

## NICHE HYPERGRAPHS OF PRODUCTS OF DIGRAPHS

MARTIN SONNTAG

*Faculty of Mathematics and Computer Science*  
*TU Bergakademie Freiberg*  
*Prüferstraße 1, D-09596 Freiberg, Germany*  
**e-mail:** sonntag@tu-freiberg.de

AND

HANNS-MARTIN TEICHERT

*Institute of Mathematics*  
*University of Lübeck*  
*Ratzeburger Allee 160, D-23562 Lübeck, Germany*  
**e-mail:** teichert@math.uni-luebeck.de

### Abstract

If  $D = (V, A)$  is a digraph, its *niche hypergraph*  $N\mathcal{H}(D) = (V, \mathcal{E})$  has the edge set  $\mathcal{E} = \{e \subseteq V \mid |e| \geq 2 \wedge \exists v \in V : e = N_D^-(v) \vee e = N_D^+(v)\}$ . Niche hypergraphs generalize the well-known niche graphs and are closely related to competition hypergraphs as well as common enemy hypergraphs. For several products  $D_1 \circ D_2$  of digraphs  $D_1$  and  $D_2$ , we investigate the relations between the niche hypergraphs of the factors  $D_1$ ,  $D_2$  and the niche hypergraph of their product  $D_1 \circ D_2$ .

**Keywords:** niche hypergraph, product of digraphs, competition hypergraph.

**2010 Mathematics Subject Classification:** 05C65, 05C76, 05C20.

### 1. INTRODUCTION AND DEFINITIONS

All hypergraphs  $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ , graphs  $G = (V(G), E(G))$  and digraphs  $D = (V(D), A(D))$  considered in the following may have isolates but no multiple edges. Moreover, in digraphs loops are forbidden. With  $N_D^-(v)$ ,  $N_D^+(v)$ ,  $d_D^-(v)$  and  $d_D^+(v)$  we denote the in-neighborhood, the out-neighborhood, the in-degree

and the out-degree of  $v \in V(D)$ , respectively. In standard terminology we follow Bang-Jensen and Gutin [1].

In 1968, Cohen [3] introduced the *competition graph*  $C(D) = (V, E(C(D)))$  of a digraph  $D = (V, A)$  representing a food web of an ecosystem. Here the vertices correspond to the species and different vertices  $v_1, v_2$  are connected by an edge if and only if they compete for a common prey  $w$ , i.e.,

$$E(C(D)) = \{\{v_1, v_2\} \mid v_1 \neq v_2 \wedge \exists w \in V : v_1 \in N_D^-(w) \wedge v_2 \in N_D^-(w)\}.$$

Surveys of the large literature around competition graphs (and its variants) can be found in [5, 6, 11]; for (a selection of) recent results see [4, 7–10, 12–17, 21].

Meanwhile the following variants of  $C(D)$  have been investigated. The *common enemy graph*  $CE(D)$  (cf. [11]) with the edge set

$$E(CE(D)) = \{\{v_1, v_2\} \mid v_1 \neq v_2 \wedge \exists w \in V : v_1 \in N_D^+(w) \wedge v_2 \in N_D^+(w)\},$$

the *double competition graph* or *competition-common enemy graph*  $DC(D)$  with the edge set  $E(DC(D)) = E(C(D)) \cap E(CE(D))$  (cf. [18]), and the *niche graph*  $N(D)$  with  $E(N(D)) = E(C(D)) \cup E(CE(D))$  (cf. [2]).

In 2004, the concept of competition hypergraphs was introduced by Sonntag and Teichert [19]. The *competition hypergraph*  $\mathcal{CH}(D)$  of a digraph  $D = (V, A)$  has the vertex set  $V$  and the edge set

$$\mathcal{E}(\mathcal{CH}(D)) = \{e \subseteq V \mid |e| \geq 2 \wedge \exists v \in V : e = N_D^-(v)\}.$$

As a second hypergraph generalization, recently Park and Sano [16] defined the *double competition hypergraph*  $DC\mathcal{H}(D)$  of a digraph  $D = (V, A)$ , which has the vertex set  $V$  and the edge set

$$\mathcal{E}(DC\mathcal{H}(D)) = \{e \subseteq V \mid |e| \geq 2 \wedge \exists v_1, v_2 \in V : e = N_D^-(v_1) \cap N_D^+(v_2)\}.$$

Our paper [5] was a third step in this direction; there we considered the *niche hypergraph*  $N\mathcal{H}(D)$  of a digraph  $D = (V, A)$ , again with the vertex set  $V$  and the edge set

$$\mathcal{E}(N\mathcal{H}(D)) = \{e \subseteq V \mid |e| \geq 2 \wedge \exists v \in V : e = N_D^-(v) \vee e = N_D^+(v)\}.$$

Note that  $N\mathcal{H}(D) = N\mathcal{H}(\overleftarrow{D})$  holds for any digraph  $D$ , if  $\overleftarrow{D}$  denotes the digraph obtained from  $D$  by reversing all arcs.

In [5] we present results on several properties of niche hypergraphs and the so-called *niche number*  $\hat{n}$  of hypergraphs. In most of the investigations in [5] the *generating digraph*  $D$  of  $N\mathcal{H}(D)$  is assumed to be acyclic.

For technical reasons, we define another hypergraph generalization. The *common enemy hypergraph*  $CE\mathcal{H}(D)$  of a digraph  $D = (V, A)$  has the vertex set  $V$  and the edge set

$$\mathcal{E}(CE\mathcal{H}(D)) = \{e \subseteq V \mid |e| \geq 2 \wedge \exists v \in V : e = N_D^+(v)\}.$$

In the hypergraphs  $\mathcal{CH}(D)$ ,  $CE\mathcal{H}(D)$  and  $N\mathcal{H}(D)$  no loops are allowed. Therefore, by definition the in-neighborhoods and out-neighborhoods of cardinality 1 in the digraph  $D$  play no role in the corresponding hypergraphs. This loss of information proved to be disadvantageous in the investigation of competition hypergraphs of products of digraphs (cf. [20]). So, considering niche hypergraphs of products of digraphs, it seems to be consequent to allow loops in niche hypergraphs, too. Therefore, we define the *l-competition hypergraph*  $\mathcal{CH}^l(D)$ , the *l-common enemy hypergraph*  $CE\mathcal{H}^l(D)$  and the *l-niche hypergraph*  $N\mathcal{H}^l(D)$  (with loops) having the edge sets

$$\begin{aligned} \mathcal{E}(\mathcal{CH}^l(D)) &= \{e \subseteq V \mid \exists v \in V : e = N_D^-(v) \neq \emptyset\}, \\ \mathcal{E}(CE\mathcal{H}^l(D)) &= \{e \subseteq V \mid \exists v \in V : e = N_D^+(v) \neq \emptyset\} \quad \text{and} \\ \mathcal{E}(N\mathcal{H}^l(D)) &= \{e \subseteq V \mid \exists v \in V : e = N_D^-(v) \neq \emptyset \vee e = N_D^+(v) \neq \emptyset\} \\ &= \mathcal{E}(\mathcal{CH}^l(D)) \cup \mathcal{E}(CE\mathcal{H}^l(D)). \end{aligned}$$

For the sake of brevity, in the following we often use the term *(l)-competition hypergraph* (sometimes in connection with the notation  $\mathcal{CH}^{(l)}(D)$ ) for the competition hypergraph  $\mathcal{CH}(D)$  as well as for the l-competition hypergraph  $\mathcal{CH}^l(D)$ , analogously for *(l)-common enemy* and *(l)-niche hypergraphs* with the notations  $CE\mathcal{H}^{(l)}(D)$  and  $N\mathcal{H}^{(l)}(D)$ , respectively.

For five products  $D_1 \circ D_2$  (*Cartesian product*  $D_1 \times D_2$ , *Cartesian sum*  $D_1 + D_2$ , *normal product*  $D_1 * D_2$ , *lexicographic product*  $D_1 \cdot D_2$  and *disjunction*  $D_1 \vee D_2$ ) of digraphs  $D_1 = (V_1, A_1)$  and  $D_2 = (V_2, A_2)$  we investigate the construction of the *(l)-niche hypergraph*  $N\mathcal{H}^{(l)}(D_1 \circ D_2) = (V, \mathcal{E}_\circ^{(l)})$  from  $N\mathcal{H}^{(l)}(D_1) = (V_1, \mathcal{E}_1^{(l)})$ ,  $N\mathcal{H}^{(l)}(D_2) = (V_2, \mathcal{E}_2^{(l)})$  and vice versa.

