(G)$ then we call the labeling a *total labeling*. For a total k-labeling $\psi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ the associated total vertex-weight of a vertex x is

$$wt_\psi(x) = \psi(x) + \sum_{xy \in E(G)} \psi(xy)$$

and the associated total edge-weight of an edge xy is

$$wt_\psi(xy) = \psi(x) + \psi(xy) + \psi(y).$$

A total k-labeling ψ is defined to be an *edge irregular total k-labeling* of the graph G if for every two different edges xy and $x'y'$ of G there is $wt_\psi(xy) \neq wt_\psi(x'y')$ and to be a *vertex irregular total k-labeling* of G if for every two distinct vertices x and y of G there is $wt_\psi(x) \neq wt_\psi(y)$. This concept was given by Bača, Jendrol', Miller and Ryan in [8].

The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of the graph G, $\text{tes}(G)$. Analogously, we define the *total vertex irregularity strength* of G, $\text{tvs}(G)$, as the minimum k for which there exists a vertex irregular total k-labeling of G.

The following lower bound on the total edge irregularity strength of a graph G is given in [8].

$$\text{tes}(G) \geq \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}, \tag{1}$$

where $\Delta(G)$ is the maximum degree of G. This lower bound is tight for paths, cycles and complete bipartite graphs of the form $K_{1,n}$.

Ivančo and Jendrol' [12] posed a conjecture that for an arbitrary graph G different from K_5 with maximum degree $\Delta(G)$, $\text{tes}(G) = \max\{\lceil(|E(G)|+2)/3\rceil, \lceil(\Delta(G)+1)/2\rceil\}$. This conjecture has been verified for complete graphs and complete bipartite graphs in [13, 14], for the categorical product of two cycles and two paths in [2, 4], for generalized Petersen graphs in [11], for generalized prisms in [9], for the corona product of a path with certain graphs in [16] and for large dense graphs with $(|E(G)|+2)/3 \le (\Delta(G)+1)/2$ in [10].

The bounds for the total vertex irregularity strength are given in [8] as follows.

$$\left\lceil \tfrac{|V(G)|+\delta(G)}{\Delta(G)+1} \right\rceil \le \text{tvs}(G) \le |V(G)| + \Delta(G) - 2\delta(G) + 1, \tag{2}$$

where $\delta(G)$ is the minimum degree of G.

Przybyło in [17] proved that $\text{tvs}(G) < 32|V(G)|/\delta(G) + 8$ in general and $\text{tvs}(G) < 8|V(G)|/r + 3$ for r-regular graphs. This was then improved by Anholcer, Kalkowski and Przybyło [5] in the following way

$$\text{tvs}(G) \le 3\left\lceil \tfrac{|V(G)|}{\delta(G)} \right\rceil + 1 \le \tfrac{3|V(G)|}{\delta(G)} + 4. \tag{3}$$

Recently, Majerski and Przybyło [15] based on a random ordering of the vertices proved that if $\delta(G) \ge (|V(G)|)^{0.5} \ln |V(G)|$, then

$$\text{tvs}(G) \le \tfrac{(2+o(1))|V(G)|}{\delta(G)} + 4. \tag{4}$$

The exact values for the total vertex irregularity strength for circulant graphs and unicyclic graphs are determined in [1, 6] and [3], respectively.

An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. Then it is said that G admits an $(H_1, H_2, \dots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-*covering*.

Let G be a graph admitting an H-covering. For the subgraph $H \subseteq G$ under the total k-labeling ψ, we define the associated H-weight as

$$wt_\psi(H) = \sum_{v \in V(H)} \psi(v) + \sum_{e \in E(H)} \psi(e).$$

A total k-labeling ψ is called to be an *H-irregular total k-labeling* of the graph G if all subgraphs of G isomorphic to H have distinct weights. The *total H-irregularity strength* of a graph G, denoted $\text{ths}(G, H)$, is the smallest integer k such that G has an H-irregular total k-labeling. This definition was introduced by Ashraf, Bača, Lascsáková and Semaničová-Feňovčíková [7]. If H is isomorphic to K_2, then the K_2-irregular total k-labeling is isomorphic to the edge irregular total k-labeling and thus the total K_2-irregularity strength of a graph G is equivalent to the total edge irregularity strength; that is $\text{ths}(G, K_2) = \text{tes}(G)$.