The products considered here have always the vertex set  $V := V_1 \times V_2$ ; using the notation  $\tilde{A} := \{((a, b), (a', b')) \mid a, a' \in V_1 \wedge b, b' \in V_2\}$  their arc sets are defined as follows:

$$\begin{aligned} A(D_1 \times D_2) &:= \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \wedge (b, b') \in A_2\}, \\ A(D_1 + D_2) &:= \{((a, b), (a', b')) \in \tilde{A} \mid ((a, a') \in A_1 \wedge b = b') \vee (a = a' \wedge (b, b') \in A_2)\}, \\ A(D_1 * D_2) &:= A(D_1 \times D_2) \cup A(D_1 + D_2), \\ A(D_1 \cdot D_2) &:= \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \vee (a = a' \wedge (b, b') \in A_2)\}, \\ A(D_1 \vee D_2) &:= \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \vee (b, b') \in A_2\}. \end{aligned}$$

It follows immediately that  $A(D_1 + D_2) \subseteq A(D_1 * D_2) \subseteq A(D_1 \cdot D_2) \subseteq A(D_1 \vee D_2)$  and  $A(D_1 \times D_2) \subseteq A(D_1 * D_2)$ . Except the lexicographic product all these products are commutative in the sense that  $D_1 \circ D_2 \simeq D_2 \circ D_1$ , where  $\circ \in \{\times, +, *, \vee\}$ .

Usually we number the vertices of  $D_1$  and  $D_2$  such that  $V_1 = \{1, 2, \dots, r\}$ ,  $V_2 = \{1, 2, \dots, s\}$  and arrange the vertices of  $V = V_1 \times V_2$  according to the places of an  $(r, s)$ -matrix.

In analogy with the rows and the columns of the described  $(r, s)$ -matrix we call the set  $Z_i = \{(i, j) \mid j \in V_2\}$  ( $i \in V_1$ ) and the set  $S_j = \{(i, j) \mid i \in V_1\}$  ( $j \in V_2$ ) the  $i$ -th row and the  $j$ -th column of  $D_1 \circ D_2$ , respectively.

Then, for each  $\circ \in \{+, *, \cdot, \vee\}$ , the subdigraph  $\langle S_j \rangle_{D_1 \circ D_2}$  of  $D_1 \circ D_2$  induced by the vertices of a column  $S_j$  is isomorphic to  $D_1$ , and, analogously, the subdigraph  $\langle Z_i \rangle_{D_1 \circ D_2}$  of  $D_1 \circ D_2$  induced by the vertices of a row  $Z_i$  is isomorphic to  $D_2$ . Moreover, if an arc  $a \in A(D_1 \circ D_2)$  consists only of vertices of one row  $Z_i$  ( $i \in V_1$ ), we refer to  $a$  as a *horizontal arc*. Analogously, an arc  $a$  containing only vertices of one column  $S_j$  ( $j \in V_2$ ) is called a *vertical arc*.

Considering  $(l)$ -niche hypergraphs, the question arises, whether or not  $N\mathcal{H}^{(l)}(D_1 \circ D_2)$  can be obtained from  $N\mathcal{H}^{(l)}(D_1)$  and  $N\mathcal{H}^{(l)}(D_2)$  and vice versa.

As an instance for competition hypergraphs  $\mathcal{CH}^{(l)}$ , we cite two results from [20].

**Theorem 1** [20]. *The  $l$ -competition hypergraph  $\mathcal{CH}^{(l)}(D_1 \times D_2) = (V, \mathcal{E}_\times^l)$  of the Cartesian product can be obtained from the  $l$ -competition hypergraphs  $\mathcal{CH}^{(l)}(D_1) = (V_1, \mathcal{E}_1^l)$  and  $\mathcal{CH}^{(l)}(D_2) = (V_2, \mathcal{E}_2^l)$  of  $D_1$  and  $D_2 : \mathcal{E}_\times^l = \{e_1 \times e_2 \mid e_1 \in \mathcal{E}_1^l \wedge e_2 \in \mathcal{E}_2^l\}$ .*

**Theorem 2** [20]. *The  $l$ -competition hypergraph  $\mathcal{CH}^{(l)}(D_1 \vee D_2) = (V, \mathcal{E}_\vee^l)$  of the disjunction can be obtained from the  $l$ -competition hypergraphs  $\mathcal{CH}^{(l)}(D_1) = (V_1, \mathcal{E}_1^l)$  and  $\mathcal{CH}^{(l)}(D_2) = (V_2, \mathcal{E}_2^l)$  of  $D_1$  and  $D_2$ , if for each of the following conditions is known whether it is true or not:*

- (a)  $\exists v_2 \in V_2 : N_2^-(v_2) = \emptyset$  and (b)  $\exists v_1 \in V_1 : N_1^-(v_1) = \emptyset$ .

*In general,  $\mathcal{CH}^{(l)}(D_1 \vee D_2)$  cannot be obtained from  $\mathcal{CH}^{(l)}(D_1)$  and  $\mathcal{CH}^{(l)}(D_2)$  without the extra information on points (a) and (b).*

Note that in some cases under certain conditions  $D_1 \circ D_2$  and even  $D_1$  and  $D_2$  can be reconstructed from  $\mathcal{CH}^{(l)}(D_1 \circ D_2)$ . For niche hypergraphs such strong results are not expectable.

The main reason why the reconstruction of  $D_1$  and  $D_2$  from  $N\mathcal{H}^{(l)}(D_1 \circ D_2)$  is much more difficult is the following. In general, for any hyperedge  $e \in \mathcal{E}(N\mathcal{H}^{(l)}(D))$  it is not possible to see whether  $e$  is a set of predecessors  $e = N_D^-(v)$  or a set of successors  $e = N_D^+(v)$  of a certain vertex  $v \in V(D)$ .

It is interesting that, in general, for the same reason also the construction of  $N\mathcal{H}^{(l)}(D_1 \circ D_2)$  from  $N\mathcal{H}^{(l)}(D_1)$  and  $N\mathcal{H}^{(l)}(D_2)$  is impossible.

2. CONSTRUCTION OF  $N\mathcal{H}^{(l)}(D_1 \circ D_2)$  FROM  $N\mathcal{H}^{(l)}(D_1)$  AND  $N\mathcal{H}^{(l)}(D_2)$ 

The digraphs  $D = (V, A)$  and  $D' = (V, A')$  are  $(l)$ -niche equivalent if and only if  $D$  and  $D'$  have the same  $(l)$ -niche hypergraph, i.e.,  $N\mathcal{H}^{(l)}(D) = N\mathcal{H}^{(l)}(D')$ .

**Theorem 3.** *Let  $D_1 = (V_1, A_1)$  and  $D_2 = (V_2, A_2)$  be digraphs. In general, for  $\circ \in \{\times, +, *, \cdot, \vee\}$ , the niche hypergraph  $N\mathcal{H}(D_1 \circ D_2) = (V, \mathcal{E}_\circ)$  of  $D_1 \circ D_2$  cannot be obtained from the  $l$ -niche hypergraphs  $N\mathcal{H}^l(D_1) = (V_1, \mathcal{E}_1^l)$  and  $N\mathcal{H}^l(D_2) = (V_2, \mathcal{E}_2^l)$  of  $D_1$  and  $D_2$ .*

**Proof.** It suffices to present digraphs  $D_1 = (V_1, A_1)$ ,  $D'_1 = (V_1, A'_1)$ ,  $D_2 = (V_2, A_2)$  such that  $D_1$  and  $D'_1$  are  $l$ -niche equivalent, but the niche hypergraphs of  $D_1 \circ D_2$  and  $D'_1 \circ D_2$  are distinct, i.e.,  $N\mathcal{H}(D_1 \circ D_2) \neq N\mathcal{H}(D'_1 \circ D_2)$ .

So let us consider the following digraphs and their niche hypergraphs:

$D_1 = (V_1, A_1)$  with  $V_1 = \{1, 2, 3, 4, 5\}$  and  $A_1 = \{(1, 2), (3, 2), (4, 5), (2, 4)\}$ ,  
 $D'_1 = (V_1, A'_1)$  with  $A'_1 = \{(1, 2), (3, 2), (4, 5)\}$  and  
 $D_2 = (V_2, A_2)$  with  $V_2 = \{1, 2, 3\}$  and  $A_2 = \{(1, 3), (2, 3)\}$ .

Obviously,  $D_1$  and  $D'_1$  are  $l$ -niche equivalent, they have the  $l$ -niche hypergraph  $N\mathcal{H}^l(D_1) = N\mathcal{H}^l(D'_1) = (V_1, \mathcal{E}_1^l)$ , where  $\mathcal{E}_1^l = \{\{1, 3\}, \{2\}, \{4\}, \{5\}\}$ .

In detail, looking at  $D_1$  we have

$$\mathcal{E}_1^l = \mathcal{E}(N\mathcal{H}^l(D_1)) = \{\{1, 3\} = N_{D_1}^-(2), \{2\} = N_{D_1}^-(4) = N_{D_1}^+(1) = N_{D_1}^+(3), \\ \{4\} = N_{D_1}^-(5) = N_{D_1}^+(2), \{5\} = N_{D_1}^+(4)\};$$

regarding  $D'_1$  we get

$$\mathcal{E}_1^l = \mathcal{E}(N\mathcal{H}^l(D'_1)) = \{\{1, 3\} = N_{D'_1}^-(2), \{2\} = N_{D'_1}^+(1) = N_{D'_1}^+(3), \{4\} = N_{D'_1}^-(5), \\ \{5\} = N_{D'_1}^+(4)\}.$$

Note that  $D_1$  and  $D'_1$  — despite having one and the same  $l$ -niche hypergraph — are significantly different in the sense that  $D'_1 \neq \overleftarrow{D}_1$ ,  $D_1 \not\cong D'_1$ , and, moreover,  $D_1$  is connected but  $D'_1$  consists of two components. Of course, using  $D_1$  and  $\overleftarrow{D}_1$  instead of  $D_1$  and  $D'_1$  could be an alternative approach for proving Theorem 3.

For the sake of completeness, we give the  $l$ -niche hypergraph  $N\mathcal{H}^l(D_2) = (V_2, \mathcal{E}_2^l)$ , with  $\mathcal{E}_2^l = \{\{1, 2\} = N_{D_2}^-(3), \{3\} = N_{D_2}^+(1) = N_{D_2}^+(2)\}$ .

Now we compare the niche hypergraphs of the products  $D_1 \circ D_2$  and  $D'_1 \circ D_2$ .

- Cartesian product  $D_1^{(\prime)} \times D_2$ .

Since the Cartesian product has not so many arcs and, consequently, its niche hypergraph  $N\mathcal{H}(D_1^{(\prime)} \times D_2)$  includes only few hyperedges, we present the whole edge sets  $\mathcal{E}(N\mathcal{H}(D_1^{(\prime)} \times D_2))$  here (in case of the other four products the edge sets of  $N\mathcal{H}(D_1^{(\prime)} \circ D_2)$  will be considerably larger, hence in these cases we will give up on writing down these sets completely).

$$\begin{aligned}\mathcal{E}(\mathcal{NH}(D_1 \times D_2)) &= \{(1, 1), (1, 2), (3, 1), (3, 2)\} = N_{D_1 \times D_2}^-((2, 3)), \\ &\quad \{(2, 1), (2, 2)\} = N_{D_1 \times D_2}^-((4, 3)), \\ &\quad \{(4, 1), (4, 2)\} = N_{D_1 \times D_2}^-((5, 3))\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}(\mathcal{NH}(D'_1 \times D_2)) &= \{(1, 1), (1, 2), (3, 1), (3, 2)\} = N_{D'_1 \times D_2}^-((2, 3)), \\ &\quad \{(4, 1), (4, 2)\} = N_{D'_1 \times D_2}^-((5, 3))\}.\end{aligned}$$

- *Cartesian sum*  $D_1^{(\prime)} + D_2$ , *normal product*  $D_1^{(\prime)} * D_2$  and *lexicographic product*  $D_1^{(\prime)} \cdot D_2$ .

Since  $D_1$  is connected, the Cartesian sum  $D_1 + D_2$ , the normal product  $D_1 * D_2$  as well as the lexicographic product  $D_1 \cdot D_2$  are connected, too. Considering the (disconnected) digraph  $D'_1$ , obviously  $D'_1 + D_2$ ,  $D'_1 * D_2$  and  $D'_1 \cdot D_2$  are disconnected. In detail, each of the products  $D'_1 \circ D_2$  ( $\circ \in \{+, *, \cdot\}$ ) consists of the two components  $\langle Z_1 \cup Z_2 \cup Z_3 \rangle_{D'_1 \circ D_2}$  and  $\langle Z_4 \cup Z_5 \rangle_{D'_1 \circ D_2}$ .

Therefore, in the niche hypergraph  $\mathcal{NH}(D'_1 \circ D_2)$  hyperedges containing vertices of both components cannot exist:

$$\forall e \in \mathcal{E}(\mathcal{NH}(D'_1 \circ D_2)) : e \cap (Z_1 \cup Z_2 \cup Z_3) = \emptyset \vee e \cap (Z_4 \cup Z_5) = \emptyset.$$

Consequently, to show  $\mathcal{NH}(D_1 \circ D_2) \neq \mathcal{NH}(D'_1 \circ D_2)$ , it suffices to find a hyperedge  $e \in \mathcal{E}(\mathcal{NH}(D_1 \circ D_2))$  such that both  $e \cap (Z_1 \cup Z_2 \cup Z_3)$  and  $e \cap (Z_4 \cup Z_5)$  are nonempty.

For each of the three products  $D_1 \circ D_2$  we will obtain such a hyperedge by considering the set of the predecessors of the vertex  $(4, 3) \in V(D_1 \circ D_2)$ , i.e.,  $e = N_{D_1 \circ D_2}^-((4, 3))$ . Clearly,  $e$  results from  $N_{D_1}^-(4) = \{2\}$  and  $N_{D_2}^-(3) = \{1, 2\}$ .

For the Cartesian sum  $D_1 + D_2$ , we have

$$e = \{(2, 3), (4, 1), (4, 2)\} = N_{D_1 + D_2}^-((4, 3)).$$

In case of the normal product  $D_1 * D_2$ , we obtain

$$e = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2)\} = N_{D_1 * D_2}^-((4, 3)).$$

It is easy to see that in the lexicographic product  $D_1 \cdot D_2$  the vertex  $(4, 3)$  has the same predecessors as in the normal product, hence

$$e = N_{D_1 \cdot D_2}^-((4, 3)) = N_{D_1 * D_2}^-((4, 3)) = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2)\}.$$

- *Disjunction*  $D_1^{(\prime)} \vee D_2$ .

Now both  $D_1 \vee D_2$  and  $D'_1 \vee D_2$  are connected. Nevertheless, as in the previous cases, we consider the predecessors of the vertex  $(4, 3)$  and get the hyperedge

$$\begin{aligned}e &= N_{D_1 \vee D_2}^-((4, 3)) \\ &= \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)\} \\ &= S_1 \cup S_2 \cup \{(2, 3)\} = S_1 \cup S_2 \cup Z_2 \in \mathcal{E}(\mathcal{NH}(D_1 \vee D_2)).\end{aligned}$$

Note that  $S_1 \cup S_2$  in  $e$  result from  $N_{D_2}^-(3) = \{1, 2\}$  and  $Z_2$  from  $N_{D_1}^-(4) = \{2\}$ .

We search for this hyperedge  $e$  in  $N\mathcal{H}(D'_1 \vee D_2)$ .

Assume  $e = N_{D'_1 \vee D_2}^+((i, j))$  or  $e = N_{D'_1 \vee D_2}^-((i, j))$ . Since  $D'_1$  and  $D_2$  are loopless digraphs, we obtain  $(i, j) \notin e$  and  $(i, j) \in \{(1, 3), (3, 3), (4, 3), (5, 3)\}$ , i.e.,  $j = 3$ .

Let  $e = N_{D'_1 \vee D_2}^+((i, 3))$ . Because of  $N_{D_2}^+(3) = \emptyset$  and  $S_1 \subseteq e$ , all vertices of  $S_1$  have to be successors of  $(i, 3)$  in  $D'_1 \vee D_2$  and  $\{1, 2, \dots, 5\} = N_{D'_1}^+(i)$ , where  $i \in \{1, 2, \dots, 5\}$ . This contradicts the fact that  $D'_1$  is loopless.

Consequently,  $e = N_{D'_1 \vee D_2}^-((i, 3))$ . Then,  $S_1 \cup S_2 \subseteq e$  holds trivially. Owing to  $(2, 3) \in e$  we get  $(2, 3) \in N_{D'_1 \vee D_2}^-((i, 3))$ , i.e.,  $2 \in N_{D'_1}^-(i)$  with  $i \in \{1, 2, \dots, 5\}$ . This contradicts  $N_{D'_1}^+(2) = \emptyset$ .

Hence,  $e \notin \mathcal{E}(N\mathcal{H}(D'_1 \vee D_2))$ , thus  $D_1 \vee D_2$  and  $D'_1 \vee D_2$  are not niche equivalent. Therefore, the niche hypergraph of the disjunction  $D_1 \vee D_2$  cannot be constructed from the niche hypergraphs of  $D_1$  and  $D_2$  in general. ■

Using Theorems 1 and 2, for the Cartesian product and the disjunction some positive construction results can be derived. For this end we have to make use of  $\mathcal{E}(N\mathcal{H}^l(D)) = \mathcal{E}(C\mathcal{H}^l(D)) \cup \mathcal{E}(CE\mathcal{H}^l(D))$  and  $CE\mathcal{H}^l(D) = C\mathcal{H}^l(\overleftarrow{D})$ .

**Remark 4.** The  $l$ -niche hypergraph  $N\mathcal{H}^l(D_1 \times D_2)$  of the Cartesian product can be obtained from the  $l$ -competition hypergraphs  $C\mathcal{H}^l(D_1)$ ,  $C\mathcal{H}^l(D_2)$  and the  $l$ -common enemy hypergraphs  $CE\mathcal{H}^l(D_1)$ ,  $CE\mathcal{H}^l(D_2)$ :

$$\begin{aligned} \mathcal{E}(N\mathcal{H}^l(D_1 \times D_2)) &= \mathcal{E}(C\mathcal{H}^l(D_1 \times D_2)) \cup \mathcal{E}(CE\mathcal{H}^l(D_1 \times D_2)) \\ &= \{e_1 \times e_2 \mid e_1 \in \mathcal{E}(C\mathcal{H}^l(D_1)) \wedge e_2 \in \mathcal{E}(C\mathcal{H}^l(D_2))\} \\ &\quad \cup \{e_1 \times e_2 \mid e_1 \in \mathcal{E}(CE\mathcal{H}^l(D_1)) \wedge e_2 \in \mathcal{E}(CE\mathcal{H}^l(D_2))\}. \end{aligned}$$

**Remark 5.** The  $l$ -niche hypergraph  $N\mathcal{H}^l(D_1 \vee D_2)$  of the disjunction can be obtained from the  $l$ -competition hypergraphs  $C\mathcal{H}^l(D_1)$ ,  $C\mathcal{H}^l(D_2)$  and the  $l$ -common enemy hypergraphs  $CE\mathcal{H}^l(D_1)$ ,  $CE\mathcal{H}^l(D_2)$  provided that each of the following conditions is known to be true or false:

- (a)  $\exists v_2 \in V_2 : N_{D_2}^-(v_2) = \emptyset$  and (b)  $\exists v_1 \in V_1 : N_{D_1}^-(v_1) = \emptyset$  and
- (c)  $\exists v_2 \in V_2 : N_{D_2}^+(v_2) = \emptyset$  and (d)  $\exists v_1 \in V_1 : N_{D_1}^+(v_1) = \emptyset$ .