The next theorem gives a lower bound for the total H-irregularity strength.

Theorem 1 [7]**.** *Let G be a graph admitting an H-covering given by t subgraphs isomorphic to H. Then*

$$\operatorname{ths}(G,H) \geq \left\lceil 1 + \tfrac{t-1}{|V(H)|+|E(H)|} \right\rceil .$$

If H is isomorphic to K_2 then from Theorem 1 the lower bound on the total edge irregularity strength given in (1) follows immediately.

The next theorem proves that the lower bound in Theorem 1 is tight.

Theorem 2 [7]**.** *Let r, s, $2 \leq s \leq r$, be positive integers. Then*

$$\operatorname{ths}(P_r, P_s) = \left\lceil \tfrac{s+r-1}{2s-1} \right\rceil .$$

In this paper, we estimate lower and upper bounds on the total H-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

2. Copies of Graphs

By the symbol mG we denote the disjoint union of m copies of a graph G. Immediately from Theorem 1 we obtain a lower bound for the H-irregularity strength of m copies of a graph G.

Corollary 3. *Let G be a graph admitting an H-covering given by t subgraphs isomorphic to H and let m be a positive integer. Then*

$$\operatorname{ths}(mG,H) \geq \left\lceil 1 + \tfrac{mt-1}{|V(H)|+|E(H)|} \right\rceil .$$

In the next theorem we give an upper bound for $\operatorname{ths}(mG,H)$.

Theorem 4. *Let G be a graph having an H-irregular total $\operatorname{ths}(G,H)$-labeling f. Let m be a positive integer. Then*

$$\operatorname{ths}(mG,H) \leq \operatorname{ths}(G,H) + (m-1) \left\lceil \tfrac{wt_f^{\max}(H) - wt_f^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil ,$$

where $wt_f^{\max}(H)$ and $wt_f^{\min}(H)$ are the largest and smallest weights of a subgraph H under a total $\operatorname{ths}(G,H)$-labeling f of G.

Proof. Let G be a graph that admits an H-covering given by t subgraphs isomorphic to H. We denote these subgraphs as $H^1, H^2, \dots, H^t$. Assume that f is an H-irregular total k-labeling of a graph G with $\operatorname{ths}(G,H) = k$. The smallest weight of a subgraph H under the total k-labeling f is denoted by the symbol $wt_f^{\min}(H)$. Evidently

$$wt_f^{\min}(H) \geq |V(H)| + |E(H)|. \tag{5}$$

Analogously, the largest weight of a subgraph H under the total k-labeling f is denoted by the symbol $wt_f^{\max}(H)$. It holds that

$$wt_f^{\max}(H) \geq wt_f^{\min}(H) + t - 1 \tag{6}$$

and

$$wt_f^{\max}(H) \leq (|V(H)| + |E(H)|)k. \tag{7}$$

Thus $f : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ and

$$\left\{wt_f(H^j) : j = 1, 2, \dots, t\right\} \subset \left\{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \dots, wt_f^{\max}(H)\right\}. \tag{8}$$

By the symbol x_i, $i = 1, 2, \dots, m$, we denote an element (a vertex or an edge) in the i^{th} copy of G, denoted by G_i, corresponding to the element x in G, i.e., $x \in V(G) \cup E(G)$. Analogously, let H_i^j, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, t$, be the subgraph in the i^{th} copy of G corresponding to the subgraph H^j in G.

Let us define the total labeling g of mG in the following way. For $i = 1, 2, \dots, m$ let

$$g(x_i) = f(x) + (i-1)\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$

Evidently, all the labels are at most

$$k + (m-1)\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$

For the weight of every subgraph H_i^j, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, t$, isomorphic to the graph H under the labeling g we have

$$\begin{aligned} wt_g(H_i^j) &= \sum_{v \in V(H_i^j)} g(v) + \sum_{e \in E(H_i^j)} g(e) \\ &= \sum_{v \in V(H^j)} \left(f(v) + (i-1)\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \\ &+ \sum_{e \in E(H^j)} \left(f(e) + (i-1)\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \end{aligned}$$

$$= \sum_{v\in V(H^j)} f(v) + \sum_{e\in E(H^j)} f(e) + |V(H)|(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$+ |E(H)|(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$= wt_f(H^j) + (|V(H)|+|E(H)|)(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil.$$

This means that in the given copy of G the H-weights are distinct.