In general,  $N\mathcal{H}^l(D_1 \vee D_2)$  cannot be obtained from  $C\mathcal{H}^l(D_1)$ ,  $C\mathcal{H}^l(D_2)$ ,  $CE\mathcal{H}^l(D_1)$  and  $CE\mathcal{H}^l(D_2)$  without the extra information on points (a)–(d).

### 3. RECONSTRUCTION OF $N\mathcal{H}^l(D_1)$ AND $N\mathcal{H}^l(D_2)$ FROM $N\mathcal{H}^l(D_1 \circ D_2)$

In the following, for a set  $e = \{\{i_1, j_1\}, \dots, \{i_k, j_k\}\} \subseteq V_1 \times V_2$  we define  $\pi_1(e) :=$

$\{i_1, \dots, i_k\}$  and  $\pi_2(e) := \{j_1, \dots, j_k\}$ , respectively, i.e.,  $\pi_i$  denotes the projection of vertices of  $N\mathcal{H}^{(l)}(D_1 \circ D_2)$  onto their  $i$ -th components, for  $i \in \{1, 2\}$ .

**Theorem 6** (Cartesian product  $D_1 \times D_2$ ).

- (a) If  $\mathcal{E}(N\mathcal{H}(D_1 \times D_2)) \neq \emptyset$ , then  $N\mathcal{H}(D_1)$  and  $N\mathcal{H}(D_2)$  can be obtained from  $N\mathcal{H}(D_1 \times D_2)$ .
- (b) If  $\mathcal{E}(N\mathcal{H}^l(D_1 \times D_2)) \neq \emptyset$ , then  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$  can be obtained from  $N\mathcal{H}^l(D_1 \times D_2)$ .

**Proof.** Note that  $\mathcal{E}(N\mathcal{H}(D_1 \times D_2)) \neq \emptyset$  implies  $A_1 \neq \emptyset \neq A_2$  and  $\max(|A_1|, |A_2|) \geq 2$ . Moreover,  $\mathcal{E}(N\mathcal{H}^l(D_1 \times D_2)) \neq \emptyset$  is equivalent to  $A_1 \neq \emptyset \neq A_2$  and, consequently, to  $\mathcal{E}(N\mathcal{H}^l(D_1)) \neq \emptyset \neq \mathcal{E}(N\mathcal{H}^l(D_2))$ .

(b) Let  $e \in \mathcal{E}(N\mathcal{H}^l(D_1 \times D_2))$ . This is equivalent to  $e \in \mathcal{E}(C\mathcal{H}^l(D_1 \times D_2))$  or  $e \in \mathcal{E}(CE\mathcal{H}^l(D_1 \times D_2))$ , i.e.,  $e = N_{D_1 \times D_2}^-(i, j)$  or  $e = N_{D_1 \times D_2}^+(i, j)$ , with a certain  $(i, j) \in V_1 \times V_2$ .

This holds if and only if there is a vertex  $(i, j) \in V_1 \times V_2$  such that

$$\pi_1(e) = N_{D_1}^-(i) \text{ and } \pi_2(e) = N_{D_2}^-(j) \text{ or } \pi_1(e) = N_{D_1}^+(i) \text{ and } \pi_2(e) = N_{D_2}^+(j),$$

which implies  $\pi_1(e) \in \mathcal{E}(N\mathcal{H}^l(D_1))$  and  $\pi_2(e) \in \mathcal{E}(N\mathcal{H}^l(D_2))$ .

Clearly, this way we can get all hyperedges  $e_1 \in \mathcal{E}(N\mathcal{H}^l(D_1))$  and  $e_2 \in \mathcal{E}(N\mathcal{H}^l(D_2))$ .

(a) An analog argumentation holds if we consider the niche hypergraphs  $N\mathcal{H}$  instead of the  $l$ -niche hypergraphs  $N\mathcal{H}^l$ , since hyperedges  $e \in \mathcal{E}(N\mathcal{H}^l(D_1 \times D_2))$  of cardinality 1 can be omitted if we are interested only in hyperedges  $e_i \in \mathcal{E}(N\mathcal{H}(D_i))$  (which have cardinality greater than 1), for  $i = 1, 2$ . ■

**Theorem 7** (Cartesian sum  $D_1 + D_2$ ).

- (a)  $N\mathcal{H}(D_1)$  and  $N\mathcal{H}(D_2)$  can be obtained from  $N\mathcal{H}(D_1 + D_2)$ .
- (b)  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$  can be obtained from  $N\mathcal{H}^l(D_1 + D_2)$ , provided that one of the following conditions is true:
  - (1)  $\mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) = \emptyset$ ;
  - (2)  $\forall e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : |\pi_1(e)| = 1$  and  $\exists e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : |\pi_2(e)| \geq 2$ ;
  - (3)  $\forall e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : |\pi_2(e)| = 1$  and  $\exists e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : |\pi_1(e)| \geq 2$ ;
  - (4)  $\exists (i, j) \in V_1 \times V_2 \forall e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : (i, j) \notin e$ .

**Proof.** (a) Let  $e \in \mathcal{E}(N\mathcal{H}(D_1 + D_2))$  and  $(i, j) \in V_1 \times V_2$  with  $e = N_{D_1 + D_2}^-(i, j)$  or  $e = N_{D_1 + D_2}^+(i, j)$ . Then  $e = \{(i, j_1), \dots, (i, j_k), (i_1, j), \dots, (i_l, j)\}$ , where  $i, i_1, \dots, i_l$  and  $j, j_1, \dots, j_k$  are pairwise distinct vertices in  $V_1$  and  $V_2$ , respectively.



To construct  $\mathcal{E}(N\mathcal{H}(D_1))$ , we need only those hyperedges  $e \in \mathcal{E}(N\mathcal{H}(D_1 + D_2))$  which contain  $l \geq 2$  vertices with one and the same second component:

$$\mathcal{E}(N\mathcal{H}(D_1)) = \left\{ \begin{array}{l} \pi_1(e) \setminus I \mid e \in \mathcal{E}(N\mathcal{H}(D_1 + D_2)) \wedge \\ e = \{(i, j_1), \dots, (i, j_k), (i_1, j), \dots, (i_l, j)\} \wedge l \geq 2 \wedge \\ I = \left\{ \begin{array}{l} \{i\}, \quad k \geq 1 \\ \emptyset, \quad k = 0 \end{array} \right\} \end{array} \right\}.$$

Analogously, we obtain  $\mathcal{E}(N\mathcal{H}(D_2))$ :

$$\mathcal{E}(N\mathcal{H}(D_2)) = \left\{ \begin{array}{l} \pi_2(e) \setminus J \mid e \in \mathcal{E}(N\mathcal{H}(D_1 + D_2)) \wedge \\ e = \{(i, j_1), \dots, (i, j_k), (i_1, j), \dots, (i_l, j)\} \wedge k \geq 2 \wedge \\ J = \left\{ \begin{array}{l} \{j\}, \quad l \geq 1 \\ \emptyset, \quad l = 0 \end{array} \right\} \end{array} \right\}.$$

(b) The proof of (1)–(3) is similar to the proof of (1)–(3) of Proposition 2 in [20].

*Case (1):*  $\mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) = \emptyset$ . Obviously,  $A(D_1 + D_2) = \emptyset = A(D_1) = A(D_2) = \mathcal{E}(N\mathcal{H}^l(D_1)) = \mathcal{E}(N\mathcal{H}^l(D_2))$ .

*Case (2):*  $\forall e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : |\pi_1(e)| = 1$  and  $\exists e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : |\pi_2(e)| \geq 2$ .

Let  $e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2))$  with  $|\pi_2(e)| \geq 2$ , i.e.,  $e = \{(i, j_1), \dots, (i, j_k)\} = N_{D_1+D_2}^-(i, j)$  or  $e = \{(i, j_1), \dots, (i, j_k)\} = N_{D_1+D_2}^+(i, j)$  with  $k \geq 2$  and suitable  $i \in V_1$ ,  $j \in V_2$  and  $j_1, \dots, j_k \in V_2$ .

We discuss only the situation  $e = N_{D_1+D_2}^-(i, j)$ , since  $e = N_{D_1+D_2}^+(i, j)$  can be proved analogously.

Clearly,  $N_{D_2}^-(j) = \{j_1, \dots, j_k\} = \pi_2(e)$ . The assumption that there are  $i' \in V_1$ ,  $l \geq 1$  and  $i'_1, \dots, i'_l \in V_1$  with  $N_{D_1}^-(i') = \{i'_1, \dots, i'_l\} \neq \emptyset$  would lead to  $e' = N_{D_1+D_2}^-(i', j) = \{(i'_1, j), \dots, (i'_l, j), (i', j_1), \dots, (i', j_k)\}$  with  $|\pi_1(e')| \geq 2$ , a contradiction.

Therefore,  $\mathcal{E}(N\mathcal{H}^l(D_1)) = \emptyset$  and  $\mathcal{E}(N\mathcal{H}^l(D_2)) = \{\pi_2(e) \mid e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2))\}$ .

*Case (3):*  $\forall e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : |\pi_2(e)| = 1$  and  $\exists e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : |\pi_1(e)| \geq 2$ .

This can be treated in the same way as Case (2).

*Case (4):*  $\exists (i, j) \in V_1 \times V_2 \forall e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : (i, j) \notin e$ . Since for every  $e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2))$  we have  $(i, j) \notin e$ , the vertex  $i \in V_1$  is an isolate in

$N\mathcal{H}^l(D_1)$  and in  $D_1$ . For the same reason,  $j \in V_2$  is an isolate in  $N\mathcal{H}^l(D_2)$  and in  $D_2$ . We discuss only the construction of  $N\mathcal{H}^l(D_2)$ , the rest follows analogously.

Since  $i$  is an isolate, in  $D_1 + D_2$  there is no arc between the  $i$ -th row  $Z_i$  and any other row. Therefore, all arcs with an initial or a terminal vertex in  $Z_i$  result from arcs in  $D_2$  and we have

$$\forall a \in A(D_1 + D_2) : V(a) \cap Z_i \neq \emptyset \Rightarrow V(a) \subseteq Z_i.$$

Hence, denoting by  $\langle Z_i \rangle_{D_1+D_2}$  and by  $\langle Z_i \rangle_{N\mathcal{H}^l(D_1+D_2)}$  the subdigraph of  $D_1 + D_2$  and the subhypergraph of  $N\mathcal{H}^l(D_1 + D_2)$  generated by the vertices of  $Z_i$ , respectively, we obtain

- $\langle Z_i \rangle_{D_1+D_2} \simeq D_2$ ,
- $\langle Z_i \rangle_{N\mathcal{H}^l(D_1+D_2)} \simeq N\mathcal{H}^l(D_2)$  and
- $\mathcal{E}(N\mathcal{H}^l(D_2)) = \{\pi_2(e) \mid e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) \wedge e \subseteq Z_i\}$ . ■

Note that, being interested in  $l$ -niche hypergraphs, loops  $e = \{(i, j)\} \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2))$  could lead to the problem that  $\{(i, j)\}$  can be a loop in  $N\mathcal{H}^l(D_1 + D_2)$  either because of  $\{i\} \in \mathcal{E}(N\mathcal{H}^l(D_1))$  and  $j$  is an isolate in  $D_2$  or because of  $i$  is an isolate in  $D_1$  and  $\{j\} \in \mathcal{E}(N\mathcal{H}^l(D_2))$  — and without further information it cannot be decided which of these cases occurs.

In comparison with Proposition 2(4) of our paper [20] we see that for the reconstruction of the  $l$ -competition graphs  $C\mathcal{H}^l(D_1)$  and  $C\mathcal{H}^l(D_2)$  from  $C\mathcal{H}^l(D_1 + D_2)$  there is another sufficient condition, namely:

$$\exists e \in \mathcal{E}(C\mathcal{H}^l(D_1 + D_2)) : |\pi_1(e)| \geq 3 \wedge |\pi_2(e)| \geq 3.$$

**Remark 8.** In general, for niche hypergraphs an analogous condition to Proposition 2(4) in [20], i.e.,

$$(\alpha) \quad \exists e \in \mathcal{E}(N\mathcal{H}^l(D_1 + D_2)) : |\pi_1(e)| \geq 3 \wedge |\pi_2(e)| \geq 3$$

is unsuited to ensure that  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$  can be reconstructed from  $N\mathcal{H}^l(D_1 + D_2)$ .

**Proof.** Without loss of generality, let  $e = \{(i, j_1), \dots, (i, j_k), (i_1, j), \dots, (i_l, j)\}$  be a hyperedge in  $N\mathcal{H}^l(D_1 + D_2)$  with  $k \geq 2$  and  $l \geq 2$ .

There are two possibilities for the hyperedge  $e$ , namely  $e = \begin{cases} N_{D_1+D_2}^-((i, j)) \\ N_{D_1+D_2}^+((i, j)) \end{cases}$ ,

i.e.,

$$\pi_1(e) \setminus \{i\} = \{i_1, \dots, i_l\} = \begin{cases} N_{D_1}^-(i) \\ N_{D_1}^+(i) \end{cases}, \text{ and}$$

$$\pi_2(e) \setminus \{j\} = \{j_1, \dots, j_k\} = \begin{cases} N_{D_2}^-(j) \\ N_{D_2}^+(j) \end{cases}.$$

Then we have  $e \in \mathcal{E}(\mathcal{CH}^l(D_1 + D_2))$ , which is equivalent to  $e = N_{D_1+D_2}^-((i, j))$ , or otherwise  $e \in \mathcal{E}(\mathcal{CEH}^l(D_1 + D_2))$ , i.e.,  $e = N_{D_1+D_2}^+((i, j))$ . In the first case it follows  $\pi_1(e) \setminus \{i\} = N_{D_1}^-(i)$  and  $\pi_2(e) \setminus \{j\} = N_{D_2}^-(j)$ , in the second case  $\pi_1(e) \setminus \{i\} = N_{D_1}^+(i)$  and  $\pi_2(e) \setminus \{j\} = N_{D_2}^+(j)$  is valid.

In both cases we obtain  $\pi_1(e) \setminus \{i\} \in \mathcal{E}(\mathcal{NH}^l(D_1))$  and  $\pi_2(e) \setminus \{j\} \in \mathcal{E}(\mathcal{NH}^l(D_2))$  and both sets  $\pi_1(e) \setminus \{i\}$  and  $\pi_2(e) \setminus \{j\}$  are hyperedges in the corresponding competition hypergraph  $\mathcal{CH}^l(D_\tau)$  ( $\tau \in \{1, 2\}$ ) or both are hyperedges in the common enemy hypergraph  $\mathcal{CEH}^l(D_\tau)$  ( $\tau \in \{1, 2\}$ ).

Our argumentation is the following.

- The above implies that, in this sense, "competition hyperedges"  $e \in \mathcal{E}(\mathcal{CH}^l(D_1 + D_2)) \subseteq \mathcal{E}(\mathcal{NH}^l(D_1 + D_2))$  include only information on "competition hyperedges" in  $\mathcal{E}(\mathcal{CH}^l(D_1)) \subseteq \mathcal{E}(\mathcal{NH}^l(D_1))$  and  $\mathcal{E}(\mathcal{CH}^l(D_2)) \subseteq \mathcal{E}(\mathcal{NH}^l(D_2))$ , respectively. The same applies to "common enemy hyperedges"  $e \in \mathcal{E}(\mathcal{CEH}^l(D_1 + D_2)) \subseteq \mathcal{E}(\mathcal{NH}^l(D_1 + D_2))$  and "common enemy hyperedges" in  $\mathcal{E}(\mathcal{CEH}^l(D_1)) \subseteq \mathcal{E}(\mathcal{NH}^l(D_1))$  and  $\mathcal{E}(\mathcal{CEH}^l(D_2)) \subseteq \mathcal{E}(\mathcal{NH}^l(D_2))$ .

- Below, we will describe the reconstruction of the hyperedges of  $\mathcal{CH}^l(D_1)$  and  $\mathcal{CH}^l(D_2)$  from  $\mathcal{CH}^l(D_1 + D_2)$  according to Case 4 of the proof of Proposition 2 in [20]. We will see that in this reconstruction procedure the conditions  $|\pi_1(e)| \geq 3$  and  $|\pi_2(e)| \geq 3$  (for a certain hyperedge  $e \in \mathcal{E}(\mathcal{CH}^l(D_1 + D_2))$ ) are essential. Obviously, an analog reconstruction procedure can be used to obtain  $\mathcal{CEH}^l(D_1)$  and  $\mathcal{CEH}^l(D_2)$  from  $\mathcal{CEH}^l(D_1 + D_2)$ , if there is a hyperedge  $e \in \mathcal{E}(\mathcal{CEH}^l(D_1 + D_2))$  with  $|\pi_1(e)| \geq 3$  and  $|\pi_2(e)| \geq 3$ . Clearly, the described reconstruction will fail if there is no such hyperedge  $e$  with the required properties.