According to (8) we get that the largest weight of a subgraph isomorphic to H under the total labeling g in the i^{th} copy of G, $i = 1, 2, \dots, m$, denoted by $wt_g^{\max}(H : H \subset G_i)$, is at most

$$wt_g^{\max}(H : H \subset G_i) \leq wt_f^{\max}(H) + (|V(H)|+|E(H)|)(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$

and the smallest weight of a subgraph isomorphic to H under the total labeling g in the $(i+1)^{\text{th}}$ copy of G, $i = 1, 2, \dots, m-1$, denoted by $wt_g^{\min}(H : H \subset G_{i+1})$, is at least

$$wt_g^{\min}(H : H \subset G_{i+1}) \geq wt_f^{\min}(H) + (|V(H)|+|E(H)|)i\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil.$$

After some manipulation we get

$$wt_g^{\min}(H : H \subset G_{i+1})$$
$$\geq wt_f^{min}(H) + (|V(H)|+|E(H)|)i\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$= wt_f^{\min}(H) + (|V(H)|+|E(H)|)(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$+ (|V(H)|+|E(H)|)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil.$$

As

$$\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil \geq \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}$$

we obtain

$$wt_g^{\min}(H : H \subset G_{i+1}) \geq wt_f^{\min}(H)$$
$$+ (|V(H)|+|E(H)|)(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$+ \big(wt_f^{\max}(H) - wt_f^{\min}(H) + 1\big)$$

$$= wt_f^{\max}(H) + (|V(H)| + |E(H)|)(i-1)\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil + 1$$
$$\geq wt_g^{\max}(H : H \subset G_i) + 1 > wt_g^{\max}(H : H \subset G_i).$$

Thus in all components the H-weights are distinct. This concludes the proof. ■

We obtain the following corollary.

Corollary 5. *Let G be a graph admitting an H-irregular total* $\operatorname{ths}(G,H)$*-labeling f. Let m be a positive integer. Then*

$$\operatorname{ths}(mG, H) \leq m \operatorname{ths}(G, H).$$

Proof. Let f be a $\operatorname{ths}(G,H)$-labeling of a graph G and let $\operatorname{ths}(G,H) = k$. As $wt_f^{\min}(H) \geq |V(H)| + |E(H)|$ and $wt_f^{\max}(H) \leq (|V(H)| + |E(H)|)k$ we get

$$\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \leq \left\lceil \frac{(|V(H)|+|E(H)|)k - (|V(H)|+|E(H)|)+1}{|V(H)|+|E(H)|} \right\rceil$$
$$= \left\lceil k - 1 + \frac{1}{|V(H)|+|E(H)|} \right\rceil = k.$$

Hence, by Theorem 4,

$$\operatorname{ths}(mG, H) \leq \operatorname{ths}(G,H) + (m-1)\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \leq k + (m-1)k = mk.$$

■

Let $\{H^1, H^2, \ldots, H^t\}$ be the set of all subgraphs of G isomorphic to H. Let f be an H-irregular total k-labeling of a graph G with $\operatorname{ths}(G,H) = k$ such that

$$\begin{aligned} &\left\{wt_f(H^j) : j = 1, 2, \ldots, t\right\} \\ &= \left\{wt_f^{\min}(H), wt_f^{\min}(H)+1, \ldots, wt_f^{\min}(H)+t-1\right\}. \end{aligned} \tag{9}$$

Evidently, if the fraction

$$\frac{wt_f^{\max}(H) - wt_f^{\min}(H)+1}{|V(H)|+|E(H)|} = \frac{t}{|V(H)|+|E(H)|}$$

is an integer then the weights of all H-weights in mG under the total labeling g of mG defined in the proof of Theorem 4 constitute the set

$$\left\{wt_f^{\min}(H), wt_f^{\min}(H)+1, \ldots, wt_f^{\min}(H)+mt-1\right\}.$$

In particular, this implies that the upper bound for $\operatorname{ths}(mG,H)$ given in Theorem 4 is tight if G is a graph that satisfies the conditions mentioned above.

Theorem 6. *Let G be a graph admitting an H-covering given by t subgraphs isomorphic to H. Let f be an H-irregular total* $\mathrm{ths}(G,H)$*-labeling of G such that*

$$\{wt_f(H^j) : j = 1, 2, \dots, t\} = \{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \dots, wt_f^{\min}(H) + t - 1\}.$$

If the fraction $\frac{t}{|V(H)|+|E(H)|}$ is an integer then

$$\mathrm{ths}(mG, H) \le \mathrm{ths}(G, H) + \tfrac{(m-1)t}{|V(H)|+|E(H)|}.$$

Moreover, if $\mathrm{ths}(G,H) = \left\lceil 1 + \frac{t}{|V(H)|+|E(H)|} \right\rceil = 1 + \frac{t}{|V(H)|+|E(H)|}$ *then*

$$\mathrm{ths}(mG, H) = \mathrm{ths}(G, H) + \tfrac{(m-1)t}{|V(H)|+|E(H)|} = 1 + \tfrac{mt}{|V(H)|+|E(H)|}.$$

Theorem 2 gives the exact value for the total P_s-irregularity strength for a path P_r. Moreover, the P_s-irregular total $(\lceil (s + r - 1)/(2s - 1)\rceil)$-labeling of P_r described in the proof of Theorem 2 in [7] has the property that the set of P_s-weights consists of t consecutive integers, where $t = r - s + 1$ is the number of all subgraphs in P_r isomorphic to P_s. As $|V(P_s)| = s$ and $|E(P_s)| = s - 1$ and if the number $(r - s + 1)/(2s - 1)$ is an integer then according to Theorem 6 we get that

$$\begin{aligned}
\mathrm{ths}(mP_r, P_s) &= \mathrm{ths}(P_r, P_s) + (m-1)\tfrac{r-s+1}{2s-1} = \left\lceil \tfrac{s+r-1}{2s-1} \right\rceil + (m-1)\tfrac{r-s+1}{2s-1} \\
&= \left\lceil \tfrac{r-s+1+2s-1-1}{2s-1} \right\rceil + (m-1)\tfrac{r-s+1}{2s-1} \\
&= \left\lceil \tfrac{r-s+1}{2s-1} + 1 - \tfrac{1}{2s-1} \right\rceil + (m-1)\tfrac{r-s+1}{2s-1} \\
&= \tfrac{r-s+1}{2s-1} + 1 + (m-1)\tfrac{r-s+1}{2s-1} = m\tfrac{r-s+1}{2s-1} + 1.
\end{aligned}$$

Thus we obtain the following result.

Corollary 7. *Let m, r, s, $m \ge 1$, $2 \le s \le r$, be positive integers. If $2s-1$ divides $r - s + 1$, then*

$$\mathrm{ths}(mP_r, P_s) = \tfrac{m(r-s+1)}{(2s-1)} + 1.$$

If H is isomorphic to K_2 then $\mathrm{ths}(G, K_2) = \mathrm{tes}(G)$. Immediately from Theorem 4 the next corollary follows.

Corollary 8. *Let m be a positive integer. Then*

$$\left\lceil \tfrac{m|E(G)|+2}{3} \right\rceil \le \mathrm{ths}(mG, K_2) = \mathrm{tes}(mG) \le \mathrm{tes}(G) + (m-1)\left\lceil \tfrac{wt_f^{\max} - wt_f^{\min} + 1}{3} \right\rceil,$$

where $wt_f^{\max}$ and $wt_f^{\min}$ are the largest and smallest edge weights under a total $\mathrm{tes}(G)$*-labeling f of G.*

3. Disjoint Union of Two Non-Isomorphic Graphs

In this section we will deal with the total H-irregularity strength of two graphs G_1 and G_2 admitting an H-covering. From Theorem 1 we immediately obtain

Corollary 9. *Let G_i, $i = 1, 2$, be a graph admitting an H-covering given by t_i subgraphs isomorphic to H. Then*

$$\text{ths}(G_1 \cup G_2, H) \geq \left\lceil 1 + \tfrac{t_1+t_2-1}{|V(H)|+|E(H)|} \right\rceil .$$

The next theorem gives an upper bound for $\text{ths}(G_1 \cup G_2, H)$.