- Now let  $D_1$  and  $D_2$  be digraphs fulfilling  $(\alpha)$ . Note that, in general, for an arbitrarily chosen hyperedge  $e$  in  $\mathcal{NH}^l(D_1 + D_2)$  it cannot be found out whether  $e$  is a "competition hyperedge", i.e.,  $e \in \mathcal{E}(\mathcal{CH}^l(D_1 + D_2))$ , or a "common enemy hyperedge", i.e.,  $e \in \mathcal{E}(\mathcal{CEH}^l(D_1 + D_2))$ .

- We additionally assume that in  $\mathcal{NH}^l(D_1 + D_2)$  all hyperedges fulfilling  $(\alpha)$  are edges of the competition hypergraph  $\mathcal{CH}^l(D_1 + D_2)$  but not edges of the common enemy hypergraph  $\mathcal{CEH}^l(D_1 + D_2)$ . Then, clearly, the reconstruction method from Proposition 2 in [20] has to fail for hyperedges in  $\mathcal{E}(\mathcal{CEH}^l(D_2)) \setminus \mathcal{E}(\mathcal{CH}^l(D_2)) \subseteq \mathcal{E}(\mathcal{NH}^l(D_2))$ .

It remains to describe the reconstruction method from Case 4 of the proof of Proposition 2 in [20].

Under the assumptions given above, let  $e \in \mathcal{E}(\mathcal{NH}^l(D_1 + D_2))$  be a hyperedge with  $(\alpha)$ , i.e.,  $e \in \mathcal{E}(\mathcal{CH}^l(D_1 + D_2))$ . Because of  $|\pi_1(e)| \geq 3$  and  $|\pi_2(e)| \geq 3$ , there are vertices  $i \in V_1$  and  $j \in V_2$  with  $k := |\{(i, j') \mid j' \in V_2\} \cap e| \geq 2$  and  $l := |\{(i', j) \mid i' \in V_1\} \cap e| \geq 2$ .

Then  $e = \{(i, j_1), \dots, (i, j_k), (i_1, j), \dots, (i_l, j)\} = N_{D_1+D_2}^-((i, j))$  and therefore  $N_{D_1}^-(i) = \{i_1, \dots, i_l\} = \pi_1(e) \setminus \{i\}$  and  $N_{D_2}^-(j) = \{j_1, \dots, j_k\} = \pi_2(e) \setminus \{j\}$ .

For each  $x \in V_1$  let  $e^x := \{(x, j_1), \dots, (x, j_k), (x_1, j), \dots, (x_{l_x}, j)\} \in \mathcal{E}(C\mathcal{H}^l(D_1 + D_2))$  with  $l_x \geq 0$ . Obviously,  $e^x = N_{D_1+D_2}^-((x, j))$  and  $N_{D_1}^-(x) = \{x_1, \dots, x_{l_x}\} = \pi_1(e^x) \setminus \{x\}$ . This way we obtain  $D_1 = (V_1, A_1)$  as well as  $\mathcal{E}(C\mathcal{H}^l(D_1)) = \{N_{D_1}^-(x) \mid x \in V_1 \wedge N_{D_1}^-(x) \neq \emptyset\}$ .

Analogously, for each  $y \in V_2$  let  $e^y := \{(i_1, y), \dots, (i_l, y), (i, y_1), \dots, (i, y_{k_y})\} \in \mathcal{E}(C\mathcal{H}^l(D_1 + D_2))$  with  $k_y \geq 0$ . Then  $e^y = N_{D_1+D_2}^-((i, y))$  and  $N_{D_2}^-(y) = \{y_1, \dots, y_{k_y}\} = \pi_2(e^y) \setminus \{y\}$ . ■

**Theorem 9** (Normal product  $D_1 * D_2$ ).

- (a)  $N\mathcal{H}(D_1)$  and  $N\mathcal{H}(D_2)$  can be obtained from  $N\mathcal{H}(D_1 * D_2)$ .
- (b) If there is a hyperedge  $e \in \mathcal{E}(N\mathcal{H}(D_1 * D_2))$  with  $|\pi_1(e)| \geq 2$  and  $|\pi_2(e)| \geq 2$ , then  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$  can be obtained from  $N\mathcal{H}(D_1 * D_2)$ .

**Proof.** (b) The existence of a hyperedge  $e \in \mathcal{E}(N\mathcal{H}(D_1 * D_2))$  with  $|\pi_1(e)| \geq 2$  and  $|\pi_2(e)| \geq 2$  is equivalent to  $A_1 \neq \emptyset \neq A_2$ . Let

$$e = \{(i, j_1), \dots, (i, j_k), (i_1, j), \dots, (i_l, j), (i_1, j_1), (i_1, j_2), \dots, (i_1, j_k), \dots, (i_l, j_1), (i_l, j_2), \dots, (i_l, j_k)\} \\ \in \mathcal{E}(N\mathcal{H}(D_1 * D_2)) = \mathcal{E}(C\mathcal{H}(D_1 * D_2)) \cup \mathcal{E}(CE\mathcal{H}(D_1 * D_2)),$$

with  $|\pi_1(e)| \geq 2$  and  $|\pi_2(e)| \geq 2$ .

We will follow the idea of the proof of Case 2 of Corollary 2 in our paper [20], where a similar result for competition hypergraphs was given.

But by contrast to Corollary 2 in [20], in the case of niche hypergraphs it is impossible to reconstruct the digraphs  $D_1$  and  $D_2$  themselves in general. The reason is the same as mentioned before for the Cartesian sum (see the proof of Remark 8). Although for a hyperedge  $e \in \mathcal{E}(N\mathcal{H}(D_1 * D_2))$  we can find out the vertex  $(i, j)$  with  $e = N_{D_1 * D_2}^-((i, j))$  or  $e = N_{D_1 * D_2}^+((i, j))$ , in general it will be impossible to determine whether  $e$  is the set of predecessors ( $e$  is a "competition hyperedge") or the set of successors ( $e$  is a "common enemy hyperedge") of the vertex  $(i, j)$  in  $D_1 * D_2$ .

Note that, in spite of the distinction of cases below, it is unnecessary to know for the actual hyperedge  $e \in \mathcal{E}(N\mathcal{H}(D_1 * D_2))$  under investigation whether or not it is a "competition hyperedge" ( $e \in \mathcal{E}(C\mathcal{H}(D_1 * D_2))$ ) or it is an "common enemy hyperedge" ( $e \in \mathcal{E}(CE\mathcal{H}(D_1 * D_2))$ ). This will become clear by the remarks to Case (2) below.

*Case (1):*  $e \in \mathcal{E}(C\mathcal{H}(D_1 * D_2))$ . With some modifications of the proof of Case 2 of Corollary 2 in [20] we get the following.

(a) Because of  $l = |\pi_1(e)| - 1 \geq 1$  and  $k = |\pi_2(e)| - 1 \geq 1$ , the vertices  $i \in V_1$  and  $j \in V_2$  with  $N_{D_1 * D_2}^-(i, j) = e$  can be identified as the only vertices which occur exactly  $k$  and  $l$  times in  $\pi_1(e)$  and  $\pi_2(e)$ , respectively. Moreover,  $\pi_1(e) \setminus \{i\} = \{i_1, \dots, i_l\} = N_{D_1}^-(i)$  and  $\pi_2(e) \setminus \{j\} = \{j_1, \dots, j_k\} = N_{D_2}^-(j)$ .

(b) Obviously, for every  $x \in V_1$  with  $N_{D_1}^-(x) \neq \emptyset$  in  $N_{D_1 * D_2}^-((x, j))$  there are at least 3 vertices:  $(x, j_1), (x', j), (x', j_1)$ , where  $x' \in N_{D_1}^-(x)$ . Therefore  $N_{D_1 * D_2}^-((x, j)) \in \mathcal{E}(\mathcal{CH}(D_1 * D_2)) \subseteq \mathcal{E}(\mathcal{NH}(D_1 * D_2))$ . Analogously, for each  $y \in V_2$  with  $N_{D_2}^-(y) \neq \emptyset$  we get  $N_{D_1 * D_2}^-((i, y)) \in \mathcal{E}(\mathcal{CH}(D_1 * D_2)) \subseteq \mathcal{E}(\mathcal{NH}(D_1 * D_2))$ .

(c) Note that if  $x \in V_1$  with  $N_{D_1}^-(x) = \emptyset$ , then  $N_{D_1 * D_2}^-((x, j)) = \{(x, j_1), \dots, (x, j_k)\}$ ; i.e.,  $N_{D_1 * D_2}^-((x, j)) \in \mathcal{E}(\mathcal{CH}(D_1 * D_2)) \subseteq \mathcal{E}(\mathcal{NH}(D_1 * D_2))$  if and only if  $k \geq 2$ . Analogously, for every  $y \in V_2$  with  $N_{D_2}^-(y) = \emptyset$  it follows  $N_{D_1 * D_2}^-((i, y)) \in \mathcal{E}(\mathcal{CH}(D_1 * D_2)) \subseteq \mathcal{E}(\mathcal{NH}(D_1 * D_2))$  if and only if  $l \geq 2$ .

Because of (b), for all vertices of  $D_1$  and  $D_2$ , respectively, with positive indegree we get their sets of predecessors applying the procedure described in (a) to all hyperedges  $e \in \mathcal{E}(\mathcal{CH}(D_1 * D_2))$  with  $|\pi_1(e)| \geq 2$  and  $|\pi_2(e)| \geq 2$ . (In general, for a vertex  $v_1 \in V_1$  and  $v_2 \in V_2$ , respectively, with positive indegree, procedure (a) will produce its set of predecessors more than once.) Trivially, each vertex for which (a) does not provide a set of predecessors has indegree 0 (cf. (c)).