Theorem 10. *Let G_i, $i = 1, 2$, be a graph having an H-irregular total* $\text{ths}(G_i, H)$*-labeling f_i. Then*

$$\begin{aligned}&\text{ths}(G_1 \cup G_2, H)\\ &\leq \min\left\{ \max\left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \left\lceil \tfrac{wt_{f_2}^{\max}(H) - wt_{f_1}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right\},\right.\\ &\qquad\left. \max\left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right\}\right\},\end{aligned}$$

where $wt_{f_i}^{\max}(H)$ and $wt_{f_i}^{\min}(H)$ are the largest and smallest weights of a subgraph H under a total $\text{ths}(G, H)$*-labeling f_i of G_i.*

Proof. Let G_i, $i = 1, 2$, be a graph that admits an H-covering given by t_i subgraphs isomorphic to H. We denote these subgraphs as $H_i^1, H_i^2, \ldots, H_i^{t_i}$. Assume that f_i is an H-irregular total k_i-labeling of a graph G_i with $\text{ths}(G_i, H) = k_i$. The smallest weight of a subgraph H under the total k_i-labeling f_i is denoted by the symbol $wt_{f_i}^{\min}(H)$. Evidently

$$wt_{f_i}^{\min}(H) \geq |V(H)| + |E(H)|. \tag{10}$$

Analogously, the largest weight of a subgraph H under the total k_i-labeling f_i is denoted by the symbol $wt_{f_i}^{\max}(H)$. It holds that

$$wt_{f_i}^{\max}(H) \geq wt_{f_i}^{\min}(H) + t_i - 1 \tag{11}$$

and

$$wt_{f_i}^{\max}(H) \leq (|V(H)| + |E(H)|)k_i. \tag{12}$$

Thus $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, k_i\}$ and

$$\left\{ wt_{f_i}(H_i^j) : j = 1, 2, \ldots, t_i \right\} \subset \left\{ wt_{f_i}^{\min}(H), wt_{f_i}^{\min}(H) + 1, \ldots, wt_{f_i}^{\max}(H) \right\}. \tag{13}$$

Let us define the total labeling g of $G_1 \cup G_2$ in the following way.

$$g(x) = f_1(x) \qquad \text{if } x \in V(G_1) \cup E(G_1),$$
$$g(x) = f_2(x) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \qquad \text{if } x \in V(G_2) \cup E(G_2).$$

Evidently, all the labels are not greater than

$$\max\left\{k_1, k_2 + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\}.$$

For the weight of the subgraph H_1^j, $j = 1, 2, \dots, t_1$, isomorphic to the graph H under the labeling g we get

$$wt_g(H_1^j) = \sum_{v \in V(H_1^j)} g(v) + \sum_{e \in E(H_1^j)} g(e) = \sum_{v \in V(H_1^j)} f_1(v) + \sum_{e \in E(H_1^j)} f_1(e) = wt_{f_1}\big(H_1^j\big).$$

For the weight of the subgraph H_2^j, $j = 1, 2, \dots, t_2$, isomorphic to the graph H under the labeling g we get

$$\begin{aligned}
wt_g(H_2^j) &= \sum_{v \in V(H_2^j)} g(v) + \sum_{e \in E(H_2^j)} g(e) \\
&= \sum_{v \in V(H_2^j)} \left(f_2(v) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \\
&+ \sum_{e \in E(H_2^j)} \left(f_2(e) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \\
&= \sum_{v \in V(H_2^j)} f_2(v) + \sum_{e \in E(H_2^j)} f_2(e) + |V(H)| \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \\
&+ |E(H)| \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \\
&= wt_{f_2}(H_2^j) + (|V(H)| + |E(H)|) \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil .
\end{aligned}$$

According to (13) we get that the largest weight of a subgraph H under the total labeling g in G_1, denoted by $wt_g^{\max}(H : H \subset G_1)$, is at most

$$wt_g^{\max}(H : H \subset G_1) = wt_{f_1}^{\max}(H)$$

and the smallest weight of a subgraph H under the total labeling g in G_2, denoted by $wt_g^{\min}(H : H \subset G_2)$, is at least

$$wt_g^{\min}(H : H \subset G_2) \geq wt_{f_2}^{\min}(H) + (|V(H)| + |E(H)|) \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil .$$

Note, that when writting H_i we only consider subgraphs of G_i isomorphic to H. As

$$\left\lceil \frac{wt_{f_1}^{\max}(H)-wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \geq \frac{wt_{f_1}^{\max}(H)-wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|}$$

we get

$$\begin{aligned} wt_g^{\min}(H : H \subset G_2) &\geq wt_{f_2}^{\min}(H) + (|V(H)| + |E(H)|)\frac{wt_{f_1}^{\max}(H)-wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \\ &\geq wt_{f_2}^{\min}(H) + \big(wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1\big) = wt_{f_1}^{\max}(H)+1 \\ &> wt_{f_1}^{\max}(H) = wt_g^{\max}(H : H \subset G_1). \end{aligned}$$

Thus all the H-weights under the labeling g in $G_1 \cup G_2$ are distinct.

Analogously we can define the total labeling h of $G_1 \cup G_2$ such that

$$\begin{aligned} h(x) &= f_2(x) && \text{if } x \in V(G_2) \cup E(G_2), \\ h(x) &= f_1(x) + \left\lceil \frac{wt_{f_2}^{\max}(H)-wt_{f_1}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil && \text{if } x \in V(G_1) \cup E(G_1). \end{aligned}$$

Using similar arguments we can also show that under the total labeling h the H-weights in $G_1 \cup G_2$ are distinct.

Thus g and h are H-irregular total labelings of G. Immediately from this fact we get

$$\begin{aligned} &\operatorname{ths}(G_1 \cup G_2, H) \\ &\leq \min\Bigg\{ \max\Bigg\{\operatorname{ths}(G_2, H), \operatorname{ths}(G_1, H) + \left\lceil \frac{wt_{f_2}^{\max}(H)-wt_{f_1}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \Bigg\}, \\ &\qquad \max\Bigg\{\operatorname{ths}(G_1, H), \operatorname{ths}(G_2, H) + \left\lceil \frac{wt_{f_1}^{\max}(H)-wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \Bigg\}\Bigg\}. \end{aligned}$$

■

Ramdani, Salman, Assiyatum, Semaničová-Feňovčíková and Bača [18] gave an upper bound for the total edge irregularity strength of the disjoint union of graphs by the following form.

Theorem 11 [18]**.** *The total edge irregularity strength of the disjoint union of graphs $G_1, G_2, \ldots, G_m$, $m \geq 2$, is*

$$\operatorname{tes}\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} \operatorname{tes}(G_i) - \left\lfloor \tfrac{m-1}{2} \right\rfloor.$$

If H is isomorphic to K_2 then from Theorem 10 it follows that

$$\begin{aligned}\mathrm{ths}(G_1 \cup G_2, K_2) &= \mathrm{tes}(G_1 \cup G_2)\\ &\le \min\left\{ \max\left\{\mathrm{tes}(G_2), \mathrm{tes}(G_1) + \left\lceil \tfrac{3\mathrm{tes}(G_2)-2}{3} \right\rceil \right\},\right.\\ &\qquad \left. \max\left\{\mathrm{tes}(G_1), \mathrm{tes}(G_2) + \left\lceil \tfrac{3\mathrm{tes}(G_1)-2}{3} \right\rceil \right\}\right\}\\ &= \mathrm{tes}(G_1) + \mathrm{tes}(G_2)\end{aligned}$$

which is equal to the result from Theorem 11.

4. Conclusion

In this paper, we have estimated lower and upper bounds for the total H-irregularity strength for the disjoint union of m copies of a graph. We have proved that if a graph G admits an H-irregular total $\mathrm{ths}(G, H)$-labeling f and m is a positive integer then

$$\mathrm{ths}(mG, H) \le \mathrm{ths}(G, H) + (m-1)\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil,$$

where $wt_f^{\max}(H)$ and $wt_f^{\min}(H)$ are the largest and smallest weights of a subgraph H under a total $\mathrm{ths}(G, H)$-labeling f of G. This upper bound is tight.

We have also proved an upper bound for the total H-irregularity strength for the disjoint union of two non-isomorphic graphs.

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