Thus we obtain the edge set  $\mathcal{E}(\mathcal{CH}^l(D_1))$  and  $\mathcal{E}(\mathcal{CH}^l(D_2))$  of the  $l$ -competition hypergraph  $\mathcal{CH}^l(D_1)$  and  $\mathcal{CH}^l(D_2)$ , respectively.

Note that we did not need hyperedges  $e \in \mathcal{E}(\mathcal{CH}^l(D_1 * D_2)) \setminus \mathcal{E}(\mathcal{CH}(D_1 * D_2))$ , i.e., hyperedges of cardinality 1.

*Case (2):*  $e \in \mathcal{E}(\mathcal{CEH}(D_1 * D_2))$ . Note that  $\mathcal{CH}(D) = \mathcal{CEH}(\overleftarrow{D})$ , for any digraph  $D$ . Applying the following substitutions to the proof of Case (1), word-for-word we obtain the verification of Case (2):

$$\begin{array}{lll} \mathcal{CH} & \hookrightarrow & \mathcal{CEH}, \\ N^- & \hookrightarrow & N^+, \\ \text{indegree} & \hookrightarrow & \text{outdegree} \quad \text{and} \\ \text{predecessor} & \hookrightarrow & \text{successor}. \end{array}$$

(a) Because of (b) it suffices to consider the case when  $A_1 = \emptyset$  or  $A_2 = \emptyset$  holds. Replacing "+" by "\*" in (1)–(3) of Theorem 7, we see that the occurrence of (1), (2) or (3) is equivalent to  $A_1 = \emptyset$  or  $A_2 = \emptyset$  and we can use an analog argumentation as in the corresponding part of the proof of Theorem 7. So using (2) we obtain  $\mathcal{E}(\mathcal{NH}^l(D_2)) = \{\pi_2(e) \mid e \in \mathcal{E}(\mathcal{NH}^l(D_1 * D_2))\}$  and  $\mathcal{E}(\mathcal{NH}(D_2)) = \{\pi_2(e) \mid e \in \mathcal{E}(\mathcal{NH}^l(D_1 * D_2)) \wedge |\pi_2(e)| \geq 2\}$ , respectively. ■

Note that  $A_1 = \emptyset$  or  $A_2 = \emptyset$  implies  $D_1 * D_2 = D_1 + D_2$ . Therefore, the last part of the above proof in connection with Theorem 7 lead to the following consequence.

**Corollary 10.**  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$  can be obtained from  $N\mathcal{H}^l(D_1 * D_2)$ , provided that one of the following conditions is true:

- (1)  $\mathcal{E}(N\mathcal{H}^l(D_1 * D_2)) = \emptyset$ ;
- (2)  $\forall e \in \mathcal{E}(N\mathcal{H}^l(D_1 * D_2)) : |\pi_1(e)| = 1$  and  $\exists e \in \mathcal{E}(N\mathcal{H}^l(D_1 * D_2)) : |\pi_2(e)| \geq 2$ ;
- (3)  $\forall e \in \mathcal{E}(N\mathcal{H}^l(D_1 * D_2)) : |\pi_2(e)| = 1$  and  $\exists e \in \mathcal{E}(N\mathcal{H}^l(D_1 * D_2)) : |\pi_1(e)| \geq 2$ .

**Theorem 11** (Lexicographic product  $D_1 \cdot D_2$ ).

- (a)  $N\mathcal{H}(D_1)$  and  $N\mathcal{H}(D_2)$  can be obtained from  $N\mathcal{H}(D_1 \cdot D_2)$ .
- (b) If  $|V_2| \geq 2$ , then  $N\mathcal{H}^l(D_1)$  can be obtained from  $N\mathcal{H}(D_1 \cdot D_2)$ .
- (c)  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$  can be obtained from  $N\mathcal{H}^l(D_1 \cdot D_2)$ .

**Proof.** First we will show (c), i.e.,  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$  can be reconstructed from  $N\mathcal{H}^l(D_1 \cdot D_2)$ . Then we obtain (b) and (a) as follows:

Since for  $|V_2| \geq 2$  every loop  $e_1 = \{i\}$  in  $N\mathcal{H}^l(D_1)$  leads to a non-loop  $e$  in  $N\mathcal{H}^l(D_1 \cdot D_2)$  (containing at least all vertices of the row  $Z_i$ ), we will see that we need no loops of  $N\mathcal{H}^l(D_1 \cdot D_2)$  in order to obtain  $N\mathcal{H}^l(D_1)$ , this includes (b).

Analogously, it is obvious that non-loops  $e_i$  of  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$ , respectively, result in non-loops in  $N\mathcal{H}^l(D_1 \cdot D_2)$ . In our considerations it will become clear that for the reconstruction of  $N\mathcal{H}(D_1)$  and  $N\mathcal{H}(D_2)$  we do not need the loops in  $N\mathcal{H}^l(D_1 \cdot D_2)$ , so we get (a).

In order to prove (c), we consider a hyperedge  $e \in \mathcal{E}(N\mathcal{H}^l(D_1 \cdot D_2))$ . Then there is a vertex  $(i, j) \in V_1 \times V_2$  such that  $e = N_{D_1 \cdot D_2}^-((i, j))$  or  $e = N_{D_1 \cdot D_2}^+((i, j))$ . In order to simplify our depictions, we write down the considerations only for the case  $e = N_{D_1 \cdot D_2}^-((i, j)) \in \mathcal{E}(N\mathcal{H}^l(D_1 \cdot D_2))$ ; the hyperedges  $e = N_{D_1 \cdot D_2}^+((i, j)) \in \mathcal{E}(N\mathcal{H}^l(D_1 \cdot D_2))$  can be treated analogously.

In  $N\mathcal{H}^l(D_1 \cdot D_2)$  there are two possibilities for the hyperedge  $e$ .

*Case 1.*  $\exists l \geq 1 \exists i_1, \dots, i_l \in V_1 : e = Z_{i_1} \cup \dots \cup Z_{i_l}$ . Without loss of generality let  $i_1, \dots, i_l$  be pairwise distinct.

Hence,  $e$  is the union of the complete rows  $Z_{i_1}, \dots, Z_{i_l}$  of  $D_1 \cdot D_2$  and from the definition of  $D_1 \cdot D_2$  it follows  $i \notin \{i_1, \dots, i_l\}$ ,  $N_{D_1}^-(i) = \{i_1, \dots, i_l\}$  and  $N_{D_2}^-(j) = \emptyset$ .

Therefore, Case 1 does not provide any hyperedges of  $N\mathcal{H}^l(D_2)$  but with  $\pi_1(e) = \{i_1, \dots, i_l\} = N_{D_1}^-(i) \in \mathcal{E}(N\mathcal{H}^l(D_1))$  we obtain a hyperedge of  $N\mathcal{H}^l(D_1)$ .

Note that the vertex  $i \in V_1$  is unknown if  $l < |V_1| - 1$ . Moreover, Case 1 occurs if and only if there exists a vertex  $j \in V_2$  with  $N_{D_2}^-(j) = \emptyset$ .

*Case 2.*  $\exists l \geq 0 \exists i_1, \dots, i_l, i' \in V_1 \exists Z' \subset Z_{i'} : e = Z_{i_1} \cup \dots \cup Z_{i_l} \cup Z' \wedge Z' \neq \emptyset$ . We get  $i = i' \in V_1 \setminus \{i_1, \dots, i_l\}$  as well as  $N_{D_1}^-(i') = \{i_1, \dots, i_l\} = \pi_1(e) \setminus \{i'\} \in$

$\mathcal{E}(N\mathcal{H}^l(D_1))$  and  $N_{D_2}^-(j) = \pi_2(e \cap Z') = \pi_2(Z') \in \mathcal{E}(N\mathcal{H}^l(D_2))$  with a certain  $j \in V_2$ . In general, if  $|Z'| < |V_2| - 1$  holds, the vertex  $j$  cannot be determined.

Again, for any hyperedge  $e \in \mathcal{E}(N\mathcal{H}^l(D_1 \cdot D_2))$  it cannot be found out whether  $e$  is a *competition hyperedge* (i.e.,  $e \in \mathcal{E}(C\mathcal{H}^l(D_1 \cdot D_2))$ ) or  $e$  is a *common enemy hyperedge* (i.e.,  $e \in \mathcal{E}(CE\mathcal{H}^l(D_1 \cdot D_2))$ ) in general. But for the reconstruction of  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$  this plays no role, since the considerations of Case 1 and Case 2 are valid for competition hyperedges (i.e., sets of predecessors) as well as, analogously, for common enemy hyperedges (i.e., sets of successors).

Moreover, we remark that Cases 1 and 2 (together with their analogs for the common enemy hyperedges) provide all hyperedges of the ( $l$ )-niche hypergraphs  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$ . ■

Now we discuss the disjunction  $D_1 \vee D_2$ . The case  $|V_1| = 1$  or  $|V_2| = 1$  implies  $D_1 \vee D_2 = D_1 \cdot D_2$ . Therefore, because of Theorem 11 it suffices to investigate the case  $|V_1|, |V_2| \geq 2$ .

**Theorem 12** (Disjunction  $D_1 \vee D_2$ ). *If  $|V_1|, |V_2| \geq 2$ , then  $N\mathcal{H}^l(D_1)$  and  $N\mathcal{H}^l(D_2)$  can be obtained from  $N\mathcal{H}(D_1 \vee D_2)$ .*

**Proof.** Since both  $V_1$  and  $V_2$  contain at least two vertices, in  $N\mathcal{H}^l(D_1 \vee D_2)$  there are no loops and  $N\mathcal{H}^l(D_1 \vee D_2) = N\mathcal{H}(D_1 \vee D_2)$ .

Moreover, for every hyperedge  $e \in \mathcal{E}(N\mathcal{H}(D_1 \vee D_2))$  it holds

$$\exists l \geq 0 \exists i_1, \dots, i_l \in V_1 \exists k \geq 0 \exists j_1, \dots, j_k \in V_2 : e = Z_{i_1} \cup \dots \cup Z_{i_l} \cup S_{j_1} \cup \dots \cup S_{j_k}$$

and, clearly,  $\min(l, k) > 0$ .

By analogy with the proof of Theorem 11 let  $(i, j) \in V_1 \times V_2$  be a vertex such that  $e = N_{D_1 \vee D_2}^-((i, j))$  or  $e = N_{D_1 \vee D_2}^+((i, j))$ . Now we follow the idea of the proof of Proposition 2 in [20], subsection 3.5, and use the abbreviations  $\mathcal{E}_1^l := \mathcal{E}(N\mathcal{H}^l(D_1))$ ,  $\mathcal{E}_2^l := \mathcal{E}(N\mathcal{H}^l(D_2))$  and  $\mathcal{E}_\vee := \mathcal{E}(N\mathcal{H}(D_1 \vee D_2))$ .

In case of  $\mathcal{E}_\vee = \emptyset$  both  $\mathcal{E}_1^l$  and  $\mathcal{E}_2^l$  are empty, too.

So let  $\mathcal{E}_\vee \neq \emptyset$ . Additionally, for an arbitrary hyperedge  $e \in \mathcal{E}_\vee$  we define  $\pi_1^j(e) := \{i \mid (i, j) \in e\}$  (for  $j \in \pi_2(e)$ ) and  $\pi_2^i(e) := \{j \mid (i, j) \in e\}$  (for  $i \in \pi_1(e)$ ).

In  $N\mathcal{H}(D_1 \vee D_2)$  we have three types of hyperedges:

$$\mathcal{A} := \{e \in \mathcal{E}_\vee \mid \pi_1(e) \subset V_1\},$$

$$\mathcal{B} := \{e \in \mathcal{E}_\vee \mid \pi_2(e) \subset V_2\} \text{ and}$$

$$\mathcal{C} := \{e \in \mathcal{E}_\vee \mid \pi_1(e) = V_1 \wedge \pi_2(e) = V_2\}.$$

We obtain

$$\mathcal{A} = \mathcal{C} = \emptyset \text{ if and only if } A_1 = \emptyset, \mathcal{E}_1^l = \emptyset \text{ and } \mathcal{E}_2^l = \{\pi_2(e) \mid e \in \mathcal{E}_\vee\};$$

$$\mathcal{B} = \mathcal{C} = \emptyset \text{ if and only if } A_2 = \emptyset, \mathcal{E}_2 = \emptyset \text{ and } \mathcal{E}_1^l = \{\pi_1(e) \mid e \in \mathcal{E}_\vee\};$$

$$\mathcal{C} \neq \emptyset \text{ if and only if } A_1 \neq \emptyset \neq A_2.$$

It remains to investigate the case  $\mathcal{C} \neq \emptyset$ . Here we see that, to determine  $\mathcal{E}_1^l$  and  $\mathcal{E}_2^l$ , it suffices to make use of the hyperedges in  $\mathcal{C}$ :

$$\mathcal{E}_1^l = \{ \{i \in V_1 \mid \pi_2^i(e) = V_2\} \mid e \in \mathcal{C} \} \text{ and } \mathcal{E}_2^l = \{ \{j \in V_2 \mid \pi_1^j(e) = V_1\} \mid e \in \mathcal{C} \}.$$

(Note that in case  $\mathcal{A} \neq \emptyset$  we have  $\mathcal{E}_1^l = \{\pi_1(e) \mid e \in \mathcal{A}\}$  and, analogously, if  $\mathcal{B} \neq \emptyset$  it follows  $\mathcal{E}_2^l = \{\pi_2(e) \mid e \in \mathcal{B}\}$ .) ■

### Acknowledgements

The authors would like to thank the referees for their valuable suggestions and remarks.

### REFERENCES

- [1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications* (Springer, London, 2001).
- [2] C. Cable, K.F. Jones, J.R. Lundgren and S. Seager, *Niche graphs*, *Discrete Appl. Math.* **23** (1989) 231–241.  
doi:10.1016/0166-218X(89)90015-2
- [3] J.E. Cohen, *Interval graphs and food webs: a finding and a problem*, RAND Corp. Document 17696-PR (Santa Monica, CA, 1968).
- [4] M. Cozzens, *Food webs, competition graphs and habitat formation*, *Math. Model. Nat. Phenom.* **6** (2011) 22–38.  
doi:10.1051/mmnp/20116602
- [5] C. Garske, M. Sonntag and H.-M. Teichert, *Niche hypergraphs*, *Discuss. Math. Graph Theory* **36** (2016) 819–832.  
doi:10.7151/dmgt.1893
- [6] S.-R. Kim, *The competition number and its variants*, in: *Quo Vadis, Graph Theory?*, Ed(s) J. Gimbel, J.W. Kennedy and L.V. Quintas (*Ann. Discrete Math.* **55**, North Holland, Amsterdam, 1993).  
doi:10.1016/S0167-5060(08)70396-0
- [7] S.-R. Kim, B. Park and Y. Sano, *The competition number of the complement of a cycle*, *Discrete Appl. Math.* **161** (2013) 1755–1760.  
doi:10.1016/j.dam.2011.10.034
- [8] S.-R. Kim, B. Park and Y. Sano, *A generalization of Opsut’s result on the competition numbers of line graphs*, *Discrete Appl. Math.* **181** (2015) 152–159.  
doi:10.1016/j.dam.2014.10.014
- [9] J. Kuhl, *Transversals and competition numbers of complete multipartite graphs*, *Discrete Appl. Math.* **161** (2013) 435–440.  
doi:10.1016/j.dam.2012.09.012
- [10] B.-J. Li and G.J. Chang, *Competition numbers of complete  $r$ -partite graphs*, *Discrete Appl. Math.* **160** (2012) 2271–2276.  
doi:10.1016/j.dam.2012.05.005



- [11] J.R. Lundgren, *Food webs, competition graphs, competition—common enemy graphs and niche graphs*, in: Applications of Combinatorics and Graph Theory to the Biological and Social Sciences, Ed(s) F. Roberts (IMA **17**, Springer, New York 1989) 221–243.
- [12] B.D. McKey, P. Schweitzer and P. Schweitzer, *Competition numbers, quasi line graphs and holes*, SIAM J. Discrete Math. **28** (2014) 77–91.  
doi:10.1137/110856277
- [13] B. Park and Y. Sano, *On the hypercompetition numbers of hypergraphs*, Ars Combin. **100** (2011) 151–159.
- [14] B. Park and Y. Sano, *The competition numbers of ternary Hamming graphs*, Appl. Math. Lett. **24** (2011) 1608–1613.  
doi:10.1016/j.aml.2011.04.012
- [15] B. Park and Y. Sano, *The competition number of a generalized line graph is at most two*, Discrete Math. Theor. Comput. Sci. **14** (2012) 1–10.
- [16] J. Park and Y. Sano, *The double competition hypergraph of a digraph*, Discrete Appl. Math. **195** (2015) 110–113.  
doi:10.1016/j.dam.2014.04.001
- [17] Y. Sano, *On hypercompetition numbers of hypergraphs with maximum degree of at most two*, Discuss. Math. Graph Theory **35** (2015) 595–598.  
doi:10.7151/dmgt.1826
- [18] D.D. Scott, *The competition-common enemy graph of a digraph*, Discrete Appl. Math. **17** (1987) 269–280.  
doi:10.1016/0166-218X(87)90030-8
- [19] M. Sonntag and H.-M. Teichert, *Competition hypergraphs*, Discrete Appl. Math. **143** (2004) 324–329.  
doi:10.1016/j.dam.2004.02.010
- [20] M. Sonntag and H.-M. Teichert, *Competition hypergraphs of products of digraphs*, Graphs Combin. **25** (2009) 611–624.  
doi:10.1007/s00373-005-0868-9
- [21] M. Sonntag and H.-M. Teichert, *Products of digraphs and their competition graphs*, Discuss. Math. Graph Theory **36** (2016) 43–58.  
doi:10.7151/dmgt.1851

Received 14 March 2017

Revised 13 March 2018

Accepted 13 March 2